Modal characterization of a first order language for topology

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Summary

- first order language $\mathcal{L}_t$ for topology
- modal language $\mathcal{L}_m$ for topology
- original results
- open questions

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- topological model:
  - given a signature $\mathcal{L} = (\{R_i\}, \{f_i\}, \{c_i\})$
  - topological model for $\mathcal{L}$
  - $\mathcal{A} = (A, \sigma, \{R_i^{\mathcal{A}}\}, \{f_i^{\mathcal{A}}\}, \{c_i^{\mathcal{A}}\})$
  - (A, σ) topological space
  - $(\{R_i^{\mathcal{A}}\}, \{f_i^{\mathcal{A}}\}, \{c_i^{\mathcal{A}}\})$ interpretation of $\mathcal{L}$ in $\mathcal{A}$

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- language $\mathcal{L}_t$:
  - two-sorted first order language
  - $x, y, \ldots$ point-sort variables
  - $U, V, \ldots$ open-sort variables
  - $=$ equality symbol
  - ε set membership symbol
  - symbols in $\mathcal{L}$
  - $\neg, \wedge$ ($\vee, \rightarrow, \leftrightarrow$ usual abbreviations)
  - $\exists x$, $\forall x$ existential/universal point-sort quantification
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First order language $\mathcal{L}_t$ for topology

- formulas of $\mathcal{L}_t$:
  - for every point-sort var.s x, y, open-sort var.s U, V and $P \in \mathcal{L}$:
    - $x = y$, $U = V$, $x \varepsilon U$, $P(x)$
  - for every formulas φ and ψ and point-sort variable x:
    - $\neg\varphi$, $\varphi \wedge \psi$, $\exists x \varphi$, $\forall x \varphi$
  - open-sort quantification in the form:
  - $\forall U(x \varepsilon U \rightarrow \varphi)$
    - U is an open-sort variable
    - x is a point-sort variable
    - all free occ.s of U in φ are positive (within an even nb. of $\neg$s)
  - $\exists U(x \varepsilon U \wedge \varphi)$
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- examples of formulas in $\mathcal{L}_t$:
  - $\forall U(x \varepsilon U \to \exists y(\neg x = y \land y \varepsilon U \land P(y)))$
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- if we allow open quantification in the form:
  - $\forall U \varphi$ with φ an arbitrary formula
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  - then we obtain $\mathcal{L}_2$
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  - $\mathcal{L}_t$ enjoys compactness and Löwenheim-Skolem thm.
  - $\mathcal{L}_t$ can express:
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    - triviality, discreteness
  - $\mathcal{L}_t$ cannot express:
    - that a space is compact/connected
  - $\mathcal{L}_t$ is decidable on the class of all T_3 spaces
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- properties of $\mathcal{L}_t$:
  - $\mathcal{L}_t$, $\mathcal{L}_2$ over *basoid models*: $\mathcal{A} = (A, \beta, \{P_i^{\mathcal{A}}\})$
    - β topological base
  - $\varphi(\overline{x}, \overline{X}) \in \mathcal{L}_2$ *invariant under changing base* if its truth value on $\mathcal{A} = (A, \beta, \{P_i^{\mathcal{A}}\})$ does not change by replacing β with a base γ that generates the same topology as β
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Modal language $\mathcal{L}_m$ for topology

- topological models $\mathcal{A} = (A, \sigma, \{P_i^{\mathcal{A}}\})$
- language $\mathcal{L}_m$:
  - $\mathcal{L} = \{P_i\}$ as propositional variables
  - $\neg, \wedge$ ($\vee, \rightarrow, \leftrightarrow$ usual abbreviations)
  - derivative operator $\langle d \rangle$:
    - $\mathcal{A}, a \models \langle d \rangle \varphi$ iff in every nbh of a exists $b \neq a$ st $\mathcal{A}, b \models \varphi$
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Modal language $\mathcal{L}_m$ for topology

- topological models $\mathcal{A} = (A, \sigma, \{P_i^{\mathcal{A}}\})$
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  - $\mathcal{L} = \{P_i\}$ as propositional variables
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Theorem 1

For every sentence $\alpha \in \mathcal{L}_m$ there is a sentence $\varphi \in \mathcal{L}_t$ such that for every T_3 model $\mathcal{A}$ we have that $\mathcal{A} \models \varphi$ if and only if $\mathcal{A} \models \alpha$

Proof.

- via usual standard translation
- $ST_x(\langle d \rangle \varphi) := \forall U(x \varepsilon U \to \exists y(\neg x = y \wedge y \varepsilon U \wedge ST_y(\varphi)))$
- $ST_x(\Diamond^n \varphi) := \exists x_0 ... \exists x_n(\bigwedge_{i \neq j} \neg x_i = x_j \wedge \bigwedge_{i \in n+1} ST_{x_i}(\varphi))$
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- $\mathcal{A} \models \varphi$ *if and only if* $\mathcal{A} \models \alpha$
- *quantifier depth of* φ = *modal depth of* α

Proof.

