Duality for sheaf representations and related decompositions of distributive lattice ordered algebras

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23 June 2015

Based on joint work with Sam van Gool and Anna Carla Russo

Direct products

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then

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$$\theta \cap \theta' = \Delta_A$$
- (Patch) $\forall a, b \in A \; \exists c \in A$ with $a\theta c$ and $b\theta' c$
or, equivalently,
$$\theta \circ \theta' = \nabla_A$$
(that is, $q_\theta \times q_{\theta'} : A/\theta \times A/\theta'$ is surjective)

Direct product decompositions of A

These correspond to pairs of factor congruences of A

- $\theta, \theta' \in Con(A)$
- $\theta \wedge \theta' = 0_{Con(A)}$ and $\theta \vee \theta' = 1_{Con(A)}$
  (complementary pair)
- $\theta \circ \theta' = \theta' \circ \theta$
  (permuting pair)

Very rarely are there enough factor congruences. A few cases:

- Finite Boolean algebras (BAs)
- Finitely generated Abelian groups

Lack of common refinement

Example: Klein four group $V = \mathbb{Z}_2 \times \mathbb{Z}_2$

It has three non-trivial proper subgroups:
$H_1 = \{(0,0),(0,1)\}, H_2 = \{(0,0),(1,0)\}, H_3 = \{(0,0),(1,1)\}$

Also

$$H_1 \times H_2 \cong V \cong H_1 \times H_3$$

but no common refinement of these two decompositions exists.

BA products have common refinement

B finite BA	$\longleftrightarrow$	X finite set	(Birkhoff duality)
$B \cong B_1 \times B_2$	$\longleftrightarrow$	$X = X_1 \dot\cup X_2$	
$B \cong B_3 \times B_4$	$\longleftrightarrow$	$X = X_3 \dot\cup X_4$	

Common refinement

$$B \cong B_{13} \times B_{14} \times B_{23} \times B_{24}$$

where B_{ij} is the dual of $X_i \cap X_j$

(BA = pure calculus of common refinement of direct products)

Product decompositions as given by elements

For BAs $B \cong B_1 \times B_2$

$\Longrightarrow$ $\exists\,!\; a \in B$ with $a \leftrightsquigarrow (1, 0)$

$\Longrightarrow$ $B_1 = {\downarrow} a$ and $B_2 = {\downarrow} \neg a$

The pairs of complementary elements of B are in one-to-one correspondence with direct product decompositions of B

(If the algebraic type has a (unary tuple) '0' and a (unary tuple) '1' with $0 = 1 \implies$ trivial algebra, then factor congruences are given by 'central' elements) [Vaggione]

Direct product decomposition of infinite BAs

The sheaf of direct factors of B

$$\begin{aligned} \Gamma : B^{op} &\longrightarrow \mathcal{BA} \\ a &\mapsto \ {\downarrow}a \end{aligned}$$

For $a \leq b$ the restriction map is given by ${\downarrow}b \to {\downarrow}a, x \mapsto a \wedge x$

(this corresponds to a sheaf over the dual space X of B; the patching property comes from the common refinement property)

Étale space incarnation: $p : E \to X$ local homeomorphism

$X =$ Stone dual space of B, $E = \bigcup_{x \in X} 2_x$

Generalization to abstract algebras

[Comer, Werner, Burris, Davey, Willard, Vaggione,...]

- For sheaf one needs $B \subseteq Con(A)$ relatively complemented distributive sublattice of pairwise permuting congruences
- BFC property: set of *all* factor congruences forms such a sublattice

Stone Representation Theorem: In a variety with BFC every algebra is representable as the global sections of a sheaf over a Boolean space with directly indecomposable stalks

(equalizers are clopen rather than just open iff for each $(a, b) \in A^2$ there is a least factor congruence θ with $(a, b) \in \theta$)

Sheaves with values in a variety $\mathcal{V}$

$\Gamma \colon \mathcal{O}(Y)^{\mathrm{op}} \to \mathcal{V}$ a functor satisfying a patching property

or

$p \colon E \to Y$ a local homeomorphism with each fiber in $\mathcal{V}$ continuously over Y

