Stable modal logics

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Introduction and Motivation

- Filtration and selective filtration are two main tools to establish the finite model property (fmp) in modal logic.
- It is a well-known result of Fine that transitive subframe logics are well-behaved with respect to selective filtration, and hence have the fmp.
- The recently introduced stable logics[1] behave well with respect to filtration, and hence provide another class of logics with the fmp.

In this talk, we continue the study of stable modal logics.

[1] [1] G. and N. Bezhanishvili, R. Iemhoff, Stable canonical rules, submitted.

Outline

Stable modal logics over K

- Let $\mathfrak{B} = (B, \Diamond_B)$ and $\mathfrak{A} = (A, \Diamond_A)$ be modal algebras. We say that $\mathfrak{B}$ is a *stable subalgebra* of $\mathfrak{A}$ if B is a Boolean subalgebra of A and

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Definition

$L \in \mathrm{NExt}(\mathbf{K})$ is called *stable* if $\mathcal{V}_L = \mathcal{V}(\mathcal{K})$ for some stable class $\mathcal{K}$.

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- For $a \in A'$ let $\Diamond' a := \bigwedge\{b \in A' \mid \Diamond a \leq b\}$. Then $\mathfrak{A}' = (A', \Diamond')$ is a stable subalgebra of $\mathfrak{A}$. Since $\Diamond a = \Diamond' a$ for all $\Diamond a \in V(\mathrm{Sub}(\varphi))$, V witnesses that $\mathfrak{A}' \not\models \varphi$.

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- Moreover, $\mathfrak{A}' \in \mathcal{K}$ since $\mathcal{K}$ is stable. So $\mathfrak{A}' \models L$.

Recap: logics axiomatized by rules

- A modal multi-conclusion rule is an expression Γ/Δ, where Γ and Δ are finite sets of modal formulas.
- $\mathfrak{A} = (A, \Diamond) \models \Gamma/\Delta$, provided for every valuation $V : \mathrm{Prop} \to A$, if $V(\varphi) = 1$ for all $\varphi \in \Gamma$, then there is $\psi \in \Delta$ with $V(\psi) = 1$.
- A set Σ of multi-conclusion rules axiomatizes a universal class of model algebras that we denote by $\mathcal{U}_\Sigma$.
- A multi-conclusion consequence relation Σ axiomatizes a logic L iff $\mathcal{V}_L = \mathcal{V}(\mathcal{U}_\Sigma)$.

Stable rules

- Let $\mathfrak{A} = (A, \Diamond)$ be a finite modal algebra. For every $a \in A$ let p_a be a propositional letter. The *stable rule associated with* $\mathfrak{A}$ is $\rho(\mathfrak{A}) = \Gamma/\Delta$, where

$$\begin{aligned}\Gamma &= \{p_{a \vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \cup \\ &\{p_{\neg a} \leftrightarrow \neg p_a : a \in A\} \cup \\ &\{\Diamond p_a \to p_{\Diamond a} : a \in A\},\end{aligned}$$

$$\text{and} \quad \Delta = \{p_a : a \neq 1\}.$$

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$$\text{and} \quad \Delta = \{p_a : a \neq 1\}.$$

Theorem

Let $\mathfrak{B} = (B, \lozenge)$ be a modal algebra.

$$\mathfrak{B} \not\models \rho(\mathfrak{A}) \quad \textit{iff} \quad \mathfrak{A} \textit{ is a stable subalgebra of } \mathfrak{B}.$$

Characterization of stable modal logics

Theorem

Let $L \in \mathrm{NExt}(\mathbf{K})$ *and let* $\mathcal{V}_L$ *be its corresponding variety of modal algebras. TFAE:*

1. L *is stable, i.e.* $\mathcal{V}_L$ *is generated by a stable class.*
2. $\mathcal{V}_L$ *is generated by a stable universal class.*
3. L *is axiomatizable by stable rules.*

Examples of stable logics and axiomatization

Recall that:

- $\mathbf{KD} = \mathbf{K} + \Box p \to \Diamond p$ is the logic of serial frames;
- $\mathbf{KT} = \mathbf{K} + p \to \Diamond p$ is the logic of reflexive frames.
- $\mathbf{S5} = \mathbf{KT} + \Diamond p \to \Box \Diamond p$ is the logic of frames with an equivalence relation.

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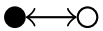
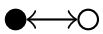
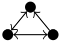
These logics are stable. In fact,

- **KD** is axiomatizable by $\rho(\bullet)$ and $\rho($  $)$,
- **KT** is axiomatizable by $\rho(\bullet)$ and $\rho($  $)$,
- **S5** is axiomatizable by $\rho(\bullet)$, $\rho($  $)$, $\rho($  $)$ and $\rho($  $)$

($\bullet$ depicts an irreflexive point, $\circ$ depicts a reflexive point and $\rho(\mathfrak{F})$ for a frame $\mathfrak{F}$ stands for the stable rule associated with the dual algebra of $\mathfrak{F}$.)

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Proposition

K4-*stable logics have the fmp.*

Stable formulas

- For a formula φ let $\Box^+\varphi := \Box\varphi \wedge \varphi$.
- Let $\mathfrak{A} = (A, \Diamond)$ be a finite subdirectly irreducible (s.i.) **K4**-algebra. The *stable formula* associated with $\mathfrak{A}$ is defined as

$$\gamma(\mathfrak{A}) := \bigwedge\{\Box^+\gamma \mid \gamma \in \Gamma\} \to \bigvee\{\Box^+\delta \mid \delta \in \Delta\},$$

where $\rho(\mathfrak{A}) = \Gamma/\Delta$.

