# Modal characterization of a first order language for topology 

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## Summary

- first order language $\mathcal{L}_{t}$ for topology
- modal language $\mathcal{L}_{m}$ for topology
- original results
- open questions


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First order language $\mathcal{L}_{t}$ for topology
－topological model：
－given a signature $\mathcal{L}=\left(\left\{R_{i}\right\},\left\{f_{i}\right\},\left\{c_{i}\right\}\right)$
－topological model for $\mathcal{L}$
－ $\mathcal{A}=\left(A, \sigma,\left\{R_{i}^{A}\right\},\left\{f_{i}^{A}\right\},\left\{C_{i}^{\mathcal{A}}\right\}\right)$
－$(A, \sigma)$ topological space
－$\left(\left\{R_{i}^{\mathcal{A}}\right\},\left\{f_{i}^{\mathcal{A}}\right\},\left\{c_{i}^{\mathcal{A}}\right\}\right)$ interpretation of $\mathcal{L}$ in $\mathcal{A}$

## First order language $\mathcal{L}_{t}$ for topology

- topological model:

```
* given a signature \mathcal{L}=({\mp@subsup{R}{i}{}},{\mp@subsup{f}{i}{}},{\mp@subsup{c}{i}{}})
- topological model for }\mathcal{L
-\mathcal{A}=(A,\sigma,{\mp@subsup{R}{i}{\mathcal{A}}},{\mp@subsup{f}{i}{\mathcal{A}}},{\mp@subsup{c}{i}{\mathcal{A}}})
* (A,\sigma) topological space
- ({晶疋}},{\mp@subsup{f}{i}{\mathcal{A}}},{\mp@subsup{c}{i}{\mathcal{A}}})\mathrm{ interpretation of }\mathcal{L}\mathrm{ in }\mathcal{A
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- topological model:
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- topological model for $\mathcal{L}$

- $(A, \sigma)$ topological space
$\Rightarrow\left(\left\{R_{i}^{\mathcal{A}}\right\},\left\{f_{i}^{\mathcal{A}}\right\},\left\{c_{i}^{\mathcal{A}}\right\}\right)$ interpretation of $\mathcal{L}$ in $\mathcal{A}$


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- topological model:
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- $(A, \sigma)$ topological space
- $\left(\left\{R_{i}^{\mathcal{A}}\right\},\left\{f_{i}^{\mathcal{A}}\right\},\left\{c_{i}^{\mathcal{A}}\right\}\right)$ interpretation of $\mathcal{L}$ in $\mathcal{A}$


## First order language $\mathcal{L}_{t}$ for topology

- language $\mathcal{L}_{t}$ :

```
- two-sorted first order language
- \(x, y, \ldots\) point-sort variables
- \(U, V, \ldots\) open-sort variables
- = equality symbol
- \(\varepsilon\) set membership symbol
- symbols in \(\mathcal{L}\)
\(\checkmark \neg, \wedge(\vee, \rightarrow\), \(\leftrightarrow\) usual abbreviations \()\)
- \(\exists x, \forall x\) existential/universal point-sort quantification
- \(\exists U, \forall U\) existential/universal open-sort quantification
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## First order language $\mathcal{L}_{t}$ for topology

- formulas of $\mathcal{L}_{t}$ :

```
> for every point-sort var.s x,y, open-sort var.s U,V and P}
    * x}=y,U=V,x\inU,P(x
* for every formulas }\varphi\mathrm{ and }v\mathrm{ , and point-sort variable }x\mathrm{ :
* open-sort quantification in the form:
> }\forallU(x\inU->\varphi
    * U}\mathrm{ is an open-sort variable
    v x is a point-sort variable
    * all free occ.s of U}\mathrm{ in }\varphi\mathrm{ are positive (within an even nb. of }\negs\mathrm{ )
> \existsU(x\inU\wedge\varphi)
    - U is an open-sort variable
    * x is a point-sort variable
    > all free occ.s of U in }\varphi\mathrm{ are negative (within an odd nb. of }\negs\mathrm{ )
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## First order language $\mathcal{L}_{t}$ for topology

- formulas of $\mathcal{L}_{t}$ :
- for every point-sort var.s $x, y$, open-sort var.s $U, V$ and $P \in \mathcal{L}$ :

- open-sort quantification in the form:
- $\forall U(x \in U \rightarrow \varphi)$
- $U$ is an open-sort variable
- $x$ is a point-sort variable
- all free occ.s of $U$ in $\varphi$ are positive (within an even $n b$. of $\neg s$ )
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## First order language $\mathcal{L}_{t}$ for topology

- formulas of $\mathcal{L}_{t}$ :
- for every point-sort var.s $x, y$, open-sort var.s $U, V$ and $P \in \mathcal{L}$ :
- $x=y$,
- for every formulas $\varphi$ and $\psi$ and point-sort variable $x$ :