(From Γ to p)

- $E = \bigcup_{y \in Y} \{y\} \times A_y$
- $A_y = \varinjlim \{\Gamma(V) \to \Gamma(U) \mid U \subseteq V \text{ both in } \mathcal{N}(y)\}$
- topology on E induced by the $s \in \Gamma(U)$ viewed as sections of p

(From p to Γ)

$\Gamma(U) = \{s : U \to E \mid s \text{ continuous section of } p\}$

Sheaves over non-Hausdorff spaces

- are pertinent when we can't decompose as full products
- if $\mathcal{N}(y) \subseteq \mathcal{N}(y')$, then $s(y)$ determines $s(y')$ for any section s
- the algebra of global sections embeds in $\Pi_{y \in \min(Y)} A_y$ if $Y = \uparrow \min(Y)$
- a DL is NOT representable with each stalk the lattice 2 over its spectral dual (leads to the work with Anna Carla)

Stably compact spaces and compact ordered spaces

Stably compact spaces

- common generalisation of spectral spaces and compact Hausdorff spaces
- precisely the continuous retracts of spectral spaces

They are easier to describe via compact ordered spaces, which are $(Y, \tau, \leq)$, with

- (Y, τ) a compact Hausdorff space
- $\leq$ a partial order on Y
- $\leq \subseteq Y \times Y$ closed in the product topology

Stably compact spaces

Given a compact ordered space $(Y, \tau, \leq)$, let

$$\tau^{\uparrow} = \tau \cap \mathcal{U}(Y, \leq) \quad \text{and} \quad \tau^{\downarrow} = \tau \cap \mathcal{D}(Y, \leq)$$

Theorem

The spaces $Y^{\uparrow} = (Y, \tau^{\uparrow})$ for $(Y, \tau, \leq)$ a compact ordered space are precisely the stably compact spaces

Moving between $Y^{\uparrow}$ and $Y^{\downarrow}$

$\mathcal{O}$ = opens
$\mathcal{C}$ = closed
$\mathcal{S}$ = saturated = intersections of opens
$\mathcal{K}$ = compacts
co- = complements

$$\mathcal{S}(Y^{\uparrow}) = \text{co-}\mathcal{S}(Y^{\downarrow}) = \mathcal{U}(Y, \leq)$$

$$\mathcal{C}(Y^{\uparrow}) = \mathcal{K}\mathcal{S}(Y^{\downarrow})$$

In particular $Y^{\uparrow}$ and $Y^{\downarrow}$ are interdefinable without reference to $(Y, \tau, \leq)$

c-soft sheaf representations of DLs

A sheaf $\Gamma : \mathcal{O}(Y^{\uparrow}) \to \mathcal{DL}$ is a sheaf representation of a distributive lattice A provided

$$A \cong \Gamma(Y)$$

A sheaf representation of A is c-soft provided each section over a compact-saturated subset of $Y^{\uparrow}$ extends to a global section

Permuting congruences from c-soft representations

From a sheaf representation $\Gamma : \mathcal{O}(Y^{\uparrow}) \to \mathcal{DL}$ of A, we get a map

$$\theta_{(\_)} : \mathcal{KS}(Y^{\uparrow})^{\mathrm{op}} \to Con(A)$$

given by $a \, \theta_K \, b$ iff $s(a) \restriction K = s(b) \restriction K$

If Γ is c-soft then we get by patch that this map is a frame homomorphism and any two congruences in the image permute with each other

Duality for c-soft representations

Let $(X, \pi, \leq)$ be the Priestley space of A, i.e. (X, π) is the Stone space of the Booleanization A^- of A, then

$$Con(A) = Con(A^-) = \mathcal{C}(X, \pi)^{\mathrm{op}} \cong \mathcal{O}(X, \pi)$$

by Stone/Priestley duality and, as we've seen,

$$\mathcal{KS}(Y^{\uparrow})^{\mathrm{op}} = \mathcal{C}(Y^{\downarrow})^{\mathrm{op}} \cong \mathcal{O}(Y^{\downarrow})$$

