Projective Unification in Intermediate and Modal Predicate Logics

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Topology, Algebra, and Categories in Logic 2015, Ischia (Italy) 21 - 26 June 2015
• Unification, unifiers and projective unifiers in logic
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• Applications to (Almost) Structural Completeness and Admissible Rules
• The scope of projective unification in predicate logics:

$$L$$ enjoys projective unification iff $$L$$ extends IP.QLC - Gödel-Dummett predicate logic with Independence of Premises

A modal predicate logic $$L$$ extending QS4 = enjoys projective unification iff $$L$$ extends $$\Box$$ IP.QS4.3 = modal predicate logic QS4.3 with Modal Independence of Premises
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  • A modal predicate logic $L$ extending QS4$_=$ enjoys projective unification iff $L$ extends □IP.QS4.3$_=$ - modal predicate logic QS4.3$_=$ with Modal Independence of Premises
A substitution $\varepsilon$ is called a unifier for a formula $A$ in a logic $L$ if $\vdash_L \varepsilon(A)$ (or equivalently if $\varepsilon(A) \in L$).

Unifiers $\varepsilon : \text{Fm} \to \{\bot, \top\}$ are called ground unifiers.

A substitution $\varepsilon$ is said to be projective for $A$ in $L$ if $A \vdash_L B \leftrightarrow \varepsilon(B)$, for each $B$.  

S.Ghilardi: projective - formula, -unifier, $F_n/\{A\}$ projective alg.

We say that the logic $L$ has projective unification if each unifiable formula has a projective unifier.

Ex. Classical PC, Modal S5, NExt S4.3
A substitution $\varepsilon$ is called a unifier for a formula $A$ in a logic $L$ if

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**Projective unifiers**

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Ex. Classical PC, Modal S5, NExt S4.3
Applications: (A)SC, Admissible rules.

A schematic (structural) rule $r : \alpha_1, \ldots, \alpha_n, / \beta$ is \textit{admissible} in $L$, if adding $r$ does not change $L$, i.e. for every substitution $\tau$:

$$\{ \tau(\alpha_1), \ldots, \tau(\alpha_n) \} \subseteq L \Rightarrow \tau(\beta) \in L$$
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$r$ is derivable in $L$, if $\alpha_1, \ldots, \alpha_n \vdash_L \beta$. 

EX. P2: $\lozenge p \land \lozenge \neg p / \bot$ is passive in S4 and its extensions,

$L$ is Almost Structurally Complete, ASC, if every admissible rule which is not passive in $L$ is also derivable in $L$; (NExt S4.3, $L_n$)

FACT: $L$ has projective unification $\Rightarrow L$ is (Almost) Structurally Complete,

A description of admissible rules.
A schematic (structural) rule \( r : \alpha_1, \ldots, \alpha_n, / \beta \) is admissible in \( L \), if adding \( r \) does not change \( L \), i.e. for every substitution \( \tau \):
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A logic \( L \) is Structurally Complete, SC, if every admissible rule in \( L \) is also derivable in \( L \); (Class PC, LC, \( Int^{\rightarrow} \), Medvedev L.)
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\( r \) is \textit{derivable} in \( \mathbf{L} \), if \( \alpha_1, \ldots, \alpha_n \vdash_{\mathbf{L}} \beta \).

A logic \( \mathbf{L} \) is \textit{Structurally Complete, SC}, if every admissible rule in \( \mathbf{L} \) is also derivable in \( \mathbf{L} \); (Class PC, LC, \( \text{Int} \rightarrow \), Medvedev \( \mathbf{L} \).)

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A description of admissible rules
The scope of projective unification

Theorem (Wronski 1995 - 2008)

An intermediate logic $L \supseteq \text{Int}$ enjoys projective unification iff

$$((y \Rightarrow z) \lor (z \Rightarrow y)) \in L$$

iff

$$LC \subseteq L$$

iff

$\lor L$ definable ($\land$, $\rightarrow$).

Corollary

Every logic $L \supseteq LC$ is SC.

Theorem (WD, P. Wojtylak 2011)

A modal logic $L \supseteq S4$ enjoys projective unification iff

$$\Box((\Box y \rightarrow \Box z) \lor \Box(\Box z \rightarrow \Box y)) \in L$$

iff

$$S4.3 \subseteq L$$

Corollary

Every logic $L \supseteq S4.3$ is ASC.