Theorem

For every **K4***-algebra* $\mathfrak{B} = (B, \Diamond)$ *TFAE:*

1. $\mathfrak{B} \not\models \gamma(\mathfrak{A})$.
2. *There is a s.i. homomorphic image* $\mathfrak{C} = (C, \Diamond)$ *of* $\mathfrak{B}$ *such that* $\mathfrak{A}$ *is a stable subalgebra of* $\mathfrak{C}$.

Characterization of **K4**-stable logics

A **K4**-algebra $\mathfrak{A}$ is called *well-connected* if $\Box^+ a \vee \Box^+ b = 1$ implies $a = 1$ or $b = 1$ for all $a, b \in \mathfrak{A}$.

Theorem

Let $L \in \mathrm{NExt}(\mathbf{K4})$, let $\mathcal{V}_L$ be the variety corresponding to L and let $\mathcal{V}_{\mathrm{wc}}$ be the well-connected members of $\mathcal{V}_L$. TFAE:

1. *L is* **K4***-stable, i.e. $\mathcal{V}_L$ is generated by a* **K4***-stable class.*
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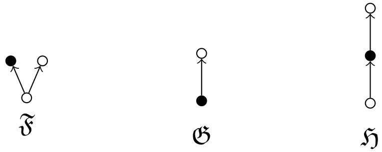
*Moreover, every **K4**-stable logic is axiomatizable by stable formulas.*

- However, not every logic $L \in \mathrm{NExt}(\mathbf{K4})$ axiomatized by stable formulas is stable. Consider:



Claim

$\mathbf{K4} + \gamma(\mathfrak{G}^*)$ *is not* **K4***-stable.*

$\mathfrak{F}^* \models \gamma(\mathfrak{G}^*)$, $\mathfrak{H}$ is a stable image of $\mathfrak{F}$, but $\mathfrak{H}^* \not\models \gamma(\mathfrak{G}^*)$.

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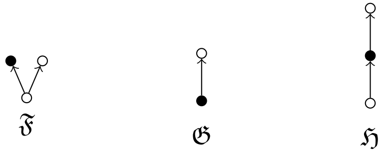

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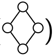
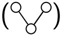
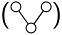
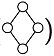
Proposition

1. *Suppose* $L = \mathbf{K4} + \{\gamma(\mathfrak{A}_i) \mid i \in I\}$ *such that for all* $i \in I$, $\mathfrak{A}_{i*}$ *has a reflexive root. Then* L *is* **K4***-stable.*
2. $L \in \mathrm{NExt}(\mathbf{S4})$ *is stable iff it is axiomatizable by stable formulas of* **S4***-algebras.*

Examples of **K4** and **S4**-stable modal logics

1. **D4** = **K4** + $\gamma(\bullet)$;
2. **K4.2** = **K4** + $\gamma($$)$;
3. **K4.3** = **K4** + $\gamma($$)$ + $\gamma($$)$;
4. **K4B** = **K4** + $\gamma($$)$;
5. **S4** = **K4** + $\gamma(\bullet)$ + $\gamma($$)$;
6. **S4.2** = **S4** + $\gamma($$)$;
7. **S4.3** = **S4** + $\gamma($$)$ + $\gamma($$)$;
8. **S5** = **S4** + $\gamma($$)$.

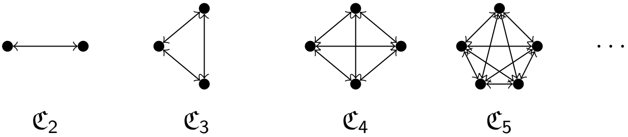
A continuum of stable logics

Theorem

1. *There is a continuum of non-transitive stable logics.*
2. *There is a continuum of* **K4***-stable logics between* **K4** *and* **S4***.*
3. *There is a continuum of* **S4***-stable logics.*

A continuum of stable logics

- Consider the following sequences of "clusters with irreflexive points"



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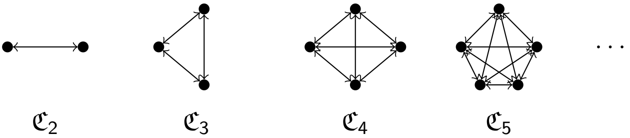

- For $I \subseteq \mathbb{N}_{\geq 2}$

$$\mathcal{X}_I := \{\mathfrak{A} = (A, \Diamond) \mid \exists n \in I \text{ and } \mathfrak{A} \text{ is a stable subalgebra of } \mathfrak{C}_n^*\}$$

- Since $\mathcal{X}_I$ is a stable class, $\mathcal{V}(\mathcal{X}_I)$ is the variety of a stable logic L_I.

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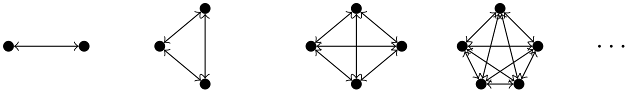

$\mathfrak{C}_2$ $\mathfrak{C}_3$ $\mathfrak{C}_4$ $\mathfrak{C}_5$ $\cdots$

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- Since $\mathcal{X}_I$ is a stable class, $\mathcal{V}(\mathcal{X}_I)$ is the variety of a stable logic L_I.

- **Claim**

  If $I \neq J \subseteq \mathbb{N}_{\geq 2}$, then $L_I \neq L_J$.

- The collection $L_I, I \subseteq \mathbb{N}_{\geq 2}$ provides a continuum of non-transitive stable logics.

Summary

- Stable logics form a large class of normal modal logics that are well-behaved with respect to the method of filtration.
- They play the same role in the theory of filtration as transitive subframe logics play in the theory of selective filtration.
- Stable logics over **K** can be axiomatized by stable rules. Stable logics over **K4** can be axiomatized by stable formulas.
- There is a continuum of non-transitive stable logics and a continuum of **K4**-stable logics.