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The logic of quasi true

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The Nature of Truth

We offer some logics as suitable tools to formalize the fuzzy concept of *quasi true*.

These logics are special extensions of the infinite-valued propositional calculus of Lukasiewicz.

The Nature of Truth

Logical truth is one of the most fundamental concept in logic, and there are different theories on its nature. Logical truths (including tautologies) are truths which are considered to be necessarily true. A logical truth is considered by some philosophers to be a statement which is true in all possible worlds.

The Nature of Truth

Considering different interpretations of the same statement leads to the notion of *truth value*.

Simplest approach to truth values means that the statement may be "true" in one case, but “false” in another.

Aristotle
(384-322 BCE)

As it is known for me, from the logical point of view, the concept of ***truth*, by its essence, was appeared in Aristotle's logical work, **On Interpretation**, where he considers the use of predicates in combination with subjects to form *propositions* or *assertions*, each of which is either **true** or **false**.

The Nature of Truth

We usually determine the truth of a proposition by reference to our experience of the reality it conveys,

but Aristotle recognized that special difficulties arise in certain circumstances.

The Nature of Truth

The problem concerning to the concept of truth has been arisen a long time ago. This problem was presented as a question: *What is truth?*

This question is related with various branches both in our reality, having different sense, and on the level of ideas. The question - What is truth? also known as Pilate Pontius question to Jesus Christ. Jesus promised: *"Seek and you will find".*

Humanly speaking, let us define truth , while waiting for a better definition,

as- a statement of the facts as they are.

Voltaire

Lukasiewicz Logic
MV-algebras

The unit interval of real numbers [0, 1] endowed with the following operations:
$x \oplus y = \min(1, x + y)$, $x \otimes y = \max(0, x + y - 1)$,
$x^* = 1 - x$, becomes an *MV* –algebra
$S=([0, 1], \oplus, \otimes, {}^*, 0, 1)$.

For $(0 \neq) m \in \omega$ we set

$S_m = (\{0, 1/m, \ldots, m\text{-}1/m, 1\}, \oplus, \otimes, {}^*, 0, 1)$.

An ***MV-algebra*** is an algebra

$$\mathrm{A} = (A, \oplus, \otimes, \neg, 0, 1),$$

where $(A, \oplus, 0)$ is an abelian monoid, and for all $x,y \in A$ the following identities hold:

$x \oplus 1 = 1, \ \neg\neg x = x,$

$\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x,$

$x \otimes y = \neg(\neg x \oplus \neg y).$

Lukasiewicz Logic

For Lukasiewicz propositional logic L we deal with sentences that can be evaluated with some truth value being in closed interval [0,1], or, roughly speaking, between true and false.

Lukasiewicz logic was originally defined in the early 20th-century by Jan Lukasiewicz as a three-valued logic. It was later generalized to *n*-valued (for all finite *n*) as well as infinitely-many valued variants, both propositional and first-order.

Lukasiewicz Logic

The original system of axioms for propositional infinite-valued Lukasiewicz logic used implication and negation as the primitive connectives as for classical logic:

- L_1. $(\alpha \to (\beta \to \alpha))$
- L_2. $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$
- L_3. $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$
- L_4. $(\neg \beta \to \neg \alpha) \to (\alpha \to \beta)$.

There is only one inference rule - *Modus Ponens*: from α and $(\alpha \to \beta)$, infer β.

Perfect *MV* -algebras

- ***Perfect*** *MV* -algebras are those *MV* –algebras generated by their infinitesimal elements or, equivalently, generated by their radical, where radical is the intersection of all maximal ideals, the radical of an MV-algebra, will be denoted by Rad(A).
- [**A. Di Nola, A. Lettieri**, *Perfect MV-algebras are Categorically Equivalent to Abelian ℓ-Groups*, Studia Logica, 88(1994), 467-490.]

Perfect *MV* -algebras

Let we have any *MV* -algebra. The least integer for which $nx = 1$ is called *the order of* x. When such an integer exists, we denote it by $ord(x)$ and say that x has finite order, otherwise we say that x has infinite order and write $ord(x) = \infty$.

An *MV*-algebra *A* is called *perfect* if for every nonzero element $x \in A$

$ord(x) = \infty$ if and only if $ord(\neg x) < \infty$.

