Quasi-primal Cornish algebras

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Outline

Quasi-primal algebras

Definition

A finite algebra **A** is called quasi-primal if (any of) the following equivalent conditions hold:

(1) the ternary discriminator operation $t\colon A^3 \to A$, given by

$$t(x,y,z) := \begin{cases} x, & \text{if } x \neq y, \\ z, & \text{if } x = y, \end{cases}$$

is a term function of **A**;

(2) the variety Var(**A**) generated by **A** is congruence distributive and congruence permutable and
   - every non-trivial subalgebra of **A** is simple;

(3) **A** has a majority term and every subuniverse of $\mathbf{A}^2$ is
   - a product $B \times C$ of two subuniverses of **A**, or
   - the graph of a partial automorphism of **A**, i.e., the graph of an isomorphism $u\colon \mathbf{B} \to \mathbf{C}$ where $\mathbf{B} \leqslant \mathbf{A}$ and $\mathbf{C} \leqslant \mathbf{A}$.

Every lattice-based algebra has a majority term:

$$m(x,y,z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x).$$

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Some quasi-primal algebras from logic

1. The two-element Boolean algebra

is quasi-primal with $t(x, y, z) := ((x \wedge z) \vee \neg y) \wedge (x \vee z)$.

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2. A finite Heyting algebra with 1 as a predicate

Let $\mathbf{A} = \langle A; \vee, \wedge, \rightarrow, f, 0, 1 \rangle$ be a finite algebra such that $\langle A; \vee, \wedge, \rightarrow, 0, 1 \rangle$ is a Heyting algebra and $f\colon A \rightarrow A$ is the characteristic function of $\{1\}$. Then $\mathbf{A}$ is quasi-primal.

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3. A finite Heyting algebra with dual pseudocomplement

Let $\mathbf{A} = \langle A; \vee, \wedge, \to, ^+, 0, 1\rangle$ be a finite algebra such that $\langle A; \vee, \wedge, \to, 0, 1\rangle$ is a Heyting algebra with $\vee$-irreducible 1 with dual pseudocomplement $^+$. Then $\mathbf{A}$ is quasi-primal.
(Via 2. with $f(x) := x^{++}$.)

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4. A finite simple relation algebra

Let $\mathbf{A} = \langle A; \vee, \wedge, {}', 0, 1, 1', \breve{}, ;\rangle$ be a finite simple relation algebra. Then $\mathbf{A}$ is quasi-primal. $\big($Via 2. with $f(x) := (1 \,;\, x \,;\, 1)'$.$\big)$

Quasi-primal algebras and decidable theories

- In the late 1940s, A. Tarski proved that the first-order theory of Boolean algebras is decidable. This can be restated as the

  the first-order theory of the variety generated by the two-element Boolean algebra is decidable.

- In the late 1970s, H. Werner extended Tarski's result by showing that every variety generated by a quasi-primal algebra has a decidable first-order theory.
- Ten years later, this result played an important role in R. McKenzie and M. Valeriote's characterisation of locally finite varieties with a decidable theory.

Outline

Ockham algebras

Definition

An algebra $\mathbf{A} = \langle A; \vee, \wedge, g^{\mathbf{A}}, 0, 1\rangle$ is an Ockham algebra if

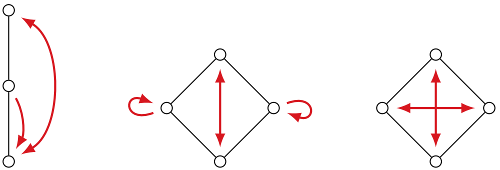
(a) $\mathbf{A}^{\flat} := \langle A; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice,

(b) $\mathbf{A}$ satisfies the De Morgan laws

$$g(x \vee y) \approx g(x) \wedge g(y) \quad \text{and} \quad g(x \wedge y) \approx g(x) \vee g(y),$$

(c) $\mathbf{A}$ satisfies $g(0) \approx 1$ and $g(1) \approx 0$.

Note that (b) and (c) say that $g^{\mathbf{A}}$ is a dual endomorphism of $\mathbf{A}^{\flat}$.

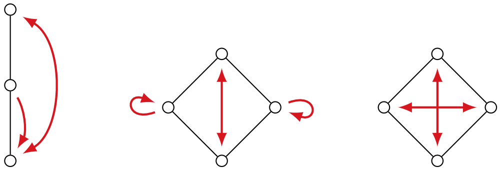


Ockham algebras

Why are they called Ockham algebras rather than De Morgan algebras?

