The finite embeddability property for some noncommutative varieties of fully-distributive residuated lattices

Nick Galatos
joint work with Riquelmi Cardona
University of Denver
ngalatos@du.edu

June, 2015

FEP and decidability

A class of algebras $\mathcal{K}$ has the *finite embeddability property (FEP)* if for every $\mathbf{A} \in \mathcal{K}$, every finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ can be (partially) embedded in a finite $\mathbf{D} \in \mathcal{K}$.

FEP and decidability

A class of algebras $\mathcal{K}$ has the *finite embeddability property (FEP)* if for every $\mathbf{A} \in \mathcal{K}$, every finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ can be (partially) embedded in a finite $\mathbf{D} \in \mathcal{K}$.

If $\mathcal{K}$ has the FEP, then every invalid universal sentence of $\mathcal{K}$ fails in a finite algebra of $\mathcal{K}$.

FEP and decidability

A class of algebras $\mathcal{K}$ has the *finite embeddability property (FEP)* if for every $\mathbf{A} \in \mathcal{K}$, every finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ can be (partially) embedded in a finite $\mathbf{D} \in \mathcal{K}$.

If $\mathcal{K}$ has the FEP, then every invalid universal sentence of $\mathcal{K}$ fails in a finite algebra of $\mathcal{K}$.

The FEP implies the FMP (finite model property), namely that every invalid equation of $\mathcal{K}$ fails in a finite algebra of $\mathcal{K}$.

FEP and decidability

A class of algebras $\mathcal{K}$ has the *finite embeddability property (FEP)* if for every $\mathbf{A} \in \mathcal{K}$, every finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ can be (partially) embedded in a finite $\mathbf{D} \in \mathcal{K}$.

If $\mathcal{K}$ has the FEP, then every invalid universal sentence of $\mathcal{K}$ fails in a finite algebra of $\mathcal{K}$.

The FEP implies the FMP (finite model property), namely that every invalid equation of $\mathcal{K}$ fails in a finite algebra of $\mathcal{K}$.

Fact. If $\mathcal{K}$ has the FEP and is finitely axiomatizable, then it's universal theory is decidable.

FEP and decidability

A class of algebras $\mathcal{K}$ has the *finite embeddability property (FEP)* if for every $\mathbf{A} \in \mathcal{K}$, every finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ can be (partially) embedded in a finite $\mathbf{D} \in \mathcal{K}$.

If $\mathcal{K}$ has the FEP, then every invalid universal sentence of $\mathcal{K}$ fails in a finite algebra of $\mathcal{K}$.

The FEP implies the FMP (finite model property), namely that every invalid equation of $\mathcal{K}$ fails in a finite algebra of $\mathcal{K}$.

Fact. If $\mathcal{K}$ has the FEP and is finitely axiomatizable, then it's universal theory is decidable.

Fact. The decidability of the universal theory implies the decidability of the quasi-equational theory, which implies the decidability of the word problem, which implies the decidability of the equational theory.

FEP and decidability

A class of algebras $\mathcal{K}$ has the *finite embeddability property (FEP)* if for every $\mathbf{A} \in \mathcal{K}$, every finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ can be (partially) embedded in a finite $\mathbf{D} \in \mathcal{K}$.

If $\mathcal{K}$ has the FEP, then every invalid universal sentence of $\mathcal{K}$ fails in a finite algebra of $\mathcal{K}$.

The FEP implies the FMP (finite model property), namely that every invalid equation of $\mathcal{K}$ fails in a finite algebra of $\mathcal{K}$.

Fact. If $\mathcal{K}$ has the FEP and is finitely axiomatizable, then it's universal theory is decidable.

Fact. The decidability of the universal theory implies the decidability of the quasi-equational theory, which implies the decidability of the word problem, which implies the decidability of the equational theory.

Fact. The FEP for a finitiely axiomatizable class $\mathcal{K}$ that forms the algebraic semantics of a finitary logical system $\vdash$, implies its *strong finite model property*:
if $\Phi \not\vdash \psi$, for finite Φ, then there is a finite counter-model.

Residuated lattices

A *residuated lattice*, is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(L, \wedge, \vee)$ is a lattice,
- $(L, \cdot, 1)$ is a monoid and
- for all $a, b, c \in L$,

$$ab \leq c \;\Leftrightarrow\; b \leq a\backslash c \;\Leftrightarrow\; a \leq c/b.$$

Residuated lattices

A *residuated lattice*, is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(L, \wedge, \vee)$ is a lattice,
- $(L, \cdot, 1)$ is a monoid and
- for all $a, b, c \in L$,

$$ab \leq c \;\Leftrightarrow\; b \leq a\backslash c \;\Leftrightarrow\; a \leq c/b.$$

Equations of the form $x^m \leq x^n$, for natural numbers m and n, are called *knotted equations*.

Residuated lattices

A *residuated lattice*, is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(L, \wedge, \vee)$ is a lattice,
- $(L, \cdot, 1)$ is a monoid and
- for all $a, b, c \in L$,

$$ab \leq c \;\Leftrightarrow\; b \leq a\backslash c \;\Leftrightarrow\; a \leq c/b.$$

Equations of the form $x^m \leq x^n$, for natural numbers m and n, are called *knotted equations*.

They define proper, non-trivial subvarieties for $m \neq n$ and $m \neq 1$, and we will assume these conditions hold.

Residuated lattices

A *residuated lattice*, is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(L, \wedge, \vee)$ is a lattice,
- $(L, \cdot, 1)$ is a monoid and
- for all $a, b, c \in L$,

$$ab \leq c \;\Leftrightarrow\; b \leq a\backslash c \;\Leftrightarrow\; a \leq c/b.$$

Equations of the form $x^m \leq x^n$, for natural numbers m and n, are called *knotted equations*.

They define proper, non-trivial subvarieties for $m \neq n$ and $m \neq 1$, and we will assume these conditions hold. Also, we will not consider the case $x^m \leq 1$, for $m > 1$, as it is equivalent to the case $m = 1$ (integrality).

FEP for RL

RL does **not** have the FEP.

FEP for RL

RL does **not** have the FEP.

RL $+ (x \leq 1)$ (integrality) has the FEP. (Blok and van Alten)

FEP for RL

RL does **not** have the FEP.

RL $+ (x \leq 1)$ (integrality) has the FEP. (Blok and van Alten)

RL $+ (x \leq 1)+$ (any equation over $\{\vee, \cdot, 1\}$) has the FEP. (G. and Jipsen)

FEP for RL

RL does **not** have the FEP.

RL $+ (x \leq 1)$ (integrality) has the FEP. (Blok and van Alten)

RL $+ (x \leq 1)+$ (any equation over $\{\vee, \cdot, 1\}$) has the FEP. (G. and Jipsen)

RL $+ (x^2 \leq x)$ (mingle) has the FEP. (Horčík)

FEP for RL

RL does **not** have the FEP.

RL $+ (x \leq 1)$ (integrality) has the FEP. (Blok and van Alten)

RL $+ (x \leq 1) +$ (any equation over $\{\vee, \cdot, 1\}$) has the FEP. (G. and Jipsen)

RL $+ (x^2 \leq x)$ (mingle) has the FEP. (Horčík)

Most varieties RL $+ (x^m \leq x)$, m-mingle, do *not* have the FEP/dWP. (Horčík)

FEP for RL

RL does **not** have the FEP.

RL $+ (x \leq 1)$ (integrality) has the FEP. (Blok and van Alten)

RL $+ (x \leq 1)+$ (any equation over $\{\vee, \cdot, 1\}$) has the FEP. (G. and Jipsen)

RL $+ (x^2 \leq x)$ (mingle) has the FEP. (Horčík)

Most varieties RL $+ (x^m \leq x)$, m-mingle, do *not* have the FEP/dWP. (Horčík)

RL $+ (x^m \leq x^n)$, for $n \neq 1$, does **not** have the FEP/dWP. (Horčík)

FEP for RL

RL does **not** have the FEP.

RL $+ (x \leq 1)$ (integrality) has the FEP. (Blok and van Alten)

RL $+ (x \leq 1)+$ (any equation over $\{\vee, \cdot, 1\}$) has the FEP. (G. and Jipsen)

RL $+ (x^2 \leq x)$ (mingle) has the FEP. (Horčík)

Most varieties RL $+ (x^m \leq x)$, m-mingle, do *not* have the FEP/dWP. (Horčík)

RL $+ (x^m \leq x^n)$, for $n \neq 1$, does **not** have the FEP/dWP. (Horčík)

The variety RL $+ (xy = yx)$ (commutativity) does **not** have the FEP. (Blok and van Alten)

FEP for RL

RL does **not** have the FEP.

RL $+ (x \leq 1)$ (integrality) has the FEP. (Blok and van Alten)

RL $+ (x \leq 1) +$ (any equation over $\{\vee, \cdot, 1\}$) has the FEP. (G. and Jipsen)

RL $+ (x^2 \leq x)$ (mingle) has the FEP. (Horčík)

Most varieties RL $+ (x^m \leq x)$, m-mingle, do *not* have the FEP/dWP. (Horčík)

RL $+ (x^m \leq x^n)$, for $n \neq 1$, does **not** have the FEP/dWP. (Horčík)

The variety RL $+ (xy = yx)$ (commutativity) does **not** have the FEP. (Blok and van Alten)

The varieties RL $+ (x^m \leq x^n) + (xy = yx)$ have the FEP. (van Alten)

FEP for RL

RL does **not** have the FEP.

RL $+ (x \leq 1)$ (integrality) has the FEP. (Blok and van Alten)

RL $+ (x \leq 1)+$ (any equation over $\{\vee, \cdot, 1\}$) has the FEP. (G. and Jipsen)

RL $+ (x^2 \leq x)$ (mingle) has the FEP. (Horčík)

Most varieties RL $+ (x^m \leq x)$, m-mingle, do *not* have the FEP/dWP. (Horčík)

RL $+ (x^m \leq x^n)$, for $n \neq 1$, does **not** have the FEP/dWP. (Horčík)

The variety RL $+ (xy = yx)$ (commutativity) does **not** have the FEP. (Blok and van Alten)

The varieties RL $+ (x^m \leq x^n) + (xy = yx)$ have the FEP. (van Alten)

The varieties RL $+ (x^m \leq x^n) + (xxy = xyx)^* +$ (any equation over $\{\vee, \cdot, 1\}$) has the FEP. (Cardona and G.)
*This is an example.

