Yet another ring-theoretic characterisation of *P*-frames

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Throughout, the term ring means a commutative ring with identity; and frame means a completely regular frame.

Let A be a ring. We denote by $\mathrm{Max}(A)$ the set of all maximal ideals of A. For any $a \in A$, we set

$$\mathfrak{M}(a) = \{M \in \mathrm{Max}(A) \mid a \in M\}.$$

The following definition comes from

G. Mason
z-Ideals and Prime Ideals
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An ideal I of a ring A is a z-ideal if

$$\mathfrak{M}(a) = \mathfrak{M}(b) \text{ and } a \in I \implies b \in I$$

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Examples

1. Every maximal ideal is a z-ideal.
2. Every minimal prime ideal is a z-ideal.
3. Intersections of z-ideals are z-ideals.
4. An ideal Q of $\mathcal{R}L$ is a z-ideal if and only if

$$\forall \alpha, \beta \in \mathcal{R}L: \ \operatorname{coz}\alpha = \operatorname{coz}\beta \ \text{and} \ \alpha \in Q \implies \beta \in Q.$$

Recall that a ring A is said to be:

- reduced (or semiprime) if it has no nonzero nilpotent elements. $\mathcal{R}L$ is reduced.
- von Neumann regular if, for every $a \in A$, there exists $b \in A$ such that $a = a^2 b$.

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An ideal I of a ring A is said to be:

- essential (or large) if it intersects every nonzero ideal nontrivially.
- a radical ideal if $I = \sqrt{I}$, where

$$\sqrt{I} = \{a \in A \mid a^n \in I \text{ for some } n \in \mathbb{N}\}.$$

  The ideal $\sqrt{I}$ is called the radical of I.

An ideal I of an ℓ-ring A is called:

- convex if, for any $a, b \in A$,

$$0 \leq a \leq b \text{ and } b \in I \implies a \in I.$$

- absolutely convex if, for any $a, b \in A$,

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Recall that a Tychonoff space X is called a *P*-space if every zero-set of X is open. These spaces have several algebraic characterizations in terms of their function rings. One of these is that:

X is a P-space if and only if $C(X)$ is von Neumann regular.

Extending this notion to pointfree topology, Ball and Walters-Wayland

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C- and C^*-quotients in pointfree topology
Dissert. Math. (Rozprawy Mat.) Vol. 412 (2002), 62pp.

define a frame L to be a P-frame if, for every $c \in \mathrm{Coz}\, L$, $c \vee c^* = 1$.

Clearly,

$$X \text{ is a } P\text{-space} \iff \mathfrak{O}X \text{ is a } P\text{-frame},$$

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Many ring-theoretic characterizations of *P*-spaces are preserved in the larger context of frames. For instance:

1. L is a P-frame iff $\mathcal{R}L$ is von Neumann regular (Banaschewski and Hong, 2003).
2. The following are equivalent for L (Dube, 2009):
   - L is a P-frame.
   - Every ideal of $\mathcal{R}L$ is pure.
   - $\langle \alpha, \beta \rangle = \langle \alpha^2 + \beta^2 \rangle$.

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There are properties of frames the proofs of whose ring-theoretic characterisations “piggyback" on their topological counterparts, via the fact that

$$\mathcal{R}^* L \cong \mathcal{R}(\beta L) \cong C(X),$$

for the Tychonoff space $X = \Sigma(\beta L)$. Typically, these properties are such that L has the property if and only if βL has the property. Being a *P*-frame is **not** one such, so frame-theoretic proofs need to be “built from scratch", so to speak.

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First, we introduce the following terminology. We say a ring A is z-good if it has the property that an ideal of A is a z-ideal if and only if its radical is a z-ideal.

Lemma

A z-good ring is von Neumann regular iff every prime ideal in it is a z-ideal.

So it remains to show that $\mathcal{R}L$ is z-good since L is a P-frame precisely when $\mathcal{R}L$ is von Neumann regular. For this we need the following lemma.

Lemma

Let L be a completely regular frame.

1. *Every positive element in $\mathcal{R}L$ has an n^{th} root, for every positive integer n.*
2. *Let $\alpha, \beta \in \mathcal{R}L$. If $|\alpha| \leq |\beta|^q$ for some $q > 1$ in $\mathbb{Q}$, then α is a multiple of β.*

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Proposition

$\mathcal{R}L$ is a z-good ring.

Proof.

(Outline)

- Let Q be an ideal of $\mathcal{R}L$ such that $\sqrt{Q}$ is a z-ideal.
- Suppose $\operatorname{coz}\beta \leq \operatorname{coz}\alpha$ for some $\alpha \in Q$ and $\beta \in \mathcal{R}L$.
- Write $\delta = \frac{\beta^+}{\mathbf{1}+\beta^+}$, and observe that $\mathbf{0} \leq \delta \leq \mathbf{1}$.
- For the function $\gamma = \sum_{n=1}^{\infty} 2^{-n}\delta^{1/n}$, we have

$$\operatorname{coz}\gamma = \bigvee_n \operatorname{coz}(2^{-n}\delta^{1/n}) = \operatorname{coz}\delta = \operatorname{coz}(\beta^+) \leq \operatorname{coz}\beta \leq \operatorname{coz}\alpha.$$

□

Proof.

(Outline continuation)

- After some algebraic calculations we obtain

$$2^{-4m^2}\delta \leq \gamma^{2m} = (\gamma^m)^2,$$

so that δ is a multiple of γ^m.

- This yields $\beta^+ \in Q$, and similarly, $\beta^- \in Q$, whence $\beta \in Q$.

□

We need one more preliminary result before stating the main theorem.

Lemma

Every radical ideal in $\mathcal{R}L$ is absolutely convex.

Putting together all the results above we arrive at:

Theorem

The following are equivalent for a completely regular frame L.

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