- De Morgan algebras were studied and named before Ockham algebras, so the name was already taken.
- A De Morgan algebra is defined to be an Ockham algebra satisfying the double negation law, that is,

$$g(g(v)) \approx v.$$



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- In 1979, Urquhart named Ockham algebras after William of Ockham who, in 1323, wrote down De Morgan's Laws, some 500 years before De Morgan did, in 1847.
  - ... the contradictory opposite of a copulative proposition is a disjunctive proposition composed of the contradictory opposites of its parts.
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    Summa totius logicae c. 1323 (transl. P. Boehner 1955)

From Ockham to Cornish

Definition

Let $F = F_+ \mathbin{\dot\cup} F_-$ be a set of unary operation symbols. An algebra $\mathbf{A} = \langle A; \vee, \wedge, F^{\mathbf{A}}, 0, 1\rangle$ is a Cornish algebra (of type F) if

- $\mathbf{A}^\flat := \langle A; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice, and
- $F^{\mathbf{A}} = \{\, f^{\mathbf{A}} \mid f \in F \,\}$ is a set of unary operations on A such that
  - $f^{\mathbf{A}}$ is an endomorphism of $\mathbf{A}^\flat$, for each $f \in F_+$, and
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- When $f \in F_-$, the operation $f^{\mathbf{A}}$ models a negation that satisfies De Morgan's laws.
- The operations $f^{\mathbf{A}}$, for $f \in F_+$, model strong modal operators which preserve $\wedge$ as well as $\vee$;
  the next operator of linear temporal logic is an example.

Ockham algebras are the special case of Cornish algebras in which $F = F_- = \{g\}$.

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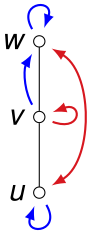

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- Cornish algebras are named in honour of the Australian mathematician William H. (“Bill") Cornish who died seven years ago.
- He introduced them in an invited lecture entitled Monoids acting on distributive lattices at the annual meeting of the Australian Mathematical Society at La Trobe University in May 1977.
- The notes from that lecture were never published but were distributed privately.

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- The notes from that lecture were never published but were distributed privately.
- The notes first appeared in print as part of Cornish’s far-reaching monograph published nine years later:
  - Cornish, W.H., *Antimorphic Action: Categories of Algebraic Structures with Involutions or Anti-endomorphisms*, Research and Exposition in Mathematics, Heldermann, Berlin (1986)

Outline

Priestley duality for Cornish algebras

$\mathbb{W} := \langle W; F^{\mathbb{W}}, \leqslant \rangle$ is a finite Cornish space (of type F) if

- $\mathbb{W}^{\flat} := \langle W; \leqslant \rangle$ is a finite ordered set, and
- $F^{\mathbb{W}} = \{ f^{\mathbb{W}} \mid f \in F \}$ is a set of unary operations on W such that
  - $f^{\mathbb{W}} \colon W \to W$ is order-preserving, for each $f \in F_+$, and
  - $g^{\mathbb{W}} \colon W \to W$ is order-reversing, for each $g \in F_-$.

Ockham spaces arise in the special case when $F = F_- = \{g\}$.

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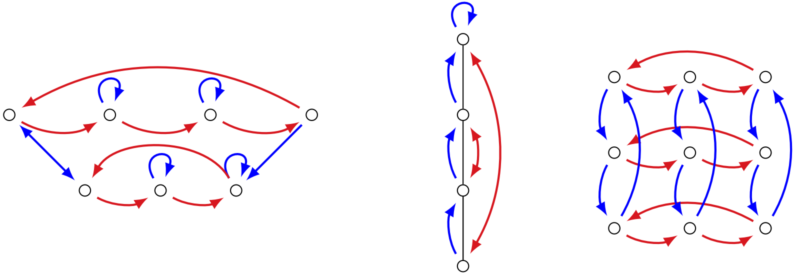

Priestley duality for Cornish algebras

- Given a finite Cornish algebra $\mathbf{A}$, there is a natural way to obtain a Cornish space $H(\mathbf{A})$ from it.
- Given a finite Cornish space $\mathbb{W}$, there is a natural way to obtain a Cornish algebra $K(\mathbb{W})$ from it.