Perfect MV -algebras do not form a variety and contains non-simple subdirectly irreducible MV–algebras. The variety generated by all perfect MV –algebras is also generated by a single MV -chain, actually the MV –algebra C, defined by Chang. The algebra C, with generator $c \in C$, is isomorphic to $\Gamma(Z \times_{lex} Z,(1, 0))$, with generator (0, 1). Let **MV(C)** be the variety generated by perfect algebras.

Perfect *MV* -algebras

Each perfect *MV*-algebra is associated with an abelian ℓ-group with a strong unit. Moreover,

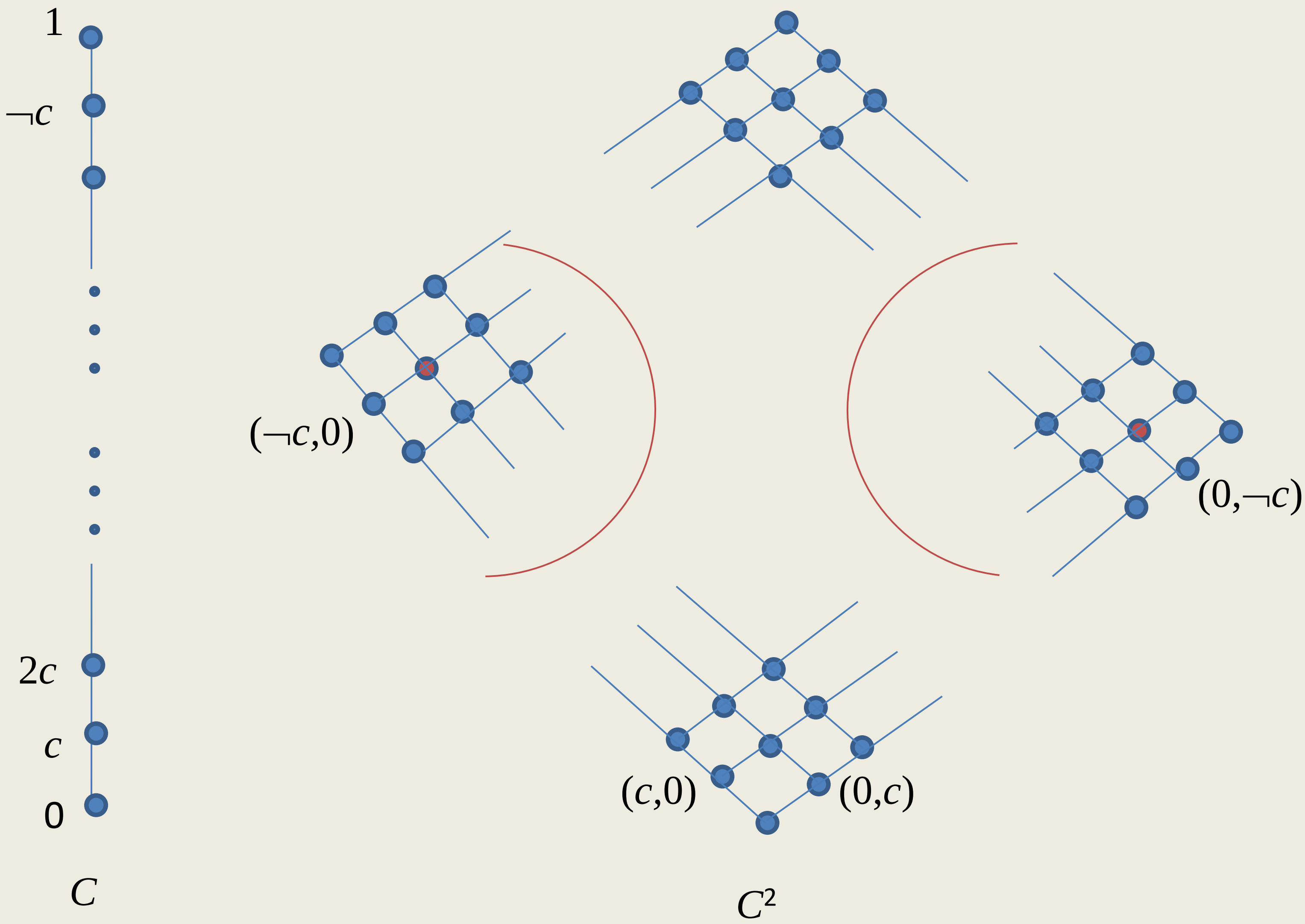
- ***the category of perfect MV–algebras is equivalent to the category of abelian ℓ-groups.***

The variety generated by all perfect *MV*-algebras, denoted by ***MV(C)***, *is also generated* by a single *MV*-chain, actually the *MV*-algebra *C*, defined by Chang.