```
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    * all free occ.s of U in }\varphi\mathrm{ are positive (within an even nb. of }\negs\mathrm{ )
* \existsU(x\inU \ \varphi)
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- $x=y, U=V, x \varepsilon U$,

```
* for every formulas }\varphi\mathrm{ and }\psi\mathrm{ and point-sort variable x:
- open-sort quantification in the form:
- }\forallU(x\varepsilonU->\varphi
    * U is an open-sort variable
    - x is a point-sort variable
    * all free occ.s of U in }\varphi\mathrm{ are positive (within an even nb. of }\negs\mathrm{ )
* \existsU(x\inU \ \varphi)
    - U is an open-sort variable
    * x is a point-sort variable
    * all free occ.s of U' in \varphi are negative (within an odd nb. of }\neg\textrm{s}\mathrm{ )
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- for every point-sort var.s $x, y$, open-sort var.s $U, V$ and $P \in \mathcal{L}$ :
- $x=y, U=V, x \varepsilon U, P(x)$

- open-sort quantification in the form:
- $\forall U(x \in U \rightarrow \varphi)$
- $U$ is an open-sort variable
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- for every formulas $\varphi$ and $\psi$ and point-sort variable $x$ :
- $\neg \varphi$,
- open-sort quantification in the form:
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－for every point－sort var．s $x, y$ ，open－sort var．s $U, V$ and $P \in \mathcal{L}$ ：
－$x=y, U=V, x \varepsilon U, P(x)$
－for every formulas $\varphi$ and $\psi$ and point－sort variable $x$ ：
－$\neg \varphi, \varphi \wedge \psi, \exists x \varphi$,
$>$ open－sort quantification in the form：
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－$x=y, U=V, x \varepsilon U, P(x)$
－for every formulas $\varphi$ and $\psi$ and point－sort variable $x$ ：
－$\neg \varphi, \varphi \wedge \psi, \exists x \varphi, \forall x \varphi$
$\Rightarrow$ open－sort quantification in the form：
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- for every formulas $\varphi$ and $\psi$ and point-sort variable $x$ :
- $\neg \varphi, \varphi \wedge \psi, \exists x \varphi, \forall x \varphi$
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- $U$ is an open-sort variable
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- open-sort quantification in the form:
- $\forall U(x \varepsilon U \rightarrow \varphi)$
- $U$ is an open-sort variable
- $x$ is a point-sort variable
- all free occ.s of $U$ in $\varphi$ are positive (within an even nb. of $\neg$ s)
- $\exists U(x \varepsilon U \wedge \varphi)$
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- all free occ.s of $U$ in $\varphi$ are negative (within an odd nb. of $\neg s$ )


## First order language $\mathcal{L}_{t}$ for topology

- formulas of $\mathcal{L}_{t}$ :
- for every point-sort var.s $x, y$, open-sort var.s $U, V$ and $P \in \mathcal{L}$ :
- $x=y, U=V, x \varepsilon U, P(x)$
- for every formulas $\varphi$ and $\psi$ and point-sort variable $x$ :
- $\neg \varphi, \varphi \wedge \psi, \exists x \varphi, \forall x \varphi$
- open-sort quantification in the form:
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## First order language $\mathcal{L}_{t}$ for topology

- examples of formulas in $\mathcal{L}_{t}$ :
- if we allow open quantification in the form:
- $\forall U \varphi$ with $\varphi$ an arbitrary formula
- $\exists U \varphi$ with $\varphi$ an arbitrary formula
- then we obtain $\mathcal{L}_{2}$
- $\mathcal{L}_{t} \subseteq \mathcal{L}_{2}$


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$\begin{aligned} & \text { - if we allow open quantification in the form: } \\ & \text { - } \forall U \varphi \text { with } \varphi \text { an arbitrary formula } \\ & \text { - } \exists U \varphi \text { with } \varphi \text { an arbitrary formula } \\ & \text { - then we obtain } \mathcal{L}_{2} \\ & \text { - } \mathcal{L}_{t} \subseteq \mathcal{L}_{2}\end{aligned}$


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## First order language $\mathcal{L}_{t}$ for topology

- properties of $\mathcal{L}_{t}$ :