- Ehrenfeucht-Fraïssé game $G_n(\mathcal{A}, \mathcal{B})$
- if Player II has a winning strategy in $G_n(\mathcal{A}, \mathcal{B})$
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- sentence $\varphi \in \mathcal{L}_t$ with quant. dep. n
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Theorem 4

1. *For every formula* $\alpha \in \mathcal{L}_m$ *there is a formula* $\varphi(x) \in \mathcal{L}_t$ *such that for every* T_3 *model* $\mathcal{A}$ *and point* $a \in \mathcal{A}$ *we have that* $\mathcal{A} \models \varphi[a]$ *if and only if* $\mathcal{A}, a \models \alpha$
2. *For every formula* $\varphi(x) \in \mathcal{L}_t$ *there is a formula* $\alpha \in \mathcal{L}_m$ *such that for every* T_3 *model* $\mathcal{A}$ *and point* $a \in \mathcal{A}$ *we have that:*
   - $\mathcal{A} \models \varphi[a]$ *if and only if* $\mathcal{A}, a \models \alpha$
   - *quantifier depth of* $\varphi(x)$ $=$ *modal depth of* α
3. *There is a computable translation between formulas* $\varphi(x)$ *in* $\mathcal{L}_t$ *and formulas in* $\mathcal{L}_m$

Original results

Theorem 5

$\mathcal{L}_m$ does not capture $\mathcal{L}_t$ on any class including all T_2 models

Proof.

- adequate notion of bisimulation for $\mathcal{L}_m$
- there is a model $\mathcal{A}$, T_2 but not T_3, bisimilar to a T_3 model $\mathcal{B}$
- $\mathcal{L}_t$ can express that a space is T_3
- if $\mathcal{L}_m$ were equivalent to $\mathcal{L}_t$ on T_2 spaces
- $\mathcal{A}$ would be T_3: contradiction
- corollary: $\mathcal{L}_m$ cannot express T_3ness on the class of all T_2 models

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Original results

Theorem 6

The $\mathcal{L}_m$-theory of the classes all T_3, T_2, T_1 models (resp.) is rec. axiomatizable

- all propositional tautologies
- $\Diamond^{n+1} p \to \Diamond^n p$
- $\Box^0(p \to q) \to (\Diamond^n p \to \Diamond^n q)$
- $\Diamond^{!0}(p \land q) \to ((\Diamond^{!n_1} p \land \Diamond^{!n_2} q) \to \Diamond^{!n_1+n_2}(p \lor q))$
- $\Box^0 p \to p$
- $\Diamond^n p \to \Box^0 \Diamond^n p$
- $[d](p \to q) \to ([d]p \to [d]q)$
- $[d]p \to [d][d]p$
- $\langle d \rangle p \to \Diamond^n p$
- $\dfrac{\varphi}{\varphi(\chi/p)}, \dfrac{\varphi, \varphi \to \psi}{\psi}, \dfrac{\varphi}{\Box^0 \varphi}, \dfrac{\varphi}{[d]\varphi}$

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Original results

Proof.

- soundness on the class of all T_1 models:
  - axioms are valid on every T_1 topological model
  - inference rules preserve validities on every T_1 topological model
- completeness on the class of all T_3 models:
  - given a maximal consistent set Γ
  - build a Kripke model validating the axioms and satisfying Γ
  - turn the Kripke model into a T_3 model satisfying Γ
- $T_1 \supseteq T_2 \supseteq T_3$

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Original results

Theorem 7

The $\mathcal{L}_m$-theory of the classes of all T_3, T_2, T_1 models (resp.) is PSPACE-complete

Proof.

- the $\mathcal{L}_m$-theory of the classes of all T_3, T_2, T_1 models (resp.) is PSPACE:
  - checking the satisfiability of $\varphi \in \mathcal{L}_m$ on the classes of all T_3, T_2, T_1 models (resp.) reduces to:
  - checking PTIME properties of a forest of
  - polynomially many
  - polynomially deep
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  - (properties) that regards
  - either the set of all root nodes
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Original results

Theorem 8

The $\mathcal{L}_m$-theory of the classes of all T_3, T_2, T_1 models (resp.) is PSPACE-complete:

Proof.

- the $\mathcal{L}_m$-theory of the classes of all T_3, T_2, T_1 models (resp.) is PSPACE-hard:
  - the $\langle d \rangle$-theory of the classes of all T_3, T_2, T_1 models (resp.) is **K4** which is PSPACE-hard
  - the $\langle d \rangle$-language $\subseteq \mathcal{L}_m$

□

Original results

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Open questions

- what is the complexity of the trans. between $\mathcal{L}_t$ and $\mathcal{L}_m$ on T_3 mod.s?
- does $\mathcal{L}_m$ capture $\mathcal{L}_t$ on classes between all T_2 and all T_3?
- does $\mathcal{L}_m$ capture $\mathcal{L}_t$ on other classes?
- can we increase $\mathcal{L}_m$ to capture $\mathcal{L}_t$ on other classes than all T_3 models?
  - hybrid logic
- How can we increase $\mathcal{L}_m$ without loosing decidability?
- How can we increase $\mathcal{L}_m$ beyond first order logic?
  - μ-calculus
    - is $\mathcal{L}_m + \mu$ decidable?
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  - metric spaces
  - $\mathbb{R}^n$ ($n \in \omega$)
  - over $\mathbb{R}$ we can split $\langle d \rangle$ and $\{\Diamond^n\}_{n\in\omega}$ in their future and past components
- If we replace $\langle d \rangle$ with $\Diamond$ in $\mathcal{L}_m$, does the new $\mathcal{L}_m$ capture $\mathcal{L}_t$ on T_3 spaces?
- What is the fragment of first-order logic that $\mathcal{L}_m$ corresponds to in the standard Kripke semantics?
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Modal characterization of a first order language for topology

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