Perfect *MV* -algebras

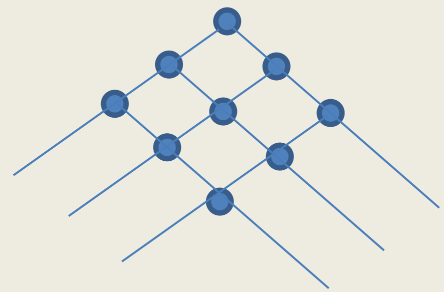
An important example of a perfect *MV*-algebra is the subalgebra *S of the Lindenbaum algebra L of the first or*der Lukasiewicz logic generated by the classes of formulas which are valid when interpreted in [0, 1] *but* non-provable.

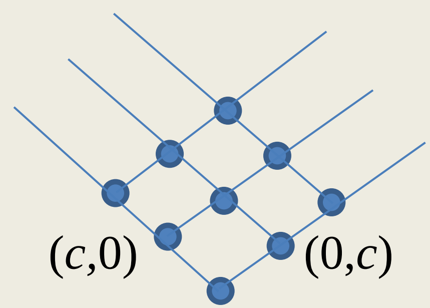
Hence perfect MV-algebras are directly connected with the very important phenomenon of incompleteness in Lukasiewicz first order logic .



1
¬c
2c
c
0
C
(¬c,0)
(0,¬c)
(c,0)
(0,c)
C²

Perfect *MV* -algebras





$\mathrm{Rad}(C^2) \cup \neg\mathrm{Rad}(C^2)$

Perfect *MV* -algebras

The *MV*-algebra *C* is the subdirectly irreducible *MV*-algebra with infinitesimals. It is generated by an atom *c*, which we can interpret as

a quasi false truth value.

The negation of *c* is

***a quasi true value*.**

Now quasi truth or quasi falsehood are vague concepts.

Perfect *MV* -algebras

About quasi truth in an MV algebra, it is reasonable to accept the following propositions:

- ***there are quasi true values which are not*** 1;
- 0 ***is not quasi true***;
- ***if x is quasi true, then*** x^2 ***is quasi true***

(where x^2 denotes the MV algebraic product of x with itself).

Perfect *MV* -algebras

In *C*, to satisfy these axioms it is enough to say that the quasi true values are the

co-infinitesimals.

Notice, that there is no notion of quasi truth in [0, 1] satisfying the previous axioms.

Perfect *MV* -algebras

Let L_{P} be the logic of perfect *MV*-algebras which coincides with the set of all Lukasiewicz formulas that are valid in all perfect *MV*-chains, or equivalently, that are valid in the *MV*-algebra *C*.

Actually, L_P is the logic obtained by adding to the axioms of Lukasiewicz sentential calculus the following axiom:

$$(x \veebar x)\ \&\ (x \veebar x) \leftrightarrow (x\ \&\ x) \veebar (x\ \&\ x)$$

[L. P. Belluce, A. Di Nola, B. Gerla, *Perfect MV -algebras and their Logic*, Applied Categorical Structures, Vol. 15, Num. 1-2 (2007), 35-151].

Notice, that the Lindenbaum algebra of L_{P} is an $MV(C)$-algebra.

Perfect *MV* -algebras

An *MV-algebra* is *MV(C)-algebra* if in addition holds

$$(2x)^2 = 2x^2 .$$

[**A. Di Nola, A. Lettieri**, *Perfect MV-algebras are Categorically Equivalent to Abelian ℓ-Groups*, Studia Logica, 88(1994), 467-490.]

The logic *CL*
CL-algebras

A *CL-algebra* is an algebra $A = (A, \oplus, \otimes, {}^*, c, 0, 1)$, where $(A; \oplus, \otimes, {}^*, 0, 1)$, is an MV-algebra and in addition satisfies the following axioms:

- 1) $2(x^2) = (2x)^2$,
- 2) $2c \otimes c^* = c$,
- 3) $c \otimes (x^* \vee x) \wedge (x \wedge x^*) = 0$,
- 4) $c \rightarrow c^* = 1$,
- 5) $(y^2 \oplus ((2y^*) \otimes (y \otimes x^*)) = 0) \Rightarrow (2x = 1)$,
- 6) $x \vee c^* = 1 \Rightarrow x = 1$.