Difference: in $LC$ - the method of ground unifiers works, in $S4.3$ it does not: projective unifiers - compositions of Löwenheim subst.'s
The scope of projective unification

Theorem (Wronski 1995 - 2008)

An intermediate logic \( L \supseteq \text{Int} \) enjoys projective unification iff \((y \Rightarrow z) \lor (z \Rightarrow y) \in L\), iff \( \text{LC} \subseteq L \) iff \( \lor \) L definable \((\land, \rightarrow)\).

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Every logic \( L \supseteq \text{LC} \) is SC.
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An intermediate logic \( L \supseteq \text{Int} \) enjoys projective unification iff
\[(y \Rightarrow z) \lor (z \Rightarrow y) \in L, \quad \text{iff} \quad \text{LC} \subseteq L \quad \text{iff} \quad \lor L \text{ definable} (\land, \rightarrow).\]

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Every logic \( L \supseteq \text{LC} \) is SC.

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A modal logic \( L \supseteq \text{S4} \) enjoys projective unification iff
\[\square(\square y \rightarrow \square z) \lor \square(\square z \rightarrow \square y) \in L, \quad \text{iff} \quad \text{S4.3} \subseteq L.\]

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Every logic \( L \supseteq \text{S4.3} \) is ASC.
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Theorem (Wronski 1995 - 2008)

An intermediate logic $L \supseteq \text{Int}$ enjoys projective unification iff

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A modal logic $L \supseteq \text{S4}$ enjoys projective unification iff

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Every logic $L \supseteq \text{S4.3}$ is ASC.

Difference: in $\text{LC}$ - the method of ground unifiers works, in $\text{S4.3}$ it does not: projective unifiers - compositions of Löwenheim subst.’s
We consider a first-order modal language without function letters.
We consider a first-order *modal* language without function letters.

- free individual variables: $a_1, a_2, a_3, \ldots$
- bound individual variables: $x_1, x_2, x_3, \ldots$
- predicate variables: $P_1, P_2, P_3, \ldots$
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The *intuitionistic* predicate language $\{\Rightarrow, \land, \lor, \bot, \forall, \Box\}$, it will also be seen as a definitional fragment of the initial modal predicate language.
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The intuitionistic predicate language \( \{ \Rightarrow, \land, \lor, \top, \forall, \Box \} \), it will also be seen as a definitional fragment of the initial modal predicate language.

Let \( Fm \) (or \( q-Fm \)) denote the set of all formulas (or quasi-formulas). \( \varphi \in Fm \) iff \( \varphi \in q-Fm \) and bounded variables in \( \varphi \) do not occur free.
Substitutions $\varepsilon : \text{q-Fm} \to \text{q-Fm}$ are mappings:

$$
\varepsilon(P(t_1, \ldots, t_k)) = \varepsilon(P(x_1, \ldots, x_k))[x_1/t_1, \ldots, x_k/t_k]
$$
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\begin{align*}
\varepsilon(P(t_1, \ldots, t_k)) &= \varepsilon(P(x_1, \ldots, x_k))[x_1/t_1, \ldots, x_k/t_k] \\
\varepsilon(\neg A) &= \neg \varepsilon(A) \\
\varepsilon(\square A) &= \square \varepsilon(A) \\
\varepsilon(\forall x A) &= \forall x \varepsilon(A) \\
\varepsilon(\exists x A) &= \exists x \varepsilon(A)
\end{align*}
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Warning: $= \text{is defined up to a correct renaming of bound variables}$ in the substituted formulas.