Proposition

A c-soft sheaf representation $\Gamma : \mathcal{O}(Y^{\uparrow}) \to \mathcal{DL}$ of a distributive lattice A yields a frame homomorphism

$$\mathcal{O}(Y^{\downarrow}) \to \mathcal{O}(X, \pi)$$

or equivalently a continuous map $q \colon (X, \pi) \to Y^{\downarrow}$

Duality for permuting congruences

Proposition

Let A be a distributive lattice with Priestley dual $(X, \pi, \leq)$, and let θ_1 and θ_2 be congruences of A corresponding to the closed subsets X_1 and X_2 of X. Then the following are equivalent:

- *the congruences θ_1 and θ_2 permute;*
- *for every $x, x' \in X$ with one in X_1 and the other in X_2, if $x \leq x'$ then there exists $x'' \in X_1 \cap X_2$ such that $x \leq x'' \leq x'$.*

In this case, $X_1 \cap X_2$ is the closed subspace dual to the congruence $\theta_1 \vee \theta_2 = \theta_1 \circ \theta_2$.

Duality for c-soft sheaf representations

Let $(X, \pi, \leq)$ be a Priestley space. We say that $q : X \to Y^{\downarrow}$ is an *interpolating decomposition* if it is continuous and for all $x, x' \in X$ with $x \leq x'$, there exists $x'' \in X$ such that $x \leq x'' \leq x'$ and $q(x), q(x') \leq q(x'')$.

Theorem

Let A be a distributive lattice with dual Priestley space X, and let Y be a compact ordered space. There is a bijective correspondence between interpolating decompositions of X over $Y^{\downarrow}$ and isomorphism classes of c-soft sheaf representations of A over $Y^{\uparrow}$.

The map k and the Dubuc-Poveda representation

(Joint work with Sam van Gool and Vincenzo Marra)

Let A be an MV-algebra, X the Priestley space of its lattice reduct

$$A \to \mathbf{KCon}_{MV}(A), \quad a \mapsto <a>$$

is a bounded lattice homomorphism and thus the Priestley dual Y of $\mathbf{KCon}_{MV}(A)$ (the MV-spectrum) is a closed subspace of X.

- There is an interpolating decomposition $k : X \longrightarrow Y^{\downarrow}$

As a consequence any MV-algebra is representable as the global sections of a c-soft sheaf over $Y^{\uparrow}$ whose stalks are MV-chains

The map m and the Filipoiu-Georgescu representation

For A, X, and Y as above, let Z be the maximal points of Y. Because $\mathbf{KCon}_{MV}(A)$ is a hereditarily normal lattice, there is a continuous map

$$m : Y \longrightarrow Z, \; y \mapsto \text{ unique maximal point above } y$$

- $\mathbf{KCon}_{MV}(A)$ is representable over Z with directly indecomposable stalks
- A is representable as the global sections of a c-soft sheaf over Z whose stalks are local MV-algebras

Other decompositions over a Priestley space

Let A be a distributive lattice and $(X, \pi, \leq)$ its dual space. Let $Y = (X, \pi)$

$$id_X : (X, \pi) \to Y$$

is not interpolation for the order on X, but it yields a Boolean sheaf representation with each stalk the 2-element lattice of A^- over Y

- A is isomorphic to the sublattice of order preserving global sections of this sheaf (this is the content of Stone/Priestley duality for DLs)
- If A is a Heyting algebra, then the same is true, but the implication on A is not stalk-wise but uses the implication of $\mathcal{U}(Y, \leq)$

Jipsen-Montagna Priestley and Esakia sums

This is a representation theory using subalgebras of Boolean sheaves

Let A be a distributive lattice and $(X, \pi, \leq)$ its dual space. Let $(Y, \tau, \leq)$ be another Priestley space. A continuous map

$$f : (X, \pi) \to (Y, \tau)$$

yields a Boolean sheaf representation of A^- over (Y, τ). We say $(X, \pi, \leq)$ is a Priestley sum over Y if in addition

$$f(x) \leq f(x') \implies x \leq x'$$

In this case $(X, \leq)$ is isomorphic to the lexicographic ordering $\bigcup_{y \in Y} \{y\} \times X_y$ where $X_y = f^{-1}(\{y\})$