On maps between Stone-Čech compactifications induced by lattice homomorphisms

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All spaces in this talk are Tychonoff (i.e., completely regular and Hausdorff), and all frames are completely regular.

In the article

S. Broverman
Homomorphisms between lattices of zero-sets
Canad. Math. Bull. **21** (1978), 1–5.

Broverman shows that any lattice homomorphism $t\colon \boldsymbol{Z}(Y) \to \boldsymbol{Z}(X)$ induces a continuous map $\tau\colon \beta X \to \beta Y$. This is how he does it. For any $p \in \beta X$, $\boldsymbol{A}^p$ is the z-ultrafilter on X given by

$$Z \in \boldsymbol{A}^p \quad \Longleftrightarrow \quad p \in \mathrm{cl}_{\beta X} Z.$$

Now $t^{\leftarrow}[\boldsymbol{A}^p]$ is a prime z-filter in Y, and is therefore contained in some unique z-ultrafilter $\boldsymbol{A}^q$ on Y. Broverman shows that the function $\tau\colon \beta X \to \beta Y$ defined by $\tau(p) = q$ is a continuous map.

Let **DLat** denote the category of bounded distributive lattices and their homomorphisms. Recall that the ideal-lattice functor $\mathfrak{J}\colon \mathbf{DLat} \to \mathbf{Frm}$ sends $A \in \mathbf{DLat}$ to the frame $\mathfrak{J}A$ of ideals of A, and sends a lattice homomorphism $\phi\colon A \to B$ to the frame homomorphism $\mathfrak{J}\phi\colon \mathfrak{J}A \to \mathfrak{J}B$ given by

$$\mathfrak{J}\phi(I) = \{b \in B \mid b \leq \phi(a) \text{ for some } a \in I\}.$$

Now let L and M be completely regular frames and $\phi\colon \operatorname{Coz} L \to \operatorname{Coz} M$ be a lattice homomorphism. Then

- ϕ preserves the completely below relation, $\prec\!\!\prec$.
- As a consequence, $\mathfrak{J}\phi(I) \in \beta M$ whenever $I \in \beta L$.

Since βL and βM are subframes of $\mathfrak{J}(\operatorname{Coz} L)$ and $\mathfrak{J}(\operatorname{Coz} M)$ respectively, it follows that the restriction of $\mathfrak{J}\phi$ to βL is a frame homomorphism into βM. We denote it by $\bar{\phi}$.

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That is

$$\bar{\phi}\colon \beta L \to \beta M$$

is the frame homomorphism given by

$$\bar{\phi}(I) = \{d \in \mathrm{Coz}\, M \mid d \leq h(c) \text{ for some } c \in \mathrm{Coz}\, L\}.$$

A lattice homomorphism $t\colon Z(Y) \to Z(X)$ induces a lattice homomorphism

$$\hat{t}\colon \mathrm{Coz}(\mathfrak{O}Y) \to \mathrm{Coz}(\mathfrak{O}X) \quad \text{by} \quad U \mapsto X \smallsetminus t(Y \smallsetminus U);$$

and a lattice homomorphism $s\colon \mathrm{Coz}(\mathfrak{O}Y) \to \mathrm{Coz}(\mathfrak{O}X)$ induces a lattice homomorphism

$$\check{s}\colon Z(Y) \to Z(X) \quad \text{by} \quad F \mapsto X \smallsetminus s(Y \smallsetminus F).$$

The correspondences $t \mapsto \hat{t}$ and $s \mapsto \check{s}$ are one-one onto, and are inverses to each other.

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$$\tilde{s}\colon \boldsymbol{Z}(Y) \to \boldsymbol{Z}(X) \quad \text{by} \quad F \mapsto X \smallsetminus s(Y \smallsetminus F).$$

The correspondences $t \mapsto \hat{t}$ and $s \mapsto \tilde{s}$ are one-one onto, and are inverses to each other.

NOTATION

We denote by $r_L\colon L \to \beta L$ the right adjoint of $\beta L \to L$.