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$\varepsilon(\neg A) = \neg \varepsilon(A)$;
$\varepsilon(A \land B) = \varepsilon(A) \land \varepsilon(B)$;
$\varepsilon(\Box A) = \Box \varepsilon(A)$;
$\varepsilon(A \lor B) = \varepsilon(A) \lor \varepsilon(B)$;
$\varepsilon(\forall x A) = \forall x \varepsilon(A)$
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\varepsilon(\Box A) & = \Box \varepsilon(A); \quad \varepsilon(A \lor B) = \varepsilon(A) \lor \varepsilon(B); \\
\varepsilon(\forall x A) & = \forall x \varepsilon(A) \quad \varepsilon(\exists x A) = \exists x \varepsilon(A)
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$$
\text{vf}(\varepsilon(A)) \subseteq \text{vf}(A)
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Substitutions for predicate variables

Substitutions $\varepsilon : \text{q-Fm} \rightarrow \text{q-Fm}$ are mappings:

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- $\varepsilon(\neg A) = \neg \varepsilon(A)$;
- $\varepsilon(A \land B) = \varepsilon(A) \land \varepsilon(B)$;
- $\varepsilon(\Box A) = \Box \varepsilon(A)$;
- $\varepsilon(A \lor B) = \varepsilon(A) \lor \varepsilon(B)$;
- $\varepsilon(\forall x A) = \forall x \varepsilon(A)$
- $\varepsilon(\exists x A) = \exists x \varepsilon(A)$

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$$vf(\varepsilon(A)) \subseteq vf(A)$$

A *predicate modal logic* is any set $L \subseteq Fm$ containing (all classical propositional tautologies, and) the predicate and modal axioms:

$$\forall_x (A \rightarrow B(x)) \rightarrow (A \rightarrow \forall_x B(x))$$  
$$\forall_x A(x) \rightarrow A[x/t]$$  
$$\Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B),$$

closed under the following inferential rules

$$MP: \frac{A \rightarrow B, A}{B} \quad \text{and} \quad RN: \frac{A}{\Box A} \quad \text{and} \quad RG: \frac{A(a)}{\forall_x A(x)}$$

and closed under substitutions.
Each unifiable formula $A$ has a projective unifier in $\mathbf{S5}$:

$$
\varepsilon(B) = \begin{cases} 
\Box A \rightarrow B & \text{if } \nu(B) = \top \\
\Box A \land B & \text{if } \nu(B) = \bot 
\end{cases}
$$

where $\nu$ is a ground unifier for $A$. 

Each unifiable formula $A$ has a projective unifier in $\textbf{S5}$:

$$
\varepsilon(B) = \begin{cases} 
\Box A \rightarrow B & \text{if } v(B) = \top \\
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where $v$ is a ground unifier for $A$.

Such substitutions are called \textit{Löwenheim substitutions} for $A$. 
Löwenheim substitutions

Theorem

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Such substitutions are called Löwenheim substitutions for $A$.

Theorem

Each unifiable formula $A$ has a projective unifier in $Q-\textbf{S5}$:

\[
\varepsilon(B) = \begin{cases} 
\forall x A(x) \rightarrow B & \text{if} \quad \nu(B) = \top \\
\forall x A(x) \land B & \text{if} \quad \nu(B) = \bot
\end{cases}
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For logics weaker than $\textbf{S5}$ the above ‘ground unifier’ method must be modified.
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one can define a sequence $\varepsilon_1, \ldots, \varepsilon_n$ of Löwenheim substitutions for $A$ and take their composition $\varepsilon = \varepsilon_1 \circ \cdots \circ \varepsilon_n$ as a unifier for $A$. 
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**Theorem**

*Each unifiable formula has a projective unifier in $\textbf{S4.3}$ which is a composition of Löwenheim substitutions.*

**Theorem**

*Each unifiable formula $A$ has a projective unifier in $\mathbf{LC}$*

$$\varepsilon(B) = \begin{cases} 
A \Rightarrow B & \text{if } v(B) = \top \\
\neg \neg A \land (A \Rightarrow B) & \text{if } v(B) = \bot
\end{cases},$$

where $v$ is a ground unifier for $A$.

Note: This is not a Löwenheim substitution.
Intermediate predicate logics

Theorem
Each unifiable formula $A$ has a projective unifier in IP.

$Q^{-L}C_{\varepsilon}(B) = \{ \bigwedge x A \Rightarrow B \text{ if } \nu(B) = \top \}
\bigwedge x A(B(x)) \land \bigwedge x A(B(x)) \Rightarrow B \}
\text{if } \nu(B) = \bot$;

where $(IP)(A \Rightarrow \bigvee x B(x)) \Rightarrow \bigvee x (A \Rightarrow B(x)))$.