```
* L L
- L\mathcal{L}
    - To, T1, T , , T3
    * triviality, discreteness
- L}\mp@subsup{\mathcal{L}}{t}{}\mathrm{ cannot express:
    * that a space is compact/connected
* L}\mp@subsup{\mathcal{L}}{t}{}\mathrm{ is decidable on the class of all }\mp@subsup{T}{3}{}\mathrm{ spaces
```



```
* Lindström thm.: there is no language on topological models
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- $\mathcal{L}_{t}$ is decidable on the class of all $T_{3}$ spaces
- $\mathcal{L}_{t}$ is not decidable on the classes of all $T_{0}, T_{1}, T_{2}$ spaces resp.
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- properties of $\mathcal{L}_{t}$ :
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- $\varphi(\bar{x}, \bar{X}) \in \mathcal{L}_{2}$ invariant under changing base if its truth value on $\mathcal{A}=\left(A, \beta,\left\{P_{i}^{\mathcal{A}}\right\}\right)$ does not change by replacing $\beta$ with a base $\gamma$ that generates the same topology as $\beta$
- $\mathcal{L}_{t}$ is invariant under changing base
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## Modal language $\mathcal{L}_{m}$ for topology

> topological models $\mathcal{A}=\left(A, \sigma,\left\{P_{i}^{\mathcal{A}}\right\}\right)$

- language $\mathcal{L}_{m}$ :
- $\mathcal{L}=\left\{P_{i}\right\}$ as propositional variables
$\triangleright \neg, \wedge(\vee, \rightarrow, \leftrightarrow$ usual abbreviations $)$
- derivative operator $\langle d\rangle$ :
- $\mathcal{A}, a \models\langle d\rangle \varphi$ iff in every nbh of a exists $b \neq a$ st $\mathcal{A}, b=\varphi$
- graded operators $\left\{\diamond^{n}\right\}_{n \in \omega}$ :
- $\mathcal{A}, a \models \diamond^{n} \varphi$ iff exist more than $n$ points $b \in A$ with $\mathcal{A}, b \models \varphi$
- [d] abbreviates $\neg\langle d\rangle \neg$
- $\square^{n}$ abbreviates $\neg \vee^{n} \neg$
- $\diamond^{!n} \varphi$ abbreviates $\diamond^{n-1} \varphi \wedge \neg \vee^{n} \varphi$ :
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## Modal language $\mathcal{L}_{m}$ for topology

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```
- \mathcal{L}={\mp@subsup{P}{i}{}}\mathrm{ as propositional variables}
\vee \neg,^ (\vee,->,\leftrightarrow usual abbreviations)
* derivative operator \langled\rangle:
    * \mathcal{A,a}\models\langled\rangle\varphi iff in every nbh of a exists b}=a\mathrm{ st }\mathcal{A},b\models
graded operators {}{\mp@subsup{\diamond}{}{n}\mp@subsup{}}{n\in\omega}{}\mathrm{ :
```