FEP for FDRL

In every residuated lattice multiplication distributes over join.

FEP for FDRL

In every residuated lattice multiplication distributes over join. A residuated lattice is called *(fully) distributive* if (multiplication and) join distributes over meet. We write (FDRL) DRL.

FEP for FDRL

In every residuated lattice multiplication distributes over join. A residuated lattice is called *(fully) distributive* if (multiplication and) join distributes over meet. We write (FDRL) DRL.

(F)DRL does **not** have the FEP. (Blok and van Alten.)

FEP for FDRL

In every residuated lattice multiplication distributes over join. A residuated lattice is called *(fully) distributive* if (multiplication and) join distributes over meet. We write (FDRL) DRL.

(F)DRL does **not** have the FEP. (Blok and van Alten.)

(F)DRL $+ (x \leq 1) +$ (any equation without divisions) has the FEP. (G. and Jipsen)

FEP for FDRL

In every residuated lattice multiplication distributes over join. A residuated lattice is called *(fully) distributive* if (multiplication and) join distributes over meet. We write (FDRL) DRL.

(F)DRL does **not** have the FEP. (Blok and van Alten.)

(F)DRL $+ (x \leq 1) +$ (any equation without divisions) has the FEP. (G. and Jipsen)

(F)DRL $+ (xy = yx)$ (commutativity) does **not** have the FEP (Blok and van Alten), or a dWP (G.)

FEP for FDRL

In every residuated lattice multiplication distributes over join. A residuated lattice is called *(fully) distributive* if (multiplication and) join distributes over meet. We write (FDRL) DRL.

(F)DRL does **not** have the FEP. (Blok and van Alten.)

(F)DRL $+ (x \leq 1) +$ (any equation without divisions) has the FEP. (G. and Jipsen)

(F)DRL $+ (xy = yx)$ (commutativity) does **not** have the FEP (Blok and van Alten), or a dWP (G.)

DRL $+ (x^m \leq x^n)$, for $m < n$, does **not** have the FEP/dWP. (Horčík)

FEP for FDRL

In every residuated lattice multiplication distributes over join. A residuated lattice is called *(fully) distributive* if (multiplication and) join distributes over meet. We write (FDRL) DRL.

(F)DRL does **not** have the FEP. (Blok and van Alten.)

(F)DRL $+ (x \leq 1) +$ (any equation without divisions) has the FEP. (G. and Jipsen)

(F)DRL $+ (xy = yx)$ (commutativity) does **not** have the FEP (Blok and van Alten), or a dWP (G.)

DRL $+ (x^m \leq x^n)$, for $m < n$, does **not** have the FEP/dWP. (Horčík)

Theorem (Cardona and G.) FDRL $+ (x^m \leq x^n) + (a) +$ (any equation without divisions) has the FEP.

FEP for FDRL

In every residuated lattice multiplication distributes over join. A residuated lattice is called *(fully) distributive* if (multiplication and) join distributes over meet. We write (FDRL) DRL.

(F)DRL does **not** have the FEP. (Blok and van Alten.)

(F)DRL $+ (x \leq 1) +$ (any equation without divisions) has the FEP. (G. and Jipsen)

(F)DRL $+ (xy = yx)$ (commutativity) does **not** have the FEP (Blok and van Alten), or a dWP (G.)

DRL $+ (x^m \leq x^n)$, for $m < n$, does **not** have the FEP/dWP. (Horčík)

Theorem (Cardona and G.) FDRL $+ (x^m \leq x^n) + (a) +$ (any equation without divisions) has the FEP.

$$xy_1xy_2 \cdots y_rx = x^{a_0}y_1x^{a_1}y_2 \cdots y_rx^{a_r}. \qquad (a)$$

Here $a = (a_0, a_1, \ldots, a_r)$ is a vector of natural numbers whose sum is $r + 1$ and product is 0.

$\mathbf{D}$ via dist. frames

Let $\mathbf{A} \in \mathcal{V}$, the above variety, and B be a finite subset of A. The algebra $\mathbf{W} = (W, \owedge, \circ, \varepsilon)$ generated by B over $\{\wedge, \cdot, 1\}$ in $\mathbf{A}$ is a (potentially infinite) $s\ell$-monoid,

$\mathbf{D}$ via dist. frames

Let $\mathbf{A} \in \mathcal{V}$, the above variety, and B be a finite subset of A. The algebra $\mathbf{W} = (W, \textcircled{\wedge}, \circ, \varepsilon)$ generated by B over $\{\wedge, \cdot, 1\}$ in $\mathbf{A}$ is a (potentially infinite) $s\ell$-monoid, and $\mathbf{D}$ will end up consisting of downsets of $\mathbf{W}$.

$\mathbf{D}$ via dist. frames

Let $\mathbf{A} \in \mathcal{V}$, the above variety, and B be a finite subset of A. The algebra $\mathbf{W} = (W, ⃝\!\!\!\!\wedge, \circ, \varepsilon)$ generated by B over $\{\wedge, \cdot, 1\}$ in $\mathbf{A}$ is a (potentially infinite) $s\ell$-monoid, and $\mathbf{D}$ will end up consisting of downsets of $\mathbf{W}$. (Further $\wedge$ in $\mathbf{D}$ will be residuated.)

$\mathbf{D}$ via dist. frames

Let $\mathbf{A} \in \mathcal{V}$, the above variety, and B be a finite subset of A. The algebra $\mathbf{W} = (W, \textcircled{\wedge}, \circ, \varepsilon)$ generated by B over $\{\wedge, \cdot, 1\}$ in $\mathbf{A}$ is a (potentially infinite) $s\ell$-monoid, and $\mathbf{D}$ will end up consisting of downsets of $\mathbf{W}$. (Further $\wedge$ in $\mathbf{D}$ will be residuated.)

$\mathbf{D}$ needs to contain residuals $u \rightarrow (x \backslash b / y)$, for $b \in B$, $u, x, y \in W$.

$\mathbf{D}$ via dist. frames

Let $\mathbf{A} \in \mathcal{V}$, the above variety, and B be a finite subset of A. The algebra $\mathbf{W} = (W, \owedge, \circ, \varepsilon)$ generated by B over $\{\wedge, \cdot, 1\}$ in $\mathbf{A}$ is a (potentially infinite) $s\ell$-monoid, and $\mathbf{D}$ will end up consisting of downsets of $\mathbf{W}$. (Further $\wedge$ in $\mathbf{D}$ will be residuated.)

$\mathbf{D}$ needs to contain residuals $u \to (x\backslash b/y)$, for $b \in B$, $u, x, y \in W$.

We represent the ideal elements $u \to (x\backslash b/y)$ by (u, x, b, y) and collect them in an index set $W' = W \times W \times B \times W$.

$\mathbf{D}$ via dist. frames

Let $\mathbf{A} \in \mathcal{V}$, the above variety, and B be a finite subset of A. The algebra $\mathbf{W} = (W, \owedge, \circ, \varepsilon)$ generated by B over $\{\wedge, \cdot, 1\}$ in $\mathbf{A}$ is a (potentially infinite) $s\ell$-monoid, and $\mathbf{D}$ will end up consisting of downsets of $\mathbf{W}$. (Further $\wedge$ in $\mathbf{D}$ will be residuated.)

$\mathbf{D}$ needs to contain residuals $u \to (x\backslash b/y)$, for $b \in B$, $u, x, y \in W$.

We represent the ideal elements $u \to (x\backslash b/y)$ by (u, x, b, y) and collect them in an index set $W' = W \times W \times B \times W$. We identify these ideal elements via their downsets in W.

$$\{w \in W : w \leq^{\mathbf{A}} u \to (x\backslash b/y)\} = \{w \in W : u \owedge (x \circ w \circ y) \leq^{\mathbf{A}} b\}.$$

$\mathbf{D}$ via dist. frames

Let $\mathbf{A} \in \mathcal{V}$, the above variety, and B be a finite subset of A. The algebra $\mathbf{W} = (W, \textcircled{\wedge}, \circ, \varepsilon)$ generated by B over $\{\wedge, \cdot, 1\}$ in $\mathbf{A}$ is a (potentially infinite) $s\ell$-monoid, and $\mathbf{D}$ will end up consisting of downsets of $\mathbf{W}$. (Further $\wedge$ in $\mathbf{D}$ will be residuated.)

$\mathbf{D}$ needs to contain residuals $u \to (x\backslash b/y)$, for $b \in B$, $u, x, y \in W$.

We represent the ideal elements $u \to (x\backslash b/y)$ by (u, x, b, y) and collect them in an index set $W' = W \times W \times B \times W$. We identify these ideal elements via their downsets in W.

$$\{w \in W : w \leq^{\mathbf{A}} u \to (x\backslash b/y)\} = \{w \in W : u \textcircled{\wedge} (x \circ w \circ y) \leq^{\mathbf{A}} b\}.$$

We extend the order to a relation $\sqsubseteq$ between W and W':

$$w \sqsubseteq (u, x, b, y) \;\Leftrightarrow\; u \textcircled{\wedge} (x \circ w \circ y) \leq^{\mathbf{A}} b.$$

For $z = (u, x, b, y) \in W'$. We define $z^{\lhd} = \{x \in W : x \sqsubseteq z\}$.

$\mathbf{D}$ via dist. frames

Let $\mathbf{A} \in \mathcal{V}$, the above variety, and B be a finite subset of A. The algebra $\mathbf{W} = (W, \bigotimes\!\!\!\!\wedge, \circ, \varepsilon)$ generated by B over $\{\wedge, \cdot, 1\}$ in $\mathbf{A}$ is a (potentially infinite) $s\ell$-monoid, and $\mathbf{D}$ will end up consisting of downsets of $\mathbf{W}$. (Further $\wedge$ in $\mathbf{D}$ will be residuated.)

$\mathbf{D}$ needs to contain residuals $u \to (x\backslash b/y)$, for $b \in B$, $u, x, y \in W$.