• Dzik, W. Chains of Structurally Complete Predicate Logics with the Application of Prucnal’s Substitution, RML 38(2004), (¬-less)

Theorem
An intermediate predicate logic $L$ enjoys projective unification iff $IP \cdot Q^{-LC} \subseteq L$ iff $\exists$ is $L$ definable by $(\forall, \rightarrow)$.

There is a chain of the type $\omega \omega + 1$ of ASC logics over IP.
**Theorem**

*Each unifiable formula A has a projective unifier in $IP.Q-\mathbf{LC}$*

\[
\varepsilon(B) = \begin{cases} 
\bigwedge_x A \Rightarrow B & \text{if } \nu(B) = \top \\
\neg\neg\bigwedge_x A(x) \land (\bigwedge_x A(x) \Rightarrow B) & \text{if } \nu(B) = \bot
\end{cases}
\]

where (IP) $(A \Rightarrow \bigvee_x B(x)) \Rightarrow \bigvee_x (A \Rightarrow B(x))$,

(Independence of Premises) and $\nu$ is a ground unifier for $A$.

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$$
\varepsilon(B) = \begin{cases} 
\bigwedge_{\bar{x}} A \Rightarrow B & \text{if } \nu(B) = \top \\
\neg \neg \bigwedge_{\bar{x}} A(\bar{x}) \land (\bigwedge_{\bar{x}} A(\bar{x}) \Rightarrow B) & \text{if } \nu(B) = \bot 
\end{cases};
$$

where (IP) $(A \Rightarrow \bigvee_{x} B(x)) \Rightarrow \bigvee_{x} (A \Rightarrow B(x))$, (Independence of Premises) and $\nu$ is a ground unifier for $A$.

- Dzik, W. Chains of Structurally Complete Predicate Logics with the Application of Prucnal’s Substitution, RML 38(2004), ($\neg$-less)

Theorem

An intermediate predicate logic $L$ enjoys projective unification iff $IP.Q-\text{LC} \subseteq L$ iff $\exists$ is $L$ definable by $(\forall, \rightarrow)$.

There is a chain of the type $\omega^\omega + 1$ of ASC logics over $IP.Q-\text{LC}$.
Let \( A(a) = (\square P(a) \vee \square Q(a)) \land (\sim P(a) \vee \sim Q(a)) \), where \( P, Q \) are monadic predicate symbols.
Let \( A(a) = (\Box P(a) \lor \Box Q(a)) \land (\sim P(a) \lor \sim Q(a)) \), where \( P, Q \) are monadic predicate symbols. There are four generalized Löwenheim substitutions for \( A \):

\[
\begin{align*}
\varepsilon_1(P(a)) &= \Box \forall_x A(x) \rightarrow P(a) & \text{and} & & \varepsilon_1(Q(a)) &= \Box \forall_x A(x) \land Q(a); \\
\varepsilon_2(P(a)) &= \Box \forall_x A(x) \land P(a) & \text{and} & & \varepsilon_2(Q(a)) &= \Box \forall_x A(x) \rightarrow Q(a); \\
\varepsilon_3(P(a)) &= \Box \forall_x A(x) \land P(a) & \text{and} & & \varepsilon_3(Q(a)) &= \Box \forall_x A(x) \land Q(a); \\
\varepsilon_4(P(a)) &= \Box \forall_x A(x) \rightarrow P(a) & \text{and} & & \varepsilon_4(Q(a)) &= \Box \forall_x A(x) \rightarrow Q(a)
\end{align*}
\]

No composition \( \varepsilon_{j_1} \circ \cdots \circ \varepsilon_{j_n} \) of \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \) is a unifier for \( A \).

On the other hand, since \( A \) is quantifier-free, it must have a projective unifier in \( Q \)-S4.3.
Let $A(a) = (\Box P(a) \lor \Box Q(a)) \land (\sim P(a) \lor \sim Q(a))$, where $P, Q$ are monadic predicate symbols. There are four generalized Löwenheim substitutions for $A$:

$\varepsilon_1(P(a)) = \Box \forall_x A(x) \rightarrow P(a)$ and $\varepsilon_1(Q(a)) = \Box \forall_x A(x) \land Q(a)$;

$\varepsilon_2(P(a)) = \Box \forall_x A(x) \land P(a)$ and $\varepsilon_2(Q(a)) = \Box \forall_x A(x) \rightarrow Q(a)$;