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Theorem

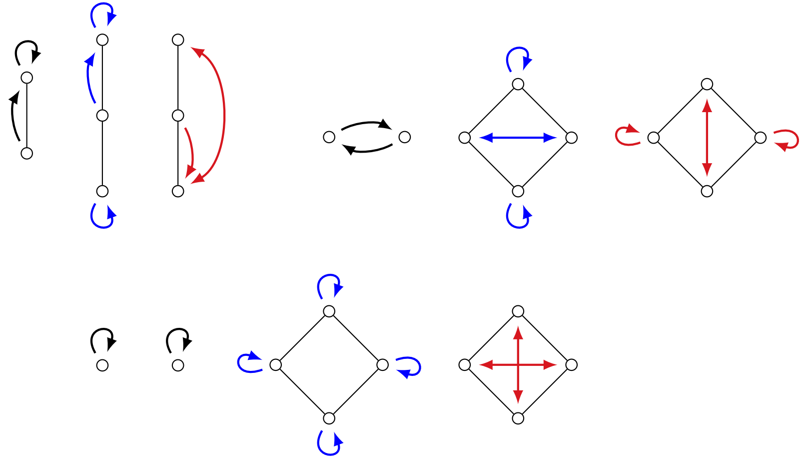
Let $\mathfrak{C}_{\mathrm{fin}}$ and $\mathfrak{Y}_{\mathrm{fin}}$ be, respectively, the categories of finite Cornish algebras and finite Cornish spaces (of some fixed type F).

(1) *Then $H\colon \mathfrak{C}_{\mathrm{fin}} \to \mathfrak{Y}_{\mathrm{fin}}$ and $K\colon \mathfrak{Y}_{\mathrm{fin}} \to \mathfrak{O}_{\mathrm{fin}}$ are well-defined functors that yield a dual category equivalence between $\mathfrak{C}_{\mathrm{fin}}$ and $\mathfrak{Y}_{\mathrm{fin}}$.*

(2) *In particular, $\mathbf{A} \cong KH(\mathbf{A})$, for every finite Cornish algebra **A**, and $\mathbb{W} \cong HK(\mathbb{W})$, for every finite Cornish space $\mathbb{W}$.*

Warning: colour is important!

Some maps on $\mathbb{W}$ are both order-preserving and order-reversing. So we need to be told whether they are encoding endomorphisms or dual endomorphisms of $\mathbf{A} = H(\mathbb{W})$.



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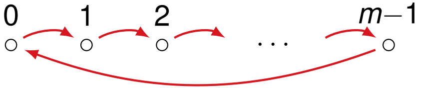
Quasi-primal Ockham algebras

Theorem (BAD, Nguyen and Pitkethly: AU 2015)

A finite Ockham algebra $\mathbf{A} = \langle A; \vee, \wedge, g^{\mathbf{A}}, 0, 1\rangle$ *is quasi-primal if and only if its Priestley dual* $H(\mathbf{A})$ *is an anti-chain of odd size and* $g^{H(\mathbf{A})}$ *is a cycle.*



(m odd)

Our results

Theorem 1: Necessary Conditions.

Let **A** *be a quasi-primal Cornish algebra of type F, say* $\mathbf{A} = K(\mathbb{W})$, *with* $\mathbb{W} = \langle W; F^{\mathbb{W}}, \leqslant \rangle$.

(1) $\mathbb{W}$ *has no proper non-empty substructures.*

(2) F_{-} *is nonempty* (*i.e., at least one of the unary maps in* $F^{\mathbb{W}}$ *must be coloured red*).

Our results

We can extend the $+/-$ labelling to the set T of unary terms of type F in the natural way. Let $t(v)$ be a unary term.

- We declare that $t(v) \in T_-$ if and only if $t(v)$ contains an odd number of operation symbols from F_-.
- For example, we would have $f^2(g(v)) \in T_-$ and $f^3(g(v)) \in T_-$, but $g(f^3(g(v))) \in T_+$.

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- For example, we would have $f^2(g(v)) \in T_{-}$ and $f^3(g(v)) \in T_{-}$, but $g(f^3(g(v))) \in T_{+}$.

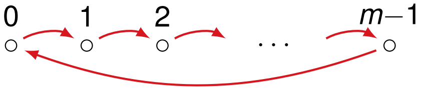
Theorem 2: Sufficient Conditions.
Let $\mathbf{A}$ be a finite Cornish algebra, say $\mathbf{A} = K(\mathbb{W})$, with $\mathbb{W} = \langle W; F^{\mathbb{W}}, \leqslant \rangle$. Assume that $\mathbb{W}$ has the following properties:

(a) *$\mathbb{W}$ has no proper non-empty substructures,*

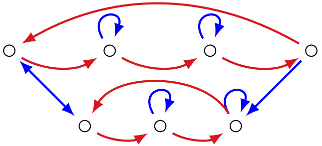
(b) *there exists a unary term $t \in T_{-}$ such that, for all $a \in W$, the orbit of a under $t^{\mathbb{W}}$ eventually reaches an odd cycle.*

Then $\mathbf{A}$ is quasi-primal.