We represent the ideal elements $u \to (x\backslash b/y)$ by (u, x, b, y) and collect them in an index set $W' = W \times W \times B \times W$. We identify these ideal elements via their downsets in W.

$$\{w \in W : w \leq^{\mathbf{A}} u \to (x\backslash b/y)\} = \{w \in W : u \bigcirc\!\!\!\!\wedge (x \circ w \circ y) \leq^{\mathbf{A}} b\}.$$

We extend the order to a relation $\sqsubseteq$ between W and W':

$$w \sqsubseteq (u, x, b, y) \ \Leftrightarrow\ u \bigcirc\!\!\!\!\wedge (x \circ w \circ y) \leq^{\mathbf{A}} b.$$

For $z = (u, x, b, y) \in W'$. We define $z^{\lhd} = \{x \in W : x \sqsubseteq z\}$. Also,

$$D = \left\{ \bigcap_{z \in Z} \{z\}^{\lhd} : Z \subseteq W' \right\} \qquad \mathbf{D} = (D, \cap, \cup_{\sqsubseteq}, \cdot_{\sqsubseteq}, \backslash, /, \varepsilon_{\sqsubseteq}).$$

Distributive residuated frames

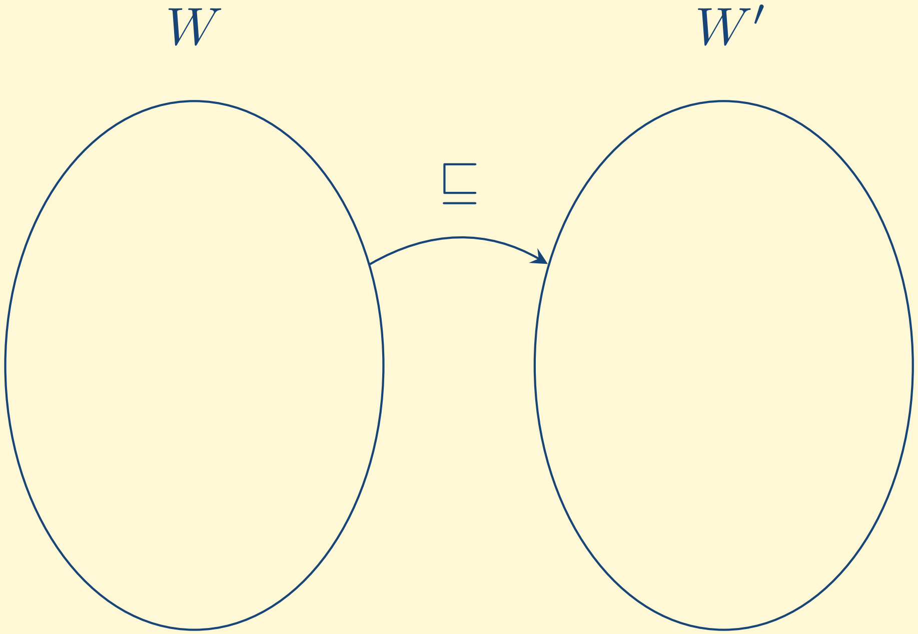
Theorem (G. and Jipsen).

- $\mathbf{D}$ is a distributive residuated lattice.
- All equations without divisions are preserved ($\mathbf{D}$ is in $\mathcal{V}$).
- The map $b \mapsto (\top, \varepsilon, b, \varepsilon)^{\lhd}$ is an embedding of the partial algebra $\mathbf{B}$ of $\mathbf{A}$ into $\mathbf{D}$.

Distributive residuated frames

Theorem (G. and Jipsen).

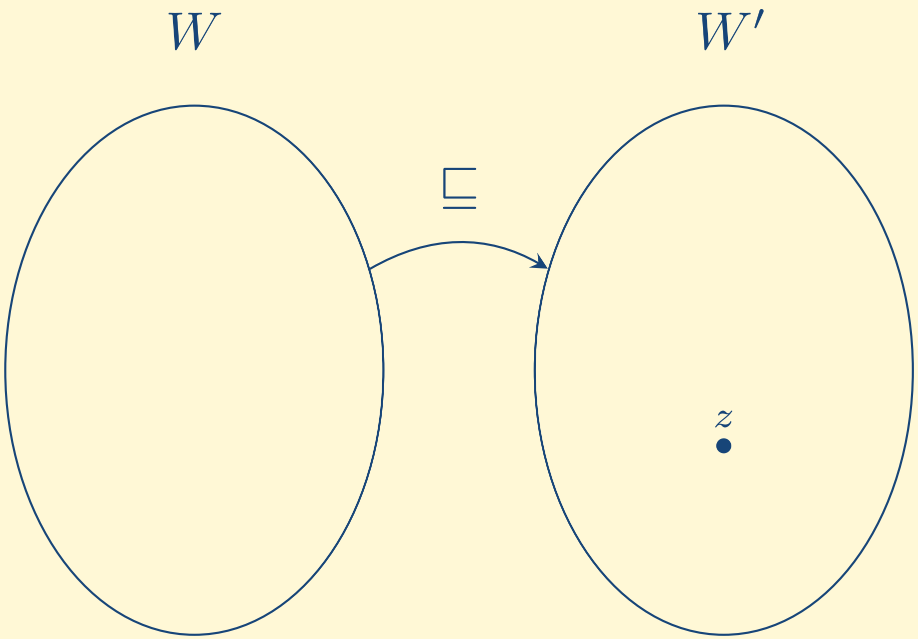
- $\mathbf{D}$ is a distributive residuated lattice.
- All equations without divisions are preserved ($\mathbf{D}$ is in $\mathcal{V}$).
- The map $b \mapsto (\top, \varepsilon, b, \varepsilon)^{\lhd}$ is an embedding of the partial algebra $\mathbf{B}$ of $\mathbf{A}$ into $\mathbf{D}$.



Distributive residuated frames

Theorem (G. and Jipsen).

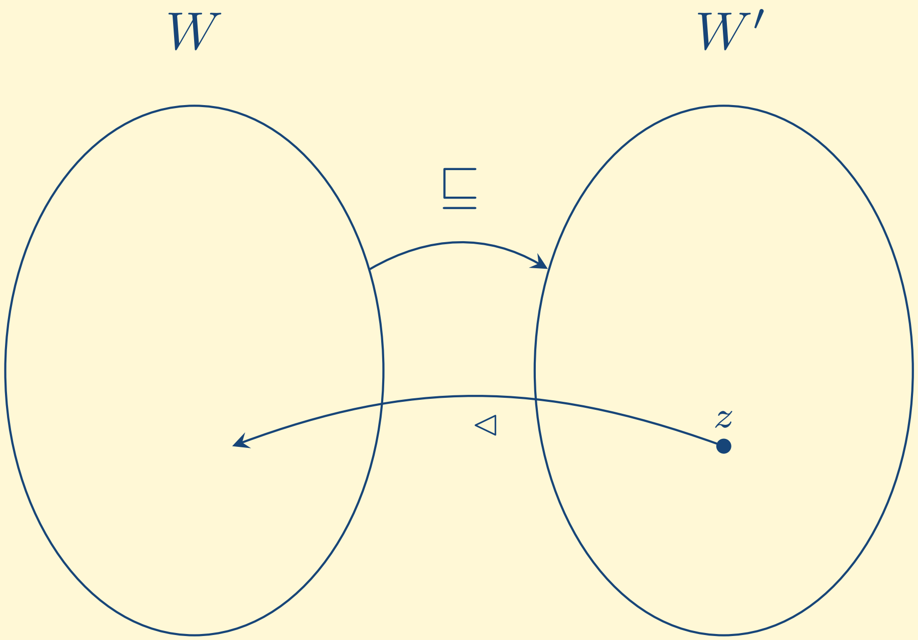
- $\mathbf{D}$ is a distributive residuated lattice.
- All equations without divisions are preserved ($\mathbf{D}$ is in $\mathcal{V}$).
- The map $b \mapsto (\top, \varepsilon, b, \varepsilon)^{\lhd}$ is an embedding of the partial algebra $\mathbf{B}$ of $\mathbf{A}$ into $\mathbf{D}$.



Distributive residuated frames

Theorem (G. and Jipsen).

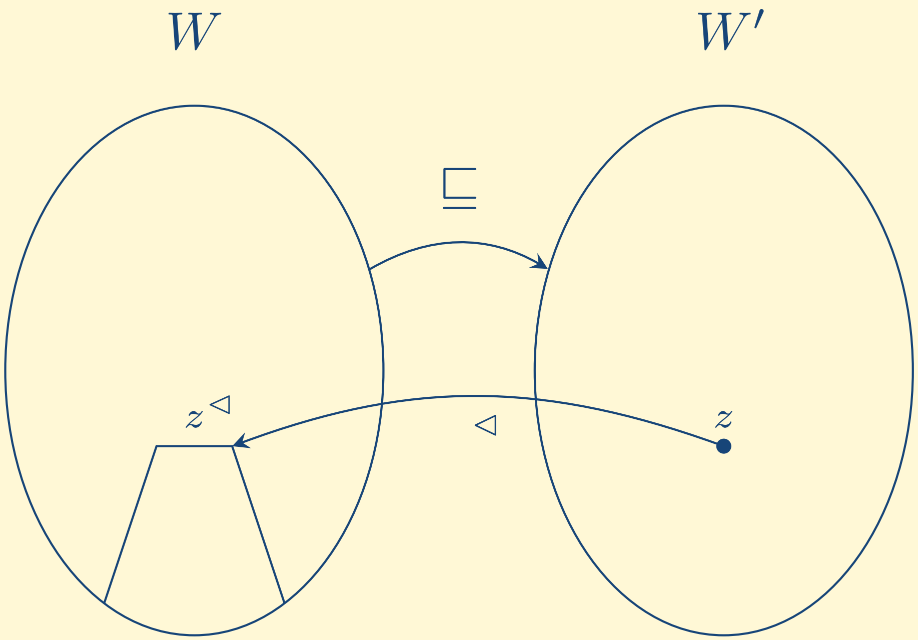
- $\mathbf{D}$ is a distributive residuated lattice.
- All equations without divisions are preserved ($\mathbf{D}$ is in $\mathcal{V}$).
- The map $b \mapsto (\top, \varepsilon, b, \varepsilon)^{\lhd}$ is an embedding of the partial algebra $\mathbf{B}$ of $\mathbf{A}$ into $\mathbf{D}$.



Distributive residuated frames

Theorem (G. and Jipsen).

- $\mathbf{D}$ is a distributive residuated lattice.
- All equations without divisions are preserved ($\mathbf{D}$ is in $\mathcal{V}$).
- The map $b \mapsto (\top, \varepsilon, b, \varepsilon)^{\lhd}$ is an embedding of the partial algebra $\mathbf{B}$ of $\mathbf{A}$ into $\mathbf{D}$.