$\varepsilon_3(P(a)) = \Box \forall_x A(x) \land P(a)$ and $\varepsilon_3(Q(a)) = \Box \forall_x A(x) \land Q(a)$;

$\varepsilon_4(P(a)) = \Box \forall_x A(x) \rightarrow P(a)$ and $\varepsilon_4(Q(a)) = \Box \forall_x A(x) \rightarrow Q(a)$

No composition $\varepsilon_{j_1} \circ \cdots \circ \varepsilon_{j_n}$ of $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ is a unifier for $A$. 
Example 1

Let \( A(a) = (\Box P(a) \lor \Box Q(a)) \land (\sim P(a) \lor \sim Q(a)) \), where \( P, Q \) are monadic predicate symbols. There are four generalized Löwenheim substitutions for \( A \):

\[
\varepsilon_1(P(a)) = \Box \forall_x A(x) \to P(a) \quad \text{and} \quad \varepsilon_1(Q(a)) = \Box \forall_x A(x) \land Q(a);
\varepsilon_2(P(a)) = \Box \forall_x A(x) \land P(a) \quad \text{and} \quad \varepsilon_2(Q(a)) = \Box \forall_x A(x) \to Q(a);
\varepsilon_3(P(a)) = \Box \forall_x A(x) \land P(a) \quad \text{and} \quad \varepsilon_3(Q(a)) = \Box \forall_x A(x) \land Q(a);
\varepsilon_4(P(a)) = \Box \forall_x A(x) \to P(a) \quad \text{and} \quad \varepsilon_4(Q(a)) = \Box \forall_x A(x) \to Q(a)
\]

No composition \( \varepsilon_{j_1} \circ \cdots \circ \varepsilon_{j_n} \) of \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \) is a unifier for \( A \). On the other hand, since \( A \) is quantifier-free, it must have a projective unifier in \( Q\text{-S4.3} \).
Example 2

Let $A(a_1, a_2) = \Box Q(a_1, a_2) \lor P(a_1) \land \neg P(a_2)$ and let $\varepsilon(P(a_i)) = (\Box \forall y A(y) \land P(a_i) \lor B(a_i)) \land (\Box \forall y A(y) \rightarrow P(a_1))$.

$\varepsilon(Q(a_1, a_2)) = \Box \forall y A(y) \rightarrow Q(a_1, a_2)$

where the formula $B(x_1)$ will be specified later on. Note that the substitution $\varepsilon$ is projective for $A$ regardless of what $B(x_1)$ is.

One can check that $\varepsilon(A)$ is valid if one takes $x_1 = a_1$ as $B(x_1)$. 
Let $A(a_1, a_2) = \Box Q(a_1, a_2) \lor P(a_1) \land \neg P(a_2)$
Let $A(a_1, a_2) = □Q(a_1, a_2) \lor P(a_1) \land \neg P(a_2)$ and let

$$\varepsilon(P(a_i)) = (□\forall \bar{y} A(\bar{y}) \land P(a_i) \lor B(a_i)) \land (□\forall \bar{y} A(\bar{y}) \rightarrow P(a_1))$$

$$\varepsilon(Q(a_1, a_2)) = □\forall \bar{y} A(\bar{y}) \rightarrow Q(a_1, a_2)$$

where the formula $B(x_1)$ will be specified later on. Note that the substitution $\varepsilon$ is projective for $A$ regardless of what $B(x_1)$ is.
Let $A(a_1, a_2) = \square Q(a_1, a_2) \lor P(a_1) \land \lnot P(a_2)$ and let

$$\varepsilon(P(a_i)) = (\square \forall \bar{y} A(\bar{y}) \land P(a_i) \lor B(a_i)) \land (\square \forall \bar{y} A(\bar{y}) \rightarrow P(a_1))$$

$$\varepsilon(Q(a_1, a_2)) = \square \forall \bar{y} A(\bar{y}) \rightarrow Q(a_1, a_2)$$

where the formula $B(x_1)$ will be specified later on. Note that the substitution $\varepsilon$ is projective for $A$ regardless of what $B(x_1)$ is. One can check that $\varepsilon(A)$ is valid if one takes $x_1 = a_1$ as $B(x_1)$. 
Main result

Note that some of the ideas are violated in the above Example. First, the unifier \( \varepsilon \) does not fulfill the condition \( \nu_f(\varepsilon(B)) \subseteq \nu_f(B) \) as we allow \( \varepsilon(P(x_1)) \) to contain the variable \( a_1 \). Second, \( \varepsilon \) is not a composition of generalized Löwenheim substitutions for \( A \).

