Uniform interpolation in modal logics

(Interpolation and proof systems)

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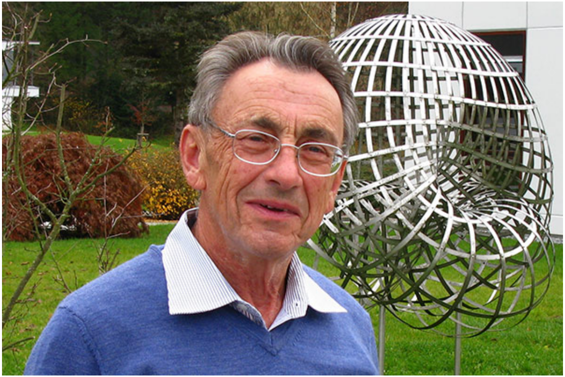
In memory of Grigori Mints (1939–2014)



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Thm (Mints, Olkhovikov, Urquhart '13)
The intermediate logic of constant domains does not have interpolation.

Dfn *A logic has uniform interpolation if the interpolant depends only on the premiss or the conclusion: For all* φ *there are formulas* $\exists p\varphi$ *and* $\forall p\varphi$ *in* $\mathcal{L}(\varphi)$ *not containing p such that for all* χ *not containing p:*

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$\exists p\varphi$ is the *post* or *right* (uniform) interpolant, $\forall p\varphi$ the *pre* or *left* interpolant.

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Show that logics without uniform interpolation cannot have calculi of that form.

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Side benefit: Modular proofs of uniform interpolation.

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Pitts uses Dyckhoff's '92 sequent calculus for IPC.

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of a rule, define the formula $\forall^{R}_{p} S_0$ *in terms of* $\forall p S_i$ $(i > 0)$.

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Likewise for $\exists$.

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where S, S_i *are sequents and* S_0 *contains exactly one formula, which is not an atom.* $(\Gamma \Rightarrow \Delta) \cdot (\Pi \Rightarrow \Sigma) = (\Gamma, \Pi \Rightarrow \Delta, \Sigma)$

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Dfn *A calculus is terminating if there exists a well-founded order $\prec$ on sequents such that in every rule the premisses come before the conclusion, and ...*

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Cor Except for the seven intermediate logics that have interpolation, no intermediate logic has a terminating sequent calculus that consists of (modal) focussed axioms and rules.

Questions

- *Extend the method to other modal logics, such as GL, KT, iGL.*
- *Find alternatives to uniform interpolation to obtain similar results.*
- *Extend the method to predicate theories.*
- *Use other proof systems than sequent calculi.*

Finis