DGN

$$\begin{array}{c} x \sqsubseteq a \quad a \sqsubseteq z \\ \hline x \sqsubseteq z \end{array} \ (\mathsf{CUT}) \qquad \begin{array}{c} \\ \hline a \sqsubseteq a \end{array} \ (\mathsf{Id})$$

$$\begin{array}{c} x \owedge (y \owedge w) \sqsubseteq z \\ \hline\hline (x \owedge y) \owedge w \sqsubseteq z \end{array} \ (\owedge a) \qquad \begin{array}{c} x \owedge y \sqsubseteq z \\ \hline y \owedge x \sqsubseteq z \end{array} \ (\owedge e)$$

$$\begin{array}{c} x \sqsubseteq z \\ \hline\hline x \owedge y \sqsubseteq z \end{array} \ (\owedge i) \qquad \begin{array}{c} x \owedge x \sqsubseteq z \\ \hline x \sqsubseteq z \end{array} \ (\owedge c)$$

$$\begin{array}{c} x \sqsubseteq a \quad b \sqsubseteq z \\ \hline x \circ (a \backslash b) \sqsubseteq z \end{array} \ (\backslash \mathsf{L}) \qquad \begin{array}{c} a \circ x \sqsubseteq b \\ \hline x \sqsubseteq a \backslash b \end{array} \ (\backslash \mathsf{R})$$

$$\begin{array}{c} x \sqsubseteq a \quad b \sqsubseteq z \\ \hline (b/a) \circ x \sqsubseteq z \end{array} \ (/\mathsf{L}) \qquad \begin{array}{c} x \circ a \sqsubseteq b \\ \hline x \sqsubseteq b/a \end{array} \ (/\mathsf{R})$$

$$\begin{array}{c} a \circ b \sqsubseteq z \\ \hline a \cdot b \sqsubseteq z \end{array} \ (\cdot \mathsf{L}) \qquad \begin{array}{c} x \sqsubseteq a \quad y \sqsubseteq b \\ \hline x \circ y \sqsubseteq a \cdot b \end{array} \ (\cdot \mathsf{R}) \qquad \begin{array}{c} \varepsilon \sqsubseteq z \\ \hline 1 \sqsubseteq z \end{array} \ (1\mathsf{L}) \qquad \begin{array}{c} \\ \hline \varepsilon \sqsubseteq 1 \end{array} \ (1\mathsf{R})$$

$$\begin{array}{c} a \owedge b \sqsubseteq z \\ \hline a \wedge b \sqsubseteq z \end{array} \ (\wedge \mathsf{L}\ell) \qquad \begin{array}{c} x \sqsubseteq a \quad x \sqsubseteq b \\ \hline x \sqsubseteq a \wedge b \end{array} \ (\wedge \mathsf{R})$$

$$\begin{array}{c} a \sqsubseteq z \quad b \sqsubseteq z \\ \hline a \vee b \sqsubseteq z \end{array} \ (\vee \mathsf{L}) \qquad \begin{array}{c} x \sqsubseteq a \\ \hline x \sqsubseteq a \vee b \end{array} \ (\vee \mathsf{R}\ell) \qquad \begin{array}{c} x \sqsubseteq b \\ \hline x \sqsubseteq a \vee b \end{array} \ (\vee \mathsf{R}r)$$

Finiteness

To prove that $\mathbf{D}$ is finite, it suffices to show that there are finitely many closed sets, which we organize by the $b \in B$:

Finiteness

To prove that $\mathbf{D}$ is finite, it suffices to show that there are finitely many closed sets, which we organize by the $b \in B$: we define $C_b = \{(u, x, b, y)^{\triangleleft} : u, x, y \in W\}$.

Finiteness

To prove that $\mathbf{D}$ is finite, it suffices to show that there are finitely many closed sets, which we organize by the $b \in B$: we define $C_b = \{(u, x, b, y)^{\lhd} : u, x, y \in W\}$. It suffices to show that each C_b is finite.

Finiteness

To prove that $\mathbf{D}$ is finite, it suffices to show that there are finitely many closed sets, which we organize by the $b \in B$: we define $C_b = \{(u, x, b, y)^{\lhd} : u, x, y \in W\}$. It suffices to show that each C_b is finite. We will do that indirectly, by showing that $(C_b, \subseteq)$ has no infinite: ascending chains,

Finiteness

To prove that $\mathbf{D}$ is finite, it suffices to show that there are finitely many closed sets, which we organize by the $b \in B$: we define $C_b = \{(u, x, b, y)^{\lhd} : u, x, y \in W\}$. It suffices to show that each C_b is finite. We will do that indirectly, by showing that $(C_b, \subseteq)$ has no infinite: ascending chains, descending chains, or

Finiteness

To prove that $\mathbf{D}$ is finite, it suffices to show that there are finitely many closed sets, which we organize by the $b \in B$: we define $C_b = \{(u, x, b, y)^{\lhd} : u, x, y \in W\}$. It suffices to show that each C_b is finite. We will do that indirectly, by showing that $(C_b, \subseteq)$ has no infinite: ascending chains, descending chains, or antichains.

Finiteness

To prove that $\mathbf{D}$ is finite, it suffices to show that there are finitely many closed sets, which we organize by the $b \in B$: we define $C_b = \{(u, x, b, y)^{\lhd} : u, x, y \in W\}$. It suffices to show that each C_b is finite. We will do that indirectly, by showing that $(C_b, \subseteq)$ has no infinite: ascending chains, descending chains, or antichains.

A poset is *(dually) well partially ordered*, *(d)wpo*,

Finiteness

To prove that $\mathbf{D}$ is finite, it suffices to show that there are finitely many closed sets, which we organize by the $b \in B$: we define $C_b = \{(u, x, b, y)^{\lhd} : u, x, y \in W\}$. It suffices to show that each C_b is finite. We will do that indirectly, by showing that $(C_b, \subseteq)$ has no infinite: ascending chains, descending chains, or antichains.

A poset is *(dually) well partially ordered*, *(d)wpo*, if it has no infinite antichains and no infinite (ascending) descending chains.

Finiteness

To prove that $\mathbf{D}$ is finite, it suffices to show that there are finitely many closed sets, which we organize by the $b \in B$: we define $C_b = \{(u, x, b, y)^{\lhd} : u, x, y \in W\}$. It suffices to show that each C_b is finite. We will do that indirectly, by showing that $(C_b, \subseteq)$ has no infinite: ascending chains, descending chains, or antichains.

A poset is *(dually) well partially ordered*, *(d)wpo*, if it has no infinite antichains and no infinite (ascending) descending chains. Equivalently if every sequence $x_1, x_2, \ldots$ has an increasing step: $i < j$ with $x_i \leq x_j$ (all sequences are *good*).

Finiteness

To prove that $\mathbf{D}$ is finite, it suffices to show that there are finitely many closed sets, which we organize by the $b \in B$: we define $C_b = \{(u, x, b, y)^{\lhd} : u, x, y \in W\}$. It suffices to show that each C_b is finite. We will do that indirectly, by showing that $(C_b, \subseteq)$ has no infinite: ascending chains, descending chains, or antichains.

A poset is *(dually) well partially ordered*, *(d)wpo*, if it has no infinite antichains and no infinite (ascending) descending chains. Equivalently if every sequence $x_1, x_2, \ldots$ has an increasing step: $i < j$ with $x_i \leq x_j$ (all sequences are *good*). Disjoint unions, products, subsets and order-preserving images of wpo's are wpo.

Finiteness

To prove that $\mathbf{D}$ is finite, it suffices to show that there are finitely many closed sets, which we organize by the $b \in B$: we define $C_b = \{(u, x, b, y)^{\lhd} : u, x, y \in W\}$. It suffices to show that each C_b is finite. We will do that indirectly, by showing that $(C_b, \subseteq)$ has no infinite: ascending chains, descending chains, or antichains.

A poset is *(dually) well partially ordered*, *(d)wpo*, if it has no infinite antichains and no infinite (ascending) descending chains. Equivalently if every sequence $x_1, x_2, \ldots$ has an increasing step: $i < j$ with $x_i \leq x_j$ (all sequences are *good*). Disjoint unions, products, subsets and order-preserving images of wpo's are wpo.

We will prove that W is wpo for $m < n$, and W is dwpo for $m > n$. This proves finiteness of C_b (it has downsets of W):

Finiteness

To prove that $\mathbf{D}$ is finite, it suffices to show that there are finitely many closed sets, which we organize by the $b \in B$: we define $C_b = \{(u, x, b, y)^{\lhd} : u, x, y \in W\}$. It suffices to show that each C_b is finite. We will do that indirectly, by showing that $(C_b, \subseteq)$ has no infinite: ascending chains, descending chains, or antichains.

A poset is *(dually) well partially ordered*, *(d)wpo*, if it has no infinite antichains and no infinite (ascending) descending chains. Equivalently if every sequence $x_1, x_2, \ldots$ has an increasing step: $i < j$ with $x_i \leq x_j$ (all sequences are *good*). Disjoint unions, products, subsets and order-preserving images of wpo's are wpo.

We will prove that W is wpo for $m < n$, and W is dwpo for $m > n$. This proves finiteness of C_b (it has downsets of W):
In both cases $\phi : (W^3, \leq) \to (C_b, \subseteq)$, $\phi(u, x, y) = (u, x, b, y)^{\lhd}$ is a surjective order-reversing map. So, if W is wpo then $(C_b, \subseteq)$ is a dwpo and if W is dwpo then $(C_b, \subseteq)$.

Finiteness

To prove that $\mathbf{D}$ is finite, it suffices to show that there are finitely many closed sets, which we organize by the $b \in B$: we define $C_b = \{(u, x, b, y)^{\lhd} : u, x, y \in W\}$. It suffices to show that each C_b is finite. We will do that indirectly, by showing that $(C_b, \subseteq)$ has no infinite: ascending chains, descending chains, or antichains.

A poset is *(dually) well partially ordered*, *(d)wpo*, if it has no infinite antichains and no infinite (ascending) descending chains. Equivalently if every sequence $x_1, x_2, \ldots$ has an increasing step: $i < j$ with $x_i \leq x_j$ (all sequences are *good*). Disjoint unions, products, subsets and order-preserving images of wpo's are wpo.