Third, to define \( \varepsilon \) one needs the equality symbol and it is not clear if our argument can be carried out without it, in \( Q\text{-S}_4 \).

Theorem □ IP \( Q\text{-S}_4 \).\( = \) enjoys projective unification.

where

\[ \Box \rightarrow \exists x \Box B(x) \rightarrow \exists x \Box (A \rightarrow B(x)) \]
Note that some of the ideas are violated in the above Example.
Note that some of the ideas are violated in the above Example. First, the unifier $\varepsilon$ does not fulfill the condition $vf(\varepsilon(B)) \subseteq vf(B)$ as we allow $\varepsilon(P(x_1))$ to contain the variable $a_1$. 

Second, $\varepsilon$ is not a composition of generalized Löwenheim substitutions for $A$. Third, to define $\varepsilon$ one needs the equality symbol and it is not clear if our argument can be carried out without it, in $Q$-S$_4$. 

**Theorem** □ $IP$.

$Q$-S$_4$. $3 = \text{enjoys projective unification.}$
Note that some of the ideas are violated in the above Example. First, the unifier \( \varepsilon \) does not fulfill the condition \( vf(\varepsilon(B)) \subseteq vf(B) \) as we allow \( \varepsilon(P(x_1)) \) to contain the variable \( a_1 \). Second, \( \varepsilon \) is not a composition of generalized Löwenheim substitutions for \( A \).
Note that some of the ideas are violated in the above Example. First, the unifier $\varepsilon$ does not fulfill the condition $\text{vf}(\varepsilon(B)) \subseteq \text{vf}(B)$ as we allow $\varepsilon(P(x_1))$ to contain the variable $a_1$. Second, $\varepsilon$ is not a composition of generalized Löwenheim substitutions for $A$. Third, to define $\varepsilon$ one needs the equality symbol and it is not clear if our argument can be carried out without it, in $Q$-$\textbf{S4.3}$. 
Note that some of the ideas are violated in the above Example. First, the unifier \( \varepsilon \) does not fulfill the condition \( \text{vf}(\varepsilon(B)) \subseteq \text{vf}(B) \) as we allow \( \varepsilon(P(x_1)) \) to contain the variable \( a_1 \). Second, \( \varepsilon \) is not a composition of generalized Löwenheim substitutions for \( A \). Third, to define \( \varepsilon \) one needs the equality symbol and it is not clear if our argument can be carried out without it, in \( Q\text{-S4.3} \).

**Theorem**

\( \square IP \cdot Q\text{-S4.3} \equiv \text{enjoys projective unification.} \)

where

\[
(\square IP) \quad \square(A \rightarrow \exists x \square B(x)) \rightarrow \exists x \square(A \rightarrow B(x))
\]
Let us consider \(\exists x \Box P(x)\) and let \(\varepsilon\) be its projective unifier in \(L\):

\[
\exists x \Box P(x) \vdash L P(a) \leftrightarrow \varepsilon(P(a)) \quad \text{and} \quad \vdash L \exists x \Box \varepsilon(P(x))
\]

Hence \(\vdash L \varepsilon(P(a)) \rightarrow (\exists x \Box P(x) \rightarrow P(a))\) and consequently \(\vdash L \exists x \Box (\exists x \Box P(x) \rightarrow P(x))\) which is equivalent (regarded as an axiom schema) to

\[
\Box IP \vdash L \Box (A \rightarrow \exists x \Box P(x)) \rightarrow \exists x \Box (A \rightarrow P(x))
\]

Theorem Any modal predicate logic \(L\) (over \(Q-S^4\)) which enjoys projective unification extends \(\Box IP\).