Theorem 1. *The class* **CL** *is a quasivariety*.

Lemma 2. *Let (A, c) be a totally ordered CL-algebra. There is no element* $x \in A$ *such that* $nc < x < (n + 1)c$ *for any* $n \in Z^+$.

Theorem 1. *The class* **LC** *is a quasivariety*.

Lemma 2. *Let (A, c) be a totally ordered CL-algebra. There is no element $x \in A$ such that $nc < x < (n + 1)c$ for some $n \in Z^+$.*

Corollary 3. *The CL-algebra (C, c) is a subalgebra of every CL-algebra (A; c).*

CL-algebras

Theorem 4. *The quasivariety* **CL** *is generated by the algebra (C, c). Moreover,* **CL** *= SP(C, c), where S is the operator of taking a subalgebra and P is the operator of taking a direct product.*

Theorem 5. *Let us suppose that A is a MV (C)-algebra and (A, c) is a CL-extension of A. Such kind extension is unique.*

CL-logic

L1. $\alpha \to (\beta \to \alpha)$,
L2. $(\alpha \to \beta) \to (((\beta \to \gamma) \to (\alpha \to \gamma))$,
L3. $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$,
L4. $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$,
Lp. $2(\alpha^2) \leftrightarrow (2\alpha)^2$,
CL. $c \to \neg c$,
CL1. $2c \otimes \neg c \leftrightarrow c$,
CL2. $(c \to (\neg\alpha \wedge \alpha)) \vee (\neg\alpha \vee \alpha)$.

Inference rules:

MP. $\alpha, \alpha \to \beta \Rightarrow \beta$,
R1. $(\neg((\beta \to \neg\beta) \to \neg((\beta \to \neg\beta) \to (\beta \to \alpha))) \Rightarrow \neg\alpha \to \alpha$,
R2. $\alpha \vee \neg\alpha \Rightarrow \alpha$.

Theorem 6. $\vdash_{CL} 2(\neg c)^n$ *for any* $n \in Z^+$.

Theorem 6. $\vdash_{CL} 2(\neg c)^n$ *for any* $n \in Z^+$.

Theorem 7. (*Completeness theorem*).
α is *1-true iff* $\vdash_{\mathrm{CL}} \alpha$.

CL- logic

We say that α is *q-true* (or *q-tautology*) iff:

$\neg\alpha \to \alpha$ is a 1-true (or 1-tautology).

Semantically, we say that α is *q*-true if

$$e(\alpha) \in \neg \mathrm{Rad}\,(C)$$

for every evaluation

$$e : \mathrm{Var} \cup \{c\} \to (C, c).$$

Theorem 8. *If* α *is q-true, then*

$$\vdash_{CL} \neg\alpha \to \alpha .$$

Corollary 10. *If* α *is q-true, then*

$$\vdash_{CL} (\neg\alpha \vee \alpha) \to \alpha .$$

CL^{+}- logic

Now let us add to the inference rules of the logic *CL* the following rule:

R3. $(\alpha \vee \neg\alpha) \rightarrow \alpha) \Rightarrow \alpha$

and denote this new logic by CL^{+}.

Theorem 11. (Completeness).

If α *is q-true, then* $\vdash_{\mathrm{CL+}} \alpha$.

CL^{+}- logic

Theorem 11. (Completeness).

If is q-true, then $\vdash_{CL+} \alpha$.

Theorem 12. (Soundness).

If $\vdash_{CL+} \alpha$ *, then* α *is q-true.*

CL^+- logic

For every formula α of the logic CL^+ define its translation tr(α) into classical logic *Cl* as follows:

(1) if α is a propositional variable *p*, then tr(α) = α;
(2) tr(*c*) = $p \wedge \neg p$;
(3) tr($\alpha \to \beta$) = tr(α) $\to$ tr(β);
(4) tr($\neg\alpha$) = $\neg$tr(α).

CL^{+}- logic

Theorem 13. $\vdash_{CL+} \alpha$ *iff* $\vdash_{Cl}$ tr(α).

Theorem 14. If $\vdash_{Cl} \alpha$, then $\vdash_{CL+} \alpha$.

THANK YOU