We will prove that W is wpo for $m < n$, and W is dwpo for $m > n$. This proves finiteness of C_b (it has downsets of W):
In both cases $\phi : (W^3, \leq) \to (C_b, \subseteq)$, $\phi(u, x, y) = (u, x, b, y)^{\lhd}$ is a surjective order-reversing map. So, if W is wpo then $(C_b, \subseteq)$ is a dwpo and if W is dwpo then $(C_b, \subseteq)$.
Further, If $(C_b, \subseteq)$ has an ascending chain $C_1 \subset C_2 \subset \ldots$ of downsets of W, then we can construct a bad sequence $w_1, w_2, \ldots$ in $(W, \leq)$ by taking $w_i \in C_{i+1} \setminus C_i$.

Finiteness

To prove that $\mathbf{D}$ is finite, it suffices to show that there are finitely many closed sets, which we organize by the $b \in B$: we define $C_b = \{(u, x, b, y)^{\lhd} : u, x, y \in W\}$. It suffices to show that each C_b is finite. We will do that indirectly, by showing that $(C_b, \subseteq)$ has no infinite: ascending chains, descending chains, or antichains.

A poset is *(dually) well partially ordered*, *(d)wpo*, if it has no infinite antichains and no infinite (ascending) descending chains. Equivalently if every sequence $x_1, x_2, \ldots$ has an increasing step: $i < j$ with $x_i \leq x_j$ (all sequences are *good*). Disjoint unions, products, subsets and order-preserving images of wpo's are wpo.

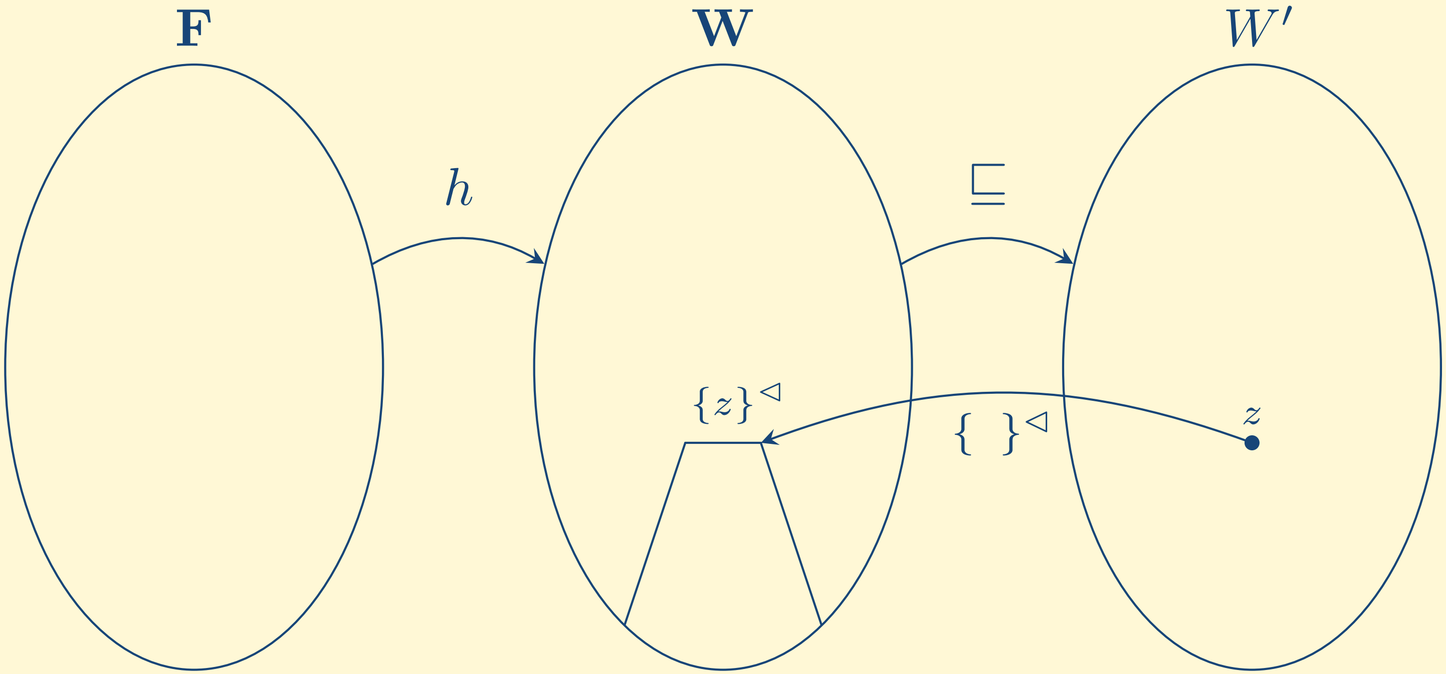
We will prove that W is wpo for $m < n$, and W is dwpo for $m > n$. This proves finiteness of C_b (it has downsets of W):
In both cases $\phi : (W^3, \leq) \to (C_b, \subseteq)$, $\phi(u, x, y) = (u, x, b, y)^{\lhd}$ is a surjective order-reversing map. So, if W is wpo then $(C_b, \subseteq)$ is a dwpo and if W is dwpo then $(C_b, \subseteq)$.
Further, If $(C_b, \subseteq)$ has an ascending chain $C_1 \subset C_2 \subset \ldots$ of downsets of W, then we can construct a bad sequence $w_1, w_2, \ldots$ in $(W, \leq)$ by taking $w_i \in C_{i+1} \setminus C_i$. For decsending: $w_i \in C_i \setminus C_{i+1}$.

Constructing $\mathbf{F}$

Since the structure of W depends a lot on the specific $\mathbf{A}$ and B, we consider a more free structure $\mathbf{F}$, and prove that it is (d)wpo and maps onto $(W, \leq)$.

Constructing $\mathbf{F}$

Since the structure of W depends a lot on the specific $\mathbf{A}$ and B, we consider a more free structure $\mathbf{F}$, and prove that it is (d)wpo and maps onto $(W, \leq)$.



Constructing $\mathbf{F}$

For that we take $\mathbf{F} = \mathcal{M}(\mathbf{H})$, the *free meet semilattice* over a (d)wpo pomonoid $\mathbf{H}$.

Constructing $\mathbf{F}$

For that we take $\mathbf{F} = \mathcal{M}(\mathbf{H})$, the *free meet semilattice* over a (d)wpo pomonoid $\mathbf{H}$. More concretely, F consists of all *finitely generated* upsets of $\mathbf{H}$ with operations $X \wedge Y = X \cup Y$ and $X \bullet Y = \uparrow (XY)$.

Constructing $\mathbf{F}$

For that we take $\mathbf{F} = \mathcal{M}(\mathbf{H})$, the *free meet semilattice* over a (d)wpo pomonoid $\mathbf{H}$. More concretely, F consists of all *finitely generated* upsets of $\mathbf{H}$ with operations $X \wedge Y = X \cup Y$ and $X \bullet Y = \uparrow (XY)$.

Lemma. If $f : (H, \cdot, \leq) \to (W, \cdot \leq)$ is an onto pomonoid homomorphism, then $h : (F, \wedge, \cdot) \to (W, \textcircled{\wedge}, \circ)$, where $h(X) = \bigwedge_{x \in X} f(x)$, is an onto semilattice monoid homomorphism.

Constructing $\mathbf{F}$

For that we take $\mathbf{F} = \mathcal{M}(\mathbf{H})$, the *free meet semilattice* over a (d)wpo pomonoid $\mathbf{H}$. More concretely, F consists of all *finitely generated* upsets of $\mathbf{H}$ with operations $X \wedge Y = X \cup Y$ and $X \bullet Y = \uparrow (XY)$.

Lemma. If $f : (H, \cdot, \leq) \to (W, \cdot \leq)$ is an onto pomonoid homomorphism, then $h : (F, \wedge, \cdot) \to (W, \textcircled{\scriptsize$\wedge$}, \circ)$, where $h(X) = \bigwedge_{x \in X} f(x)$, is an onto semilattice monoid homomorphism.

Lemma. If $\mathbf{H}$ is dwpo, then $\mathcal{M}(\mathbf{H})$ is dwpo.

Constructing $\mathbf{F}$

For that we take $\mathbf{F} = \mathcal{M}(\mathbf{H})$, the *free meet semilattice* over a (d)wpo pomonoid $\mathbf{H}$. More concretely, F consists of all *finitely generated* upsets of $\mathbf{H}$ with operations $X \wedge Y = X \cup Y$ and $X \bullet Y = \uparrow(XY)$.

Lemma. If $f : (H, \cdot, \leq) \to (W, \cdot \leq)$ is an onto pomonoid homomorphism, then $h : (F, \wedge, \cdot) \to (W, \owedge, \circ)$, where $h(X) = \bigwedge_{x \in X} f(x)$, is an onto semilattice monoid homomorphism.

Lemma. If $\mathbf{H}$ is dwpo, then $\mathcal{M}(\mathbf{H})$ is dwpo.

Lemma. If $\mathbf{H}$ is a finite unions of finite products of *linear* wpo's, then $\mathcal{M}(\mathbf{H})$ is wpo.

Constructing $\mathbf{F}$

For that we take $\mathbf{F} = \mathcal{M}(\mathbf{H})$, the *free meet semilattice* over a (d)wpo pomonoid $\mathbf{H}$. More concretely, F consists of all *finitely generated* upsets of $\mathbf{H}$ with operations $X \wedge Y = X \cup Y$ and $X \bullet Y = \uparrow (XY)$.

Lemma. If $f : (H, \cdot, \leq) \to (W, \cdot \leq)$ is an onto pomonoid homomorphism, then $h : (F, \wedge, \cdot) \to (W, \owedge, \circ)$, where $h(X) = \bigwedge_{x \in X} f(x)$, is an onto semilattice monoid homomorphism.

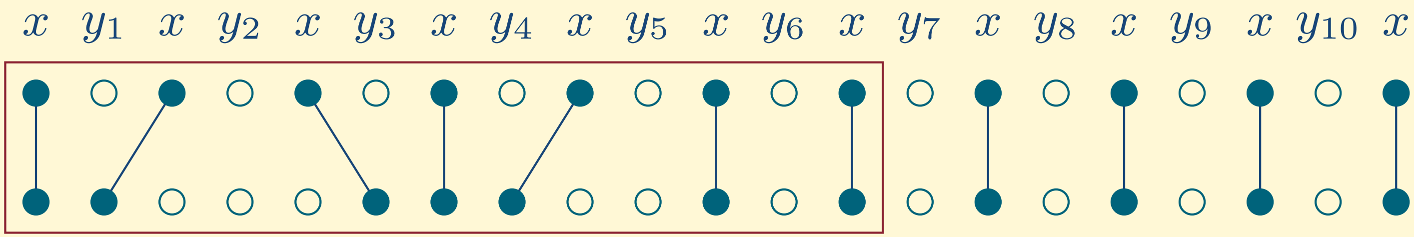
Lemma. If $\mathbf{H}$ is dwpo, then $\mathcal{M}(\mathbf{H})$ is dwpo.