Corollary Any modal predicate logic \(L\) with equality (over \(Q-S^4\)) enjoys projective unification iff \(L\) extends \(\Box IP\).
Let us consider $\exists_x \Box P(x)$
Let us consider $\exists_x \square P(x)$ and let $\varepsilon$ be its projective unifier in $L$: 
Let us consider $\exists x \Box P(x)$ and let $\varepsilon$ be its projective unifier in $L$:

$$\exists x \Box P(x) \vdash_L P(a) \leftrightarrow \varepsilon(P(a)) \quad \text{and} \quad \vdash_L \exists x \Box \varepsilon(P(x)).$$
Let us consider $\exists x \square P(x)$ and let $\varepsilon$ be its projective unifier in $L$:

$\exists x \square P(x) \vdash_L P(a) \leftrightarrow \varepsilon(P(a))$ and $\vdash_L \exists x \square \varepsilon(P(x))$.

Hence $\vdash_L \varepsilon(P(a)) \rightarrow (\exists x \square P(x) \rightarrow P(a))$
Let us consider $\exists x \Box P(x)$ and let $\varepsilon$ be its projective unifier in $L$:

$$\exists x \Box P(x) \vdash_L P(a) \leftrightarrow \varepsilon(P(a)) \quad \text{and} \quad \vdash_L \exists x \Box \varepsilon(P(x)).$$

Hence $\vdash_L \varepsilon(P(a)) \rightarrow (\exists x \Box P(x) \rightarrow P(a))$ and consequently

$$\vdash_L \exists x \Box (\exists x \Box P(x) \rightarrow P(x)).$$
Let us consider $\exists_x \Box P(x)$ and let $\varepsilon$ be its projective unifier in $L$:

$$\exists_x \Box P(x) \vdash_L P(a) \leftrightarrow \varepsilon(P(a)) \quad \text{and} \quad \vdash_L \exists_x \Box \varepsilon(P(x)).$$

Hence $\vdash_L \varepsilon(P(a)) \rightarrow (\exists_x \Box P(x) \rightarrow P(a))$ and consequently

$$\vdash_L \exists_x \Box (\exists_x \Box P(x) \rightarrow P(x))$$

which is equivalent (regarded as an axiom schema) to

$$(\Box IP) \quad \vdash_L \Box (A \rightarrow \exists_x \Box P(x)) \rightarrow \exists_x \Box (A \rightarrow P(x)).$$
Let us consider $\exists x \Box P(x)$ and let $\varepsilon$ be its projective unifier in $L$:

$$\exists x \Box P(x) \vdash_L P(a) \leftrightarrow \varepsilon(P(a)) \quad \text{and} \quad \vdash_L \exists x \Box \varepsilon(P(x)).$$

Hence $\vdash_L \varepsilon(P(a)) \rightarrow (\exists x \Box P(x) \rightarrow P(a))$ and consequently

$$\vdash_L \exists x \Box (\exists x \Box P(x) \rightarrow P(x))$$

which is equivalent (regarded as an axiom schema) to

$$(\Box IP) \quad \vdash_L \Box (A \rightarrow \exists x \Box P(x)) \rightarrow \exists x \Box (A \rightarrow P(x)).$$

**Theorem**

*Any modal predicate logic $L$ (over $Q-S4$) which enjoys projective unification extends $\Box IP$. Q-S4.3.*
Let us consider $\exists_x \Box P(x)$ and let $\varepsilon$ be its projective unifier in $L$:

$$\exists_x \Box P(x) \vdash_L P(a) \leftrightarrow \varepsilon(P(a)) \quad \text{and} \quad \vdash_L \exists_x \Box \varepsilon(P(x)).$$

Hence $\vdash_L \varepsilon(P(a)) \rightarrow (\exists_x \Box P(x) \rightarrow P(a))$ and consequently

$$\vdash_L \exists_x \Box (\exists_x \Box P(x) \rightarrow P(x))$$

which is equivalent (regarded as an axiom schema) to

$$(\Box IP) \quad \vdash_L \Box (A \rightarrow \exists_x \Box P(x)) \rightarrow \exists_x \Box (A \rightarrow P(x)).$$

Theorem

Any modal predicate logic $L$ (over $Q$-$S4$) which enjoys projective unification extends $\Box IP . Q$-$S4.3$.

Corollary

Any modal predicate logic $L_=$ with equality (over $Q$-$S4_=$) enjoys projective unification iff $L_=$ extends $\Box IP . Q$-$S4.3_=$.