Lemma

Let $\phi\colon \mathrm{Coz}\, L \to \mathrm{Coz}\, M$ be a lattice homomorphism, and $\bar{\phi}\colon \beta L \to \beta M$ be the frame homomorphism as above. The right adjoint of $\bar{\phi}$ is given by

$$\begin{aligned} \bar{\phi}_*(J) &= \bigvee\{r_L(c) \mid c \in \mathrm{Coz}\, L \text{ and } \phi(c) \in J\} \\ &= \bigcup\{r_L(c) \mid c \in \mathrm{Coz}\, L \text{ and } \phi(c) \in J\}. \end{aligned}$$

Let us view βX (and also βY) as the space $\Sigma\beta(\mathfrak{O}X)$, where the spectrum is taken as the set of prime elements of $\beta(\mathfrak{O}X)$, that is, the maximal regular ideals of $\mathrm{Coz}(\mathfrak{O}X)$.

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Theorem

Let $t\colon \mathbf{Z}(Y) \to \mathbf{Z}(X)$ *be a lattice homomorphism. Write* h *for the map* $\bar{\hat{t}}\colon \beta(\mathfrak{O}Y) \to \beta(\mathfrak{O}X)$. *Then* $\tau = \Sigma h$.

Theorem

Let $s\colon \mathrm{Coz}(\mathfrak{O}Y) \to \mathrm{Coz}(\mathfrak{O}X)$ *be a lattice homomorphism. Denote by* $t\colon \mathbf{Z}(Y) \to \mathbf{Z}(X)$ *the lattice homomorphism it induces. Consider the commutative square*

$$\begin{array}{ccc} \beta(\mathfrak{O}Y) & \xrightarrow{\bar{s}} & \beta(\mathfrak{O}X) \\ \cong\downarrow & & \downarrow\cong \\ \mathfrak{O}(\beta Y) & \xrightarrow{\bar{s}} & \mathfrak{O}(\beta X) \end{array}$$

Then $\mathfrak{O}\tau = \bar{s}$.

We extend the meaning of the term “dense" by defining a lattice homomorphism to be dense if the zero of its domain is the only element mapped to zero. Be reminded that a frame homomorphism between compact regular frames is dense precisely when it is one-one.

Proposition

The homomorphism $\bar{\phi}\colon \beta L \to \beta M$ is one-one iff ϕ is dense.

We say a lattice homomorphism $t\colon Z(Y) \to Z(X)$ is codense if, for any $K \in Z(Y)$, $t(K) = X$ implies $K = Y$. Clearly, t is codense if and only if the associated $\hat{t}\colon \mathrm{Coz}(\mathfrak{O}Y) \to \mathrm{Coz}(\mathfrak{O}X)$ is dense.

Corollary

Let $t\colon Z(Y) \to Z(X)$ be a lattice homomorphism. The induced map $\tau\colon \beta X \to \beta Y$ is surjective iff t is codense.

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Corollary

Let $t\colon \boldsymbol{Z}(Y) \to \boldsymbol{Z}(X)$ be a lattice homomorphism. The induced map $\tau\colon \beta X \to \beta Y$ is surjective iff t is codense.

Borrowing terminology from

R.N. Ball, A.W. Hager and J. Walters-Wayland
An intrinsic characterization of monomorphisms in regular Lindelöf locales
Appl. Categor. Struct. **15** (2007), 109–118.

we formulate the following definition:

Definition

(a) A lattice homomorphism $\phi\colon \operatorname{Coz} L \to \operatorname{Coz} M$ is uplifting if whenever $u \vee v = 1$ in $\operatorname{Coz} M$, then there exist $a, b \in \operatorname{Coz} L$ such that $a \vee b = 1$, $\phi(a) \leq u$ and $\phi(b) \leq v$.