Lemma. If $\mathbf{H}$ is a finite unions of finite products of *linear* wpo's, then $\mathcal{M}(\mathbf{H})$ is wpo.

So it is enough to construct a pomonoid $\mathbf{H}$ with nice order properties (dwpo, or $\cup P(wo)$) which maps homomorphically onto W.

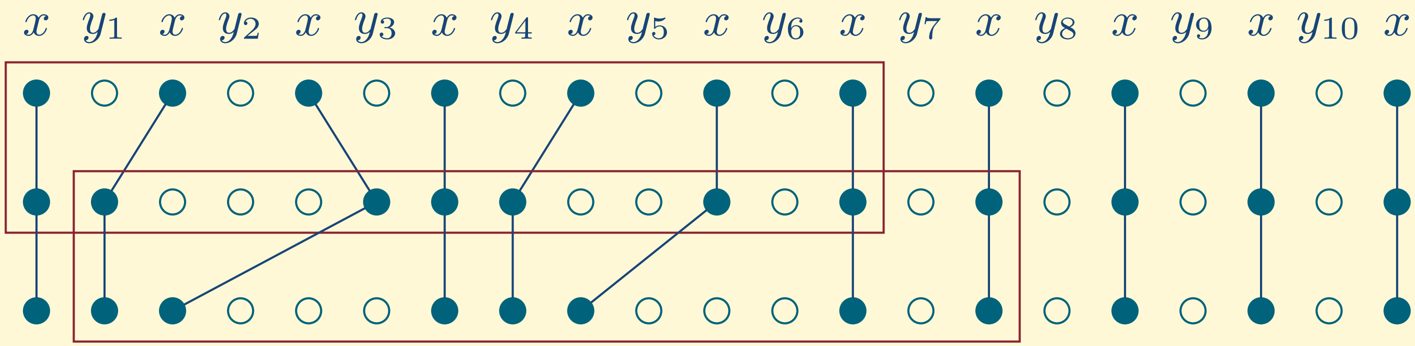
FEP and decidability

$$xy_1x\cdots xy_6x = x^2y_1y_2y_3x^3y_4y_5xy_6x$$



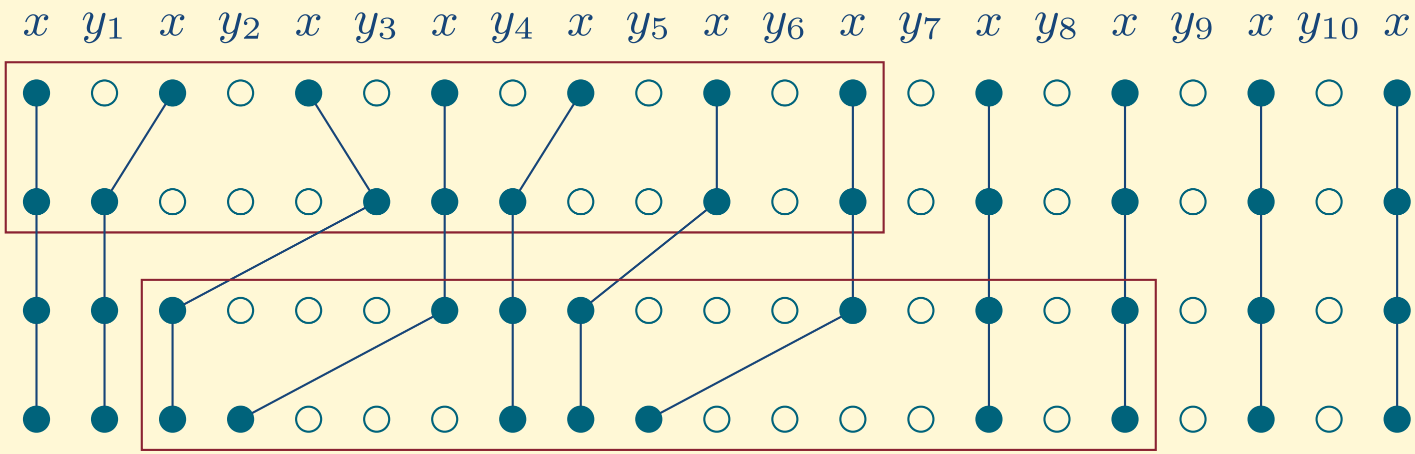
FEP and decidability

$$xy_1x\cdots xy_6x = x^2y_1y_2y_3x^3y_4y_5xy_6x$$



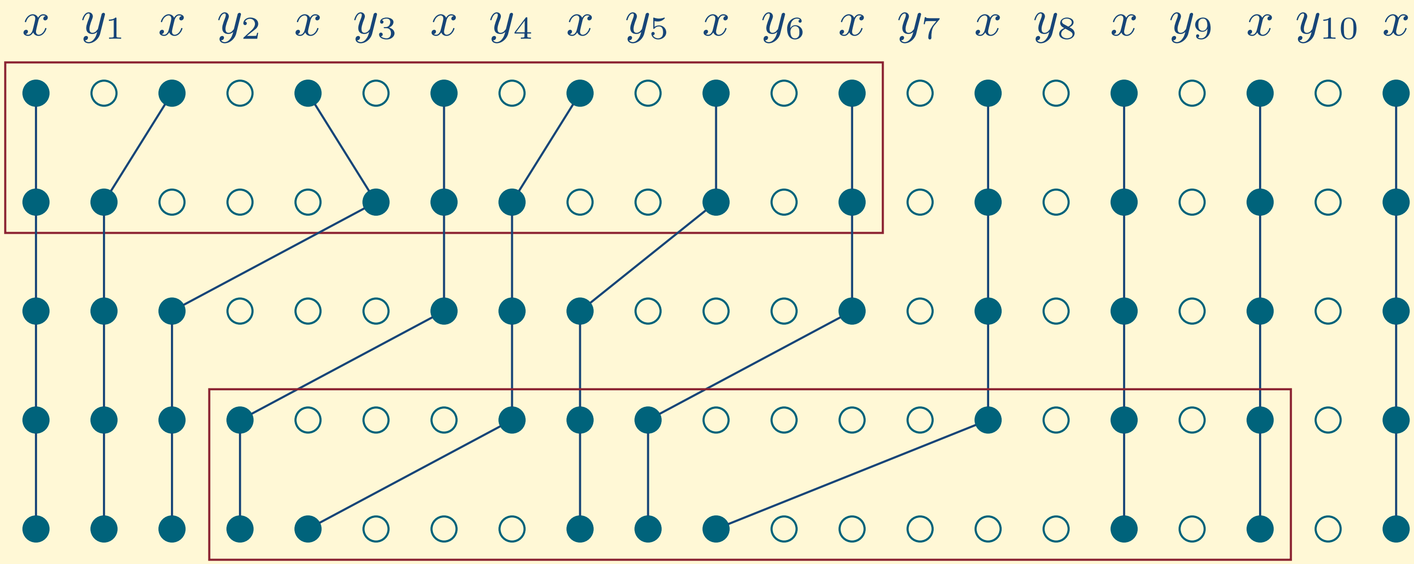
FEP and decidability

$$xy_1x\cdots xy_6x = x^2y_1y_2y_3x^3y_4y_5xy_6x$$



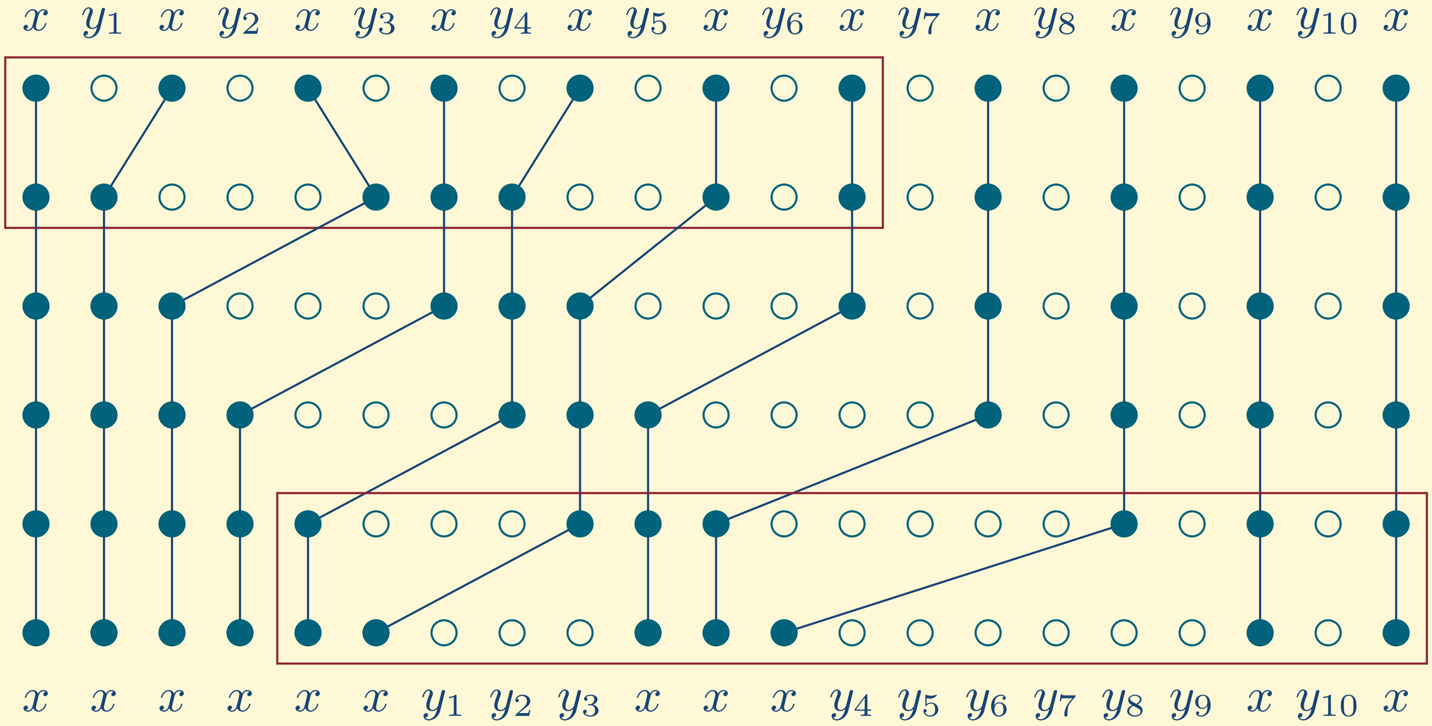
FEP and decidability

$$xy_1x\cdots xy_6x = x^2y_1y_2y_3x^3y_4y_5xy_6x$$

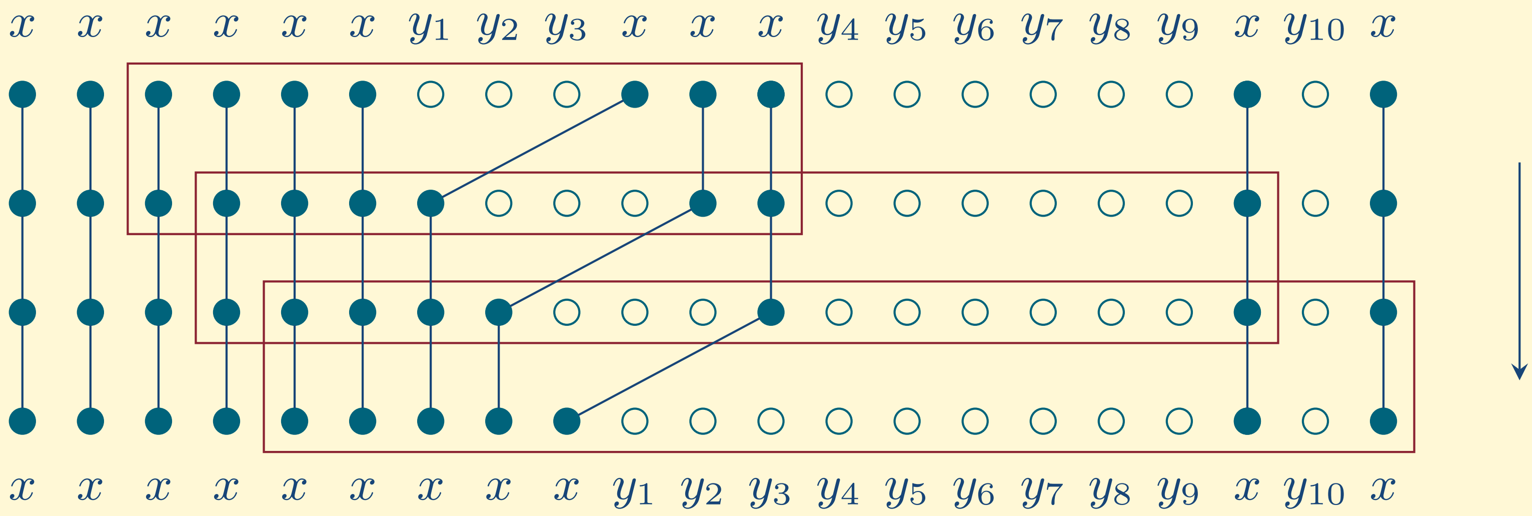