```
* [d] abbreviates }\neg\langled\rangle
- }\mp@subsup{\square}{}{n}\mathrm{ abbreviates }\neg\mp@subsup{\}{}{n}
> }\mp@subsup{\Delta}{}{!n}\varphi\mathrm{ abbreviates }\mp@subsup{\Delta}{}{n-1}\varphi\wedge\neg\mp@subsup{\nabla}{}{n}\varphi
* \mathcal{A,a}=\mp@subsup{\diamond}{}{!n}\varphi\mathrm{ iff exist exactly n points }b\inA\mathrm{ with }\mathcal{A},b\models\varphi
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－ $\mathcal{L}=\left\{P_{i}\right\}$ as propositional variables
－$\neg, \wedge(\vee, \rightarrow, \leftrightarrow$ usual abbreviations）
－derivative operator $\langle d\rangle$ ：
－ $\mathcal{A}, a \models\langle d\rangle \varphi$ iff in every nbh of $a$ exists $b \neq a$ st $\mathcal{A}, b \models \varphi$
－graded operators $\left\{\diamond^{n}\right\}_{n \in \omega}$ ：
－ $\mathcal{A}, a \models \diamond^{n} \varphi$ iff exist more than $n$ points $b \in A$ with $\mathcal{A}, b \models \varphi$
－［d］abbreviates $\neg\langle d\rangle \neg$
－$\square^{n}$ abbreviates $\neg \diamond^{n} \neg$
－$\diamond^{!n} \varphi$ abbreviates $\diamond^{n-1} \varphi \wedge \neg \diamond^{n} \varphi$ ：
－ $\mathcal{A}, a \models \diamond^{!n} \varphi$ iff exist exactly $n$ points $b \in A$ with $\mathcal{A}, b \models \varphi$

## Modal language $\mathcal{L}_{m}$ for topology

- formulas of $\mathcal{L}_{m}$ :
- for every $P \in \mathcal{L}$ :
- for every formulas $\varphi$ and $\psi$ :
- for every formula $\varphi$ and graded operator $\diamond^{n}$ :
> truth of $\Delta^{n} \varphi, \neg \nabla^{n} \varphi$ is independent from the point of evaluation - call them sentences of $\mathcal{L}_{m}$


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## Original results

Theorem 1
For every sentence $\alpha \in \mathcal{L}_{m}$ there is a sentence $\varphi \in \mathcal{L}_{t}$ such that for every $T_{3}$ model $\mathcal{A}$ we have that $\mathcal{A} \models \varphi$ if and only if $\mathcal{A} \models \alpha$

Proof.

- via usual standard translation
- $S T_{x}(\langle d\rangle \varphi):=\forall U\left(x \varepsilon U \rightarrow \exists y\left(\neg x=y \wedge y \in U \wedge S T_{y}(\varphi)\right)\right)$
$\Rightarrow S T_{x}\left(\diamond^{n} \varphi\right):=\exists x_{0} \ldots \exists x_{n}\left(\bigwedge_{i \neq j} \neg x_{i}=x_{j} \wedge \bigwedge_{i \in n+1} S T_{x_{i}}(\varphi)\right)$
- both formulas are in $\mathcal{L}_{t}$


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## Original results

Theorem 2
For every sentence $\varphi$ of $\mathcal{L}_{t}$ there is a sentence $\alpha \in \mathcal{L}_{m}$ such that for every $T_{3}$ model $\mathcal{A}$ we have that:
$\Rightarrow \mathcal{A}=\varphi$ if and only if $\mathcal{A}=\alpha$

- quantifier depth of $\varphi=$ modal depth of $\alpha$
- Ehrenfeucht-Fraïssé game $G_{n}(\mathcal{A}, \mathcal{B})$
- if Player II has a winning strategy in $G_{n}(A, B)$
$\Rightarrow$ then $\mathcal{A}$ and $\mathcal{B}$ agree on the sentences of $\mathcal{L}_{t}$ with quant. dep. $n$
- if $T_{3}$ mod.s $\mathcal{A}$ and $\mathcal{B}$ agree on the sent.s of $\mathcal{L}_{m}$ with mod. dep. $n$
- than Player II has a winning strategy in $G_{n}(\mathcal{A}, \mathcal{B})$
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## Original results

Theorem 3
There is a computable trans. between sent.s in $\mathcal{L}_{t}$ and sent.s in $\mathcal{L}_{m}$