(b) We say a lattice homomorphism $t\colon \boldsymbol{Z}(Y) \to \boldsymbol{Z}(X)$ is deflating if, for any disjoint zero-sets E and F of X, there are disjoint zero-sets G and H of Y such that $E \subseteq t(G)$ and $F \subseteq t(H)$.

Theorem

The following conditions are equivalent for a lattice homomorphism $\phi\colon \operatorname{Coz} L \to \operatorname{Coz} M$.

1. $\bar{\phi}\colon \beta L \to \beta M$ *is onto.*
2. ϕ *is uplifting.*
3. *Whenever* $u \prec\!\!\prec v$ *in* $\operatorname{Coz} M$, *there are elements* $a \prec\!\!\prec b$ *in* $\operatorname{Coz} L$ *such that* $u \leq \phi(a)$ *and* $\phi(b) \leq v$.

Recall that if $f\colon X \to Y$ is a continuous map between Tychonoff spaces, then $\mathfrak{O}f\colon \mathfrak{O}Y \to \mathfrak{O}X$ is one-one if and only if f is the inclusion of a subspace.

Corollary

Let $t\colon Z(Y) \to Z(X)$ *be a lattice homomorphism. The induced map* $\tau\colon \beta X \to \beta Y$ *is the inclusion of a subspace iff* t *is deflating.*

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In their study of patch-generated frames,

A.W. Hager and J. Martínez
Patch-generated frames and projectable hulls
Appl. Categor. Struct. **15** (2007), 49–80.

Hager and Martínez call a frame homomorphism $h\colon L \to M$ $*$-dense if, for any $b \in M$,

$$h_*(b) = 0 \quad \Longrightarrow \quad b = 0.$$

Continuous maps $f\colon X \to Y$ for which $\mathfrak{O}f\colon \mathfrak{O}Y \to \mathfrak{O}X$ is $*$-dense occur quite naturally. Indeed, recall that a surjective continuous map is called irreducible if it sends no proper closed subset of its domain onto its codomain. Since, for any continuous map $f\colon X \to Y$ and any $U \in \mathfrak{O}X$,

$$(\mathfrak{O}f)_*(U) = Y \smallsetminus \overline{f[X \smallsetminus U]},$$

it follows that

> *a closed continuous surjection $f\colon X \to Y$ is irreducible iff the frame homomorphism $\mathfrak{O}f\colon \mathfrak{O}Y \to \mathfrak{O}X$ is $*$-dense.*

Definition

A lattice homomorphism $\psi\colon A \to B$ inverse-dense if, for any ideal J of B, $\psi^{\leftarrow}[J] = \{0\}$ implies $J = \{0\}$.

Theorem

The map $\bar{\phi}\colon \beta L \to \beta M$ is $*$-dense iff ϕ is inverse-dense.

Corollary

Let $t\colon \mathbf{Z}(Y) \to \mathbf{Z}(X)$ be a codense lattice homomorphism. The induced map $\tau\colon \beta X \to \beta Y$ is irreducible iff for every nontrivial z-filter $\mathcal{F}$ in X, there is a zero-set $Z \neq Y$ of Y such that $t(Z) \in \mathcal{F}$.

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Theorem

A lattice homomorphism between cozero parts of pseudocompact frames contracts maximal ideals to maximal ideals iff it is a σ-frame homomorphism.

Pseudocompactness is a "conservative" notion. That is, a Tychonoff space is pseudocompact if and only if the frame of its open sets is pseudocompact. Therefore we have the following corollary.

Corollary

Let X and Y be pseudocompact Tychonoff spaces. The following conditions are equivalent for a lattice homomorphism $t\colon Z(Y) \to Z(X)$.

1. $t^{\leftarrow}[\boldsymbol{A}^p] = \boldsymbol{A}^{\tau(p)}$ *for every* $p \in \beta X$.
2. $\mathrm{cl}_{\beta X}\, t(Z) = \tau^{\leftarrow}[\mathrm{cl}_{\beta Y}\, Z]$ *for every* $Z \in Z(Y)$.
3. *t is a σ-homomorphism.*

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THANK YOU