FEP and decidability

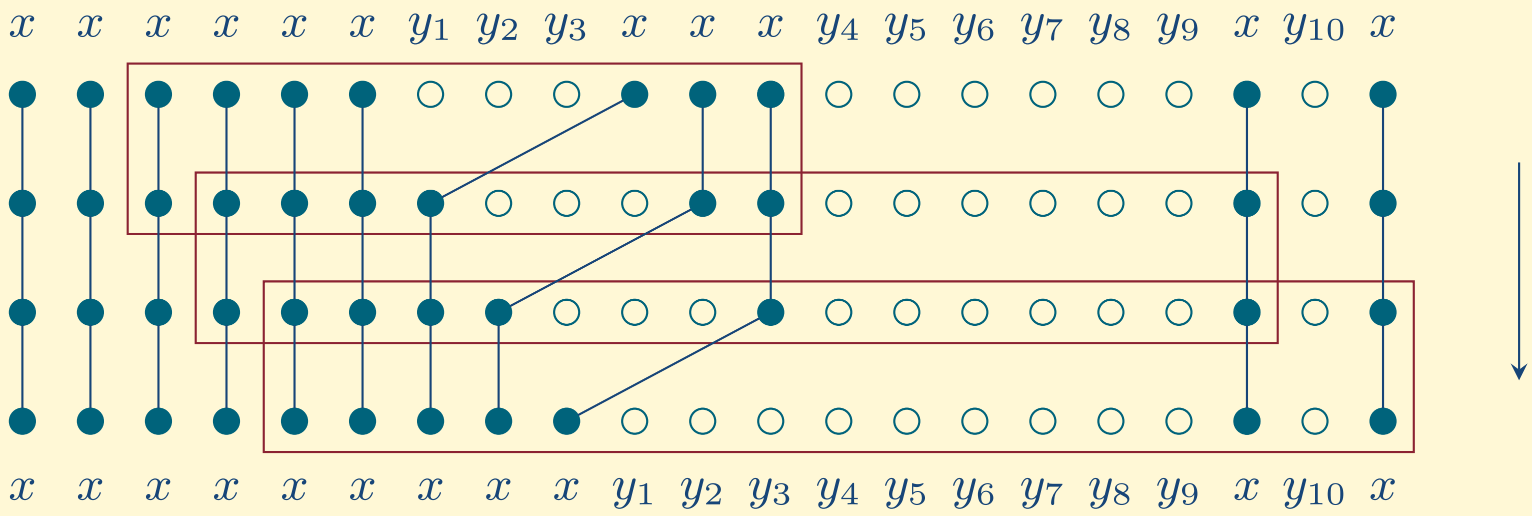
$$xy_1x \cdots xy_6x = x^2y_1y_2y_3x^3y_4y_5xy_6x$$



Using (a)

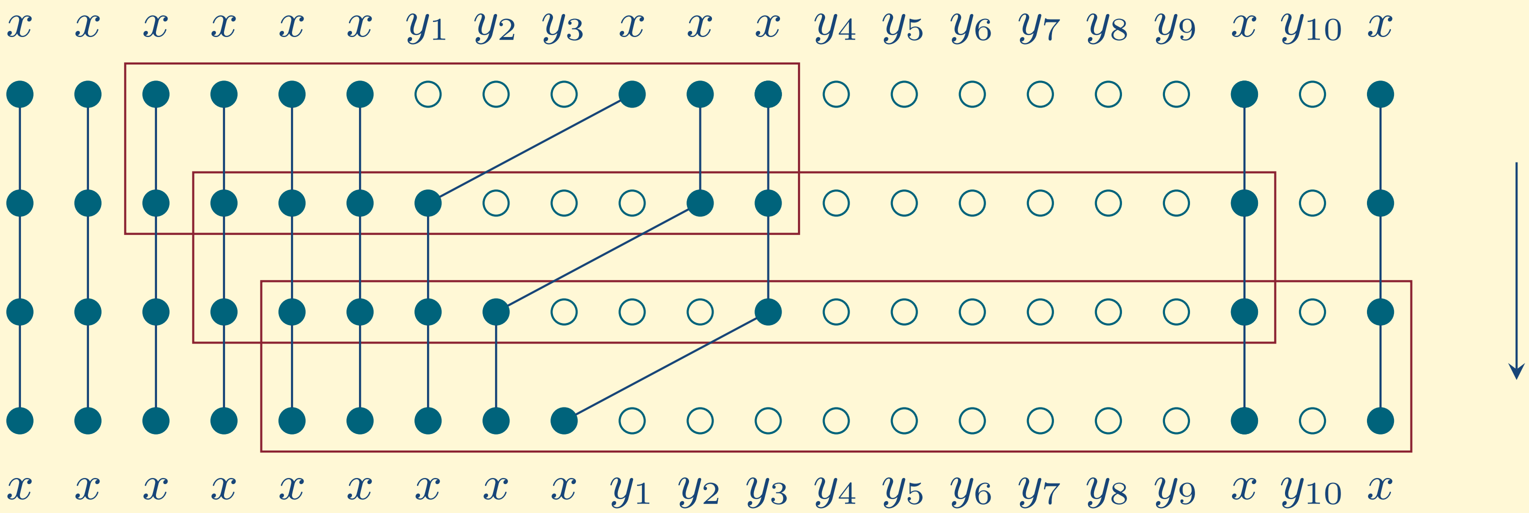

$x \ x \ x \ x \ x \ x \ y_1 \ y_2 \ y_3 \ x \ x \ x \ y_4 \ y_5 \ y_6 \ y_7 \ y_8 \ y_9 \ x \ y_{10} \ x$
$x \ x \ x \ x \ x \ x \ x \ x \ x \ y_1 \ y_2 \ y_3 \ y_4 \ y_5 \ y_6 \ y_7 \ y_8 \ y_9 \ x \ y_{10} \ x$

Using (a)



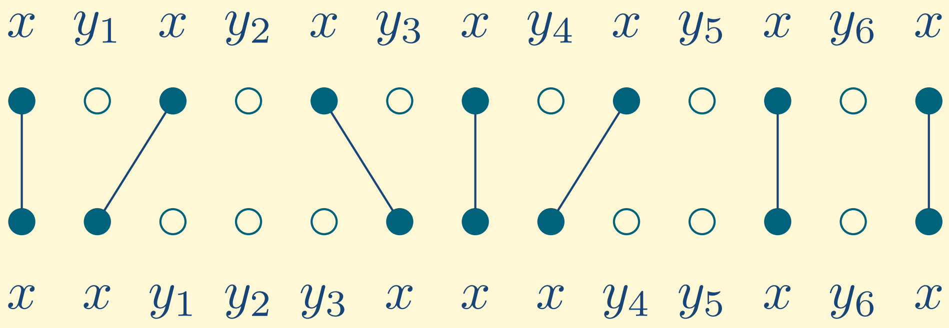
$$\alpha_N(xy_1xy_2xy_3xy_4xy_5xy_6xy_7xy_8xy_9xy_{10}x) = xx^8y_1y_2y_3y_4y_5y_6y_7y_8y_9xy_{10}x$$

Using (a)



$$\alpha_N(xy_1xy_2xy_3xy_4xy_5xy_6xy_7xy_8xy_9xy_{10}x) = xx^8y_1y_2y_3y_4y_5y_6y_7y_8y_9xy_{10}x$$

In (a) we have $\ell_1 = 1$ many x's in the 'front', $\ell_2 = 2$ in the end.