- sentence $\varphi \in \mathcal{L}_{t}$ with quant. dep. $n$
- there is a sentence in $\mathcal{R}_{m}$ with modal depth $n$ equivalent to $\varphi$ on $T_{3}$ models
- finitely many candidates $\alpha \in \mathcal{L}_{m}$
- $\mathcal{C}_{t}$ decidable on the class of all $T_{3}$ models
$\Rightarrow$ for every cand. $\alpha$, check $\varphi \leftrightarrow \alpha$ on the class of all $T_{3}$ models


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- $\mathcal{L}_{t}$ decidable on the class of all $\mathrm{T}_{3}$ models
- for every cand. $\alpha$, check $\varphi \leftrightarrow \alpha$ on the class of all $\mathrm{T}_{3}$ models


## Original results

## Theorem 4

1. For every formula $\alpha \in \mathcal{L}_{m}$ there is a formula $\varphi(x) \in \mathcal{L}_{t}$ such that for every $T_{3}$ model $\mathcal{A}$ and point $a \in \mathcal{A}$ we have that $\mathcal{A} \models \varphi[a]$ if and only if $\mathcal{A}, a \models \alpha$
2. For every formula $\varphi(x) \in \mathcal{L}_{t}$ there is a formula $\alpha \in \mathcal{L}_{m}$ such that for every $T_{3}$ model $\mathcal{A}$ and point $a \in \mathcal{A}$ we have that:

- $\mathcal{A} \models \varphi[a]$ if and only if $\mathcal{A}, a \models \alpha$
- quantifier depth of $\varphi(x)=$ modal depth of $\alpha$

3. There is a computable translation between formulas $\varphi(x)$ in $\mathcal{L}_{t}$ and formulas in $\mathcal{L}_{m}$

## Original results

Theorem 5
$\mathcal{L}_{m}$ does not capture $\mathcal{L}_{t}$ on any class including all $T_{2}$ models

- adequate notion of bisimulation for $\mathcal{L}_{m}$
$\Delta$ there is a model $1, T_{2}$ but not $T_{3}$, hisimilar to a $T_{3}$ model $\mathcal{B}$
- $\mathcal{L}_{t}$ can express that a space is $T_{3}$
- if $\mathcal{L}_{m}$ were equivalent to $\mathcal{L}_{t}$ on $\mathrm{T}_{2}$ spaces
- $A$ would be $T_{3}$ : contradiction
- corollary: $\mathcal{L}_{m}$ cannot express $T_{3}$ ness on the class of all $T_{2}$ models


## Original results

Theorem 5
$\mathcal{L}_{m}$ does not capture $\mathcal{L}_{t}$ on any class including all $T_{2}$ models Proof.

- adequate notion of bisimulation for $\mathcal{L}_{m}$
v there is a model $\mathcal{A}, T_{2}$ but not $T_{3}$, bisimilar to a $T_{3}$ model $\mathcal{B}$
- $\mathcal{L}_{t}$ can express that a space is $\mathrm{T}_{3}$
- if $\mathcal{L}_{m}$ were equivalent to $\mathcal{L}_{+}$on $T_{2}$ spaces
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Proof.

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- $\mathcal{L}_{t}$ can express that a space is $T_{3}$
- if $\mathcal{L}_{m}$ were equivalent to $\mathcal{L}_{t}$ on $T_{2}$ spaces
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$\mathcal{L}_{m}$ does not capture $\mathcal{L}_{t}$ on any class including all $T_{2}$ models
Proof．
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－ $\mathcal{L}_{t}$ can express that a space is $\mathrm{T}_{3}$
－if $\mathcal{L}_{m}$ were equivalent to $\mathcal{L}_{t}$ on $T_{2}$ spaces
－ $\mathcal{A}$ would be $\mathrm{T}_{3}$ ：contradiction
－corollary： $\mathcal{L}_{m}$ cannot express $T_{3}$ ness on the class of all $T_{2}$ models

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- corollary: $\mathcal{L}_{m}$ cannot express $\mathrm{T}_{3}$ ness on the class of all $\mathrm{T}_{2}$ models


## Original results

Theorem 6
The $\mathcal{L}_{m}$-theory of the classes all $T_{3}, T_{2}, T_{1}$ models (resp.) is rec. axiomatizable

- all propositional tautologies



## Original results

## Theorem 6

The $\mathcal{L}_{m}$-theory of the classes all $T_{3}, T_{2}, T_{1}$ models (resp.) is rec. axiomatizable

- all propositional tautologies
- $\diamond^{n+1} p \rightarrow \diamond^{n} p$
- $\square^{0}(p \rightarrow q) \rightarrow\left(\diamond^{n} p \rightarrow \diamond^{n} q\right)$
$-\nabla^{!0}(p \wedge q) \rightarrow\left(\left(\diamond^{!n_{1}} p \wedge \delta^{!n_{2}} q\right) \rightarrow \diamond^{!n_{1}+n_{2}}(p \vee q)\right)$
- $\square^{0} p \rightarrow p$
- $\diamond^{n} p \rightarrow \square^{0} \diamond^{n} p$
- $[d](p \rightarrow q) \rightarrow([d] p \rightarrow[d] q)$
- $[d] p \rightarrow[d][d] p$
- $\langle d\rangle p \rightarrow \diamond^{n} p$
$-\frac{\varphi}{\varphi(\chi / p)}, \frac{\varphi, \varphi \rightarrow \psi}{\psi}, \frac{\varphi}{\square^{0} \varphi}, \frac{\varphi}{[d] \varphi}$


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## Original results

Proof.

- soundness on the class of all $T_{1}$ models:
- axioms are valid on every $\mathrm{T}_{1}$ topological model
- inference rules preserve validities on every $\mathrm{T}_{1}$ topological model
- completeness on the class of all $T_{3}$ models:
> given a maximal consistent set 「
- build a Kripke model validating the axioms and satisfying $\Gamma$
- turn the Kripke model into a $\mathrm{T}_{3}$ model satisfying $\Gamma$
- $\mathrm{T}_{1} \supseteq \mathrm{~T}_{2} \supseteq \mathrm{~T}_{3}$


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## Original results

Theorem 7
The $\mathcal{L}_{m}$-theory of the classes of all $T_{3}, T_{2}, T_{1}$ models (resp.) is PSPACE-complete
$\Rightarrow$ the $\mathcal{L}_{m}$-theory of the classes of all $\mathrm{T}_{3}, \mathrm{~T}_{2}, \mathrm{~T}_{1}$ models (resp.)
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- checking the satisfiability of $\varphi \in \mathcal{L}_{m}$ on the classes of all $T_{3}$,
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- checking PTIME properties of a forest of
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- the $\mathcal{L}_{m}$-theory of the classes of all $\mathrm{T}_{3}, \mathrm{~T}_{2}, \mathrm{~T}_{1}$ models (resp.) is PSPACE-hard:
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## Open questions

$\Rightarrow$ what is the complexity of the trans. between $\mathcal{L}_{t}$ and $\mathcal{L}_{m}$ on $\mathrm{T}_{3}$ mod.s?

- does $\mathcal{L}_{m}$ canture $\mathcal{L}_{t}$ on classes between all $T_{2}$ and all $T_{3}$ ?
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## Open questions

- study of the $\mathcal{L}_{m}$-theory of particular (classes of) $T_{3}$ spaces
- metric spaces
- $\mathbb{R}^{n}(n \in \omega)$
- over $\mathbb{R}$ we can split $\langle d\rangle$ and $\left\{\left\langle^{n}\right\}_{n \in w}\right.$ in their future and past components
$\rightarrow$ If we replace $\langle d\rangle$ with $\diamond$ in $\mathcal{L}_{m}$, does the new $\mathcal{L}_{m}$ capture $\mathcal{L}_{t}$ on $T_{3}$ snaces?
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- metric spaces
- $\mathbb{R}^{n}(n \in \omega)$
- over $\mathbb{R}$ we can split $\langle d\rangle$ and $\left\{\diamond^{n}\right\}_{n \in \omega}$ in their future and past components
- If we replace $\langle d\rangle$ with $\diamond$ in $\mathcal{L}_{m}$, does the new $\mathcal{L}_{m}$ capture $\mathcal{L}_{t}$ on $\mathrm{T}_{3}$ spaces?
- What is the fragment of first-order logic that $\mathcal{L}_{m}$ corresponds to in the standard Kripke semantics?
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# Modal characterization of a first order language for topology 

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