The structure of $\mathbf{H}$

Let $X_k = \{x_1, \ldots, x_k\}$ be a set of variables. The equation (a) implies (in the theory of monoids) the equation $\alpha_N^{\ell(a)}(s) = s$, for all $s \in X_k^*$, for some $\ell(a)$.

The structure of $\mathbf{H}$

Let $X_k = \{x_1, \ldots, x_k\}$ be a set of variables. The equation (a) implies (in the theory of monoids) the equation $\alpha_N^{\ell(a)}(s) = s$, for all $s \in X_k^*$, for some $\ell(a)$.

For a given $\ell = (\ell_0, \ell_1, d_\ell)$, with $\ell_1 + d_\ell < \ell_0$, the function $\alpha_N^\ell : X_k* \to X_k^*$ is defined as follows:

The structure of $\mathbf{H}$

Let $X_k = \{x_1, \ldots, x_k\}$ be a set of variables. The equation (a) implies (in the theory of monoids) the equation $\alpha_N^{\ell(a)}(s) = s$, for all $s \in X_k^*$, for some $\ell(a)$.

For a given $\ell = (\ell_0, \ell_1, d_\ell)$, with $\ell_1 + d_\ell < \ell_0$, the function $\alpha_N^\ell : X_k* \to X_k^*$ is defined as follows: $\alpha_N(s)$ is obtained from s by moving next to the (ℓ_1)th occurrence of x_i the $(\ell_1 + 1)$th, the $(\ell_1 + 2)$th, and up to the $(\ell_1 + d_\ell)$th occurrence of x_i, simultaneously for each x_i with more than ℓ_0-many occurrences in s. Thus by collecting all these consecutive occurrences next to the (ℓ_1)th occurrence of x_i we obtain a power of x_i.

The structure of $\mathbf{H}$

Let $X_k = \{x_1, \ldots, x_k\}$ be a set of variables. The equation (a) implies (in the theory of monoids) the equation $\alpha_N^{\ell(a)}(s) = s$, for all $s \in X_k^*$, for some $\ell(a)$.

For a given $\ell = (\ell_0, \ell_1, d_\ell)$, with $\ell_1 + d_\ell < \ell_0$, the function $\alpha_N^\ell : X_k* \to X_k^*$ is defined as follows: $\alpha_N(s)$ is obtained from s by moving next to the (ℓ_1)th occurrence of x_i the $(\ell_1 + 1)$th, the $(\ell_1 + 2)$th, and up to the $(\ell_1 + d_\ell)$th occurrence of x_i, simultaneously for each x_i with more than ℓ_0-many occurrences in s. Thus by collecting all these consecutive occurrences next to the (ℓ_1)th occurrence of x_i we obtain a power of x_i.

If we further we truncate the exponent of this power to be at most d_ℓ, for each x_i, then we obtain the element $\alpha_D(s)$. Clearly $\alpha_D[X_k^*]$ is finite, as we control the length of the words.
We also define $H = \alpha_N[X_k^*]$ with multiplication given by $\alpha_N(xy)$. It turns out that H is bijective with a subset of $\mathbb{N}^k \times \alpha_D[X_k^*]$, under the map $\psi(s) = (|s|_{x_1}, \ldots, |s|_{x_k}, \alpha_D(s))$, where $|s|_x$ denotes the number of occurrences of x in s.

The order per exponent

We have reduced the issue to a direct product of finitely many factors (we control the length of the words).

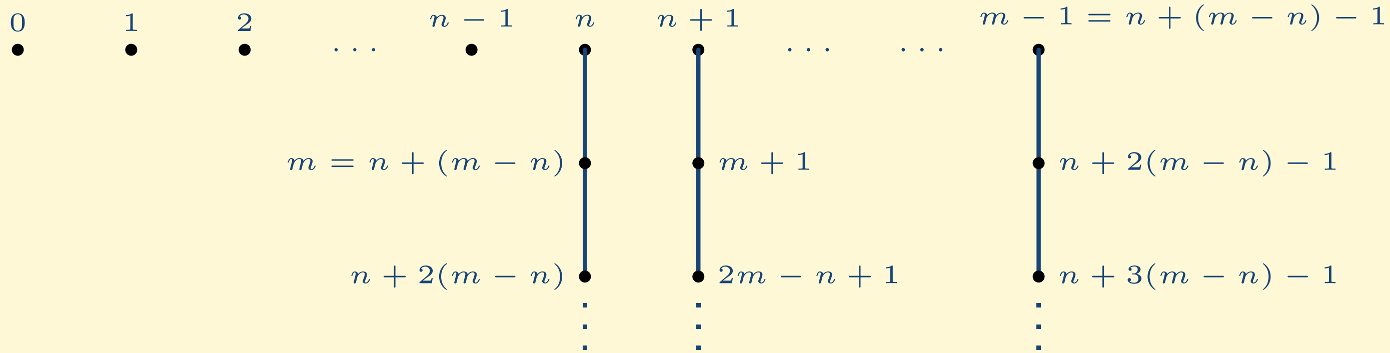
The order per exponent

We have reduced the issue to a direct product of finitely many factors (we control the length of the words). We focus on the structure of the exponents.

The order per exponent

We have reduced the issue to a direct product of finitely many factors (we control the length of the words). We focus on the structure of the exponents.

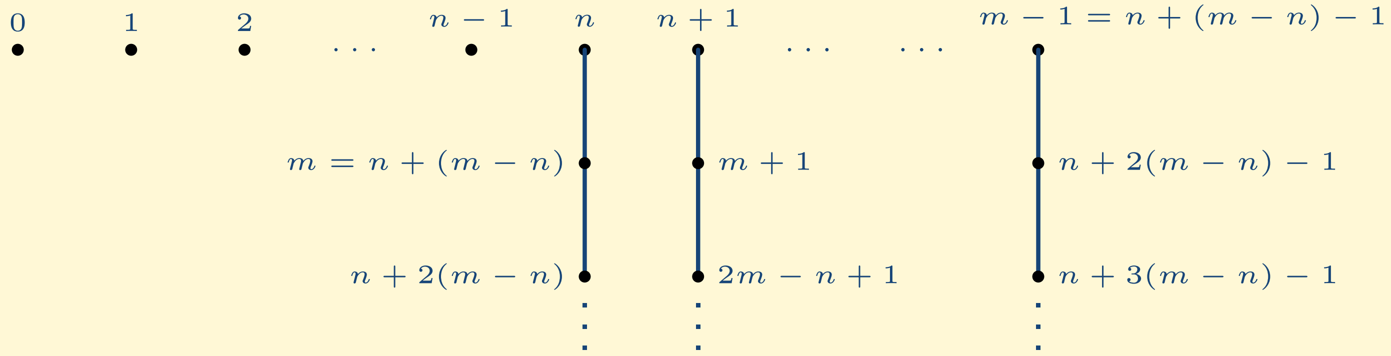
Given a knotted inequality $x^m \leq x^n$, and the above bijection, we can endow H with an order under which it becomes a pomonoid. In particular, the order on the component $\alpha_D[X_k^*]$ is discrete while the order $\leq_n^m$ on each component $\mathbb{N}$ is given as follows for the easier case $m > n$: $u \leq_n^m v$ if and only if $u = v$, or $n \leq v < u$ and $u \equiv v \pmod{m-n}$.



The order per exponent

We have reduced the issue to a direct product of finitely many factors (we control the length of the words). We focus on the structure of the exponents.

Given a knotted inequality $x^m \leq x^n$, and the above bijection, we can endow H with an order under which it becomes a pomonoid. In particular, the order on the component $\alpha_D[X_k^*]$ is discrete while the order $\leq_n^m$ on each component $\mathbb{N}$ is given as follows for the easier case $m > n$: $u \leq_n^m v$ if and only if $u = v$, or $n \leq v < u$ and $u \equiv v \pmod{m-n}$.

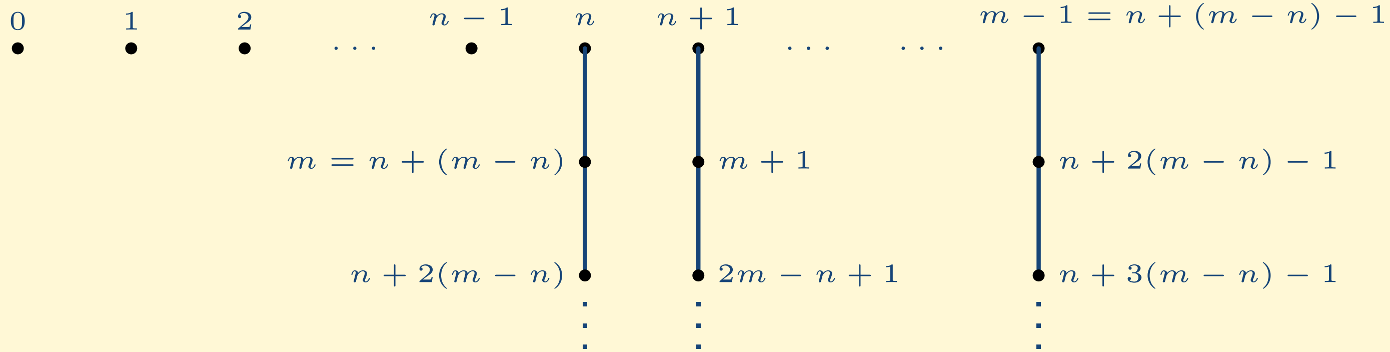


So, $\mathbf{H}$ is a dwpo.

The order per exponent

We have reduced the issue to a direct product of finitely many factors (we control the length of the words). We focus on the structure of the exponents.

Given a knotted inequality $x^m \leq x^n$, and the above bijection, we can endow H with an order under which it becomes a pomonoid. In particular, the order on the component $\alpha_D[X_k^*]$ is discrete while the order $\leq_n^m$ on each component $\mathbb{N}$ is given as follows for the easier case $m > n$: $u \leq_n^m v$ if and only if $u = v$, or $n \leq v < u$ and $u \equiv v \pmod{m-n}$.



So, $\mathbf{H}$ is a dwpo. For the second case $m < n$, we prove that $\mathbf{H}$ is a isomorphic to a finite union of finite products of well-ordered chains.