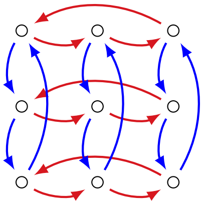
Some quasi-primal Cornish algebras



(m odd) $t(v) := g(v)$



$t(v) := f^2(g(v))$



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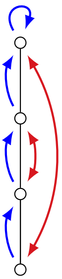
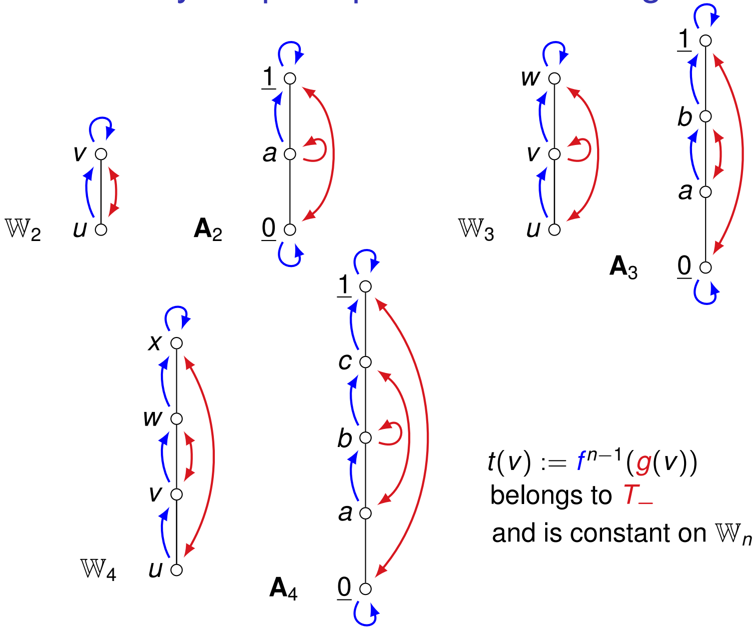


$t(v) := f^3(g(v))$

An infinite family of quasi-primal Cornish algebras



$t(v) := f^{n-1}(g(v))$
belongs to T_-
and is constant on $\mathbb{W}_n$

Outline

A useful fact

Lemma (BAD, Nguyen and Pitkethly: AU 2015)

Let $\mathbb{X} = \langle X; g, \leqslant \rangle$ *be an Ockham space. Then the union* Y *of the odd cycles of* g *is an antichain in* $\mathbb{X}$*.*

Skip proof

A useful fact

Lemma (BAD, Nguyen and Pitkethly: AU 2015)

Let $\mathbb{X} = \langle X; g, \leqslant \rangle$ be an Ockham space. Then the union Y of the odd cycles of g is an antichain in $\mathbb{X}$.

Proof.

Let $c, d \in Y$ with c in an m-cycle of g and d in an n-cycle of g, for some odd m and n. Assume that $c \leqslant d$ in $\mathbb{X}$.

- As m and n are odd and g is order-reversing, we have

$$d = g^n(d) \leqslant g^n(c) \quad \text{and} \quad g^m(d) \leqslant g^m(c) = c.$$

- As $m + n$ is even, it now follows that

$$d \leqslant g^n(c) = g^{m+n}(c) \leqslant g^{m+n}(d) = g^m(d) \leqslant c.$$

- Thus $c = d$. Hence Y is an antichain. □

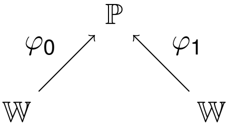
An external characterisation of quasi-primality

Theorem 3: Necessary and Sufficient Conditions.

Let $\mathbf{A}$ *be a finite Cornish algebra, say* $\mathbf{A} = K(\mathbb{W})$*, for some finite Cornish space* $\mathbb{W}$*. Then the following are equivalent:*

(1) $\mathbf{A}$ *is quasi-primal;*

(2) $\mathbb{W}$ *has no proper non-empty substructures and, for every pair of jointly surjective morphisms* $\varphi_0, \varphi_1 \colon \mathbb{W} \to \mathbb{P}$*, if* $\varphi_0(a) \leqslant \varphi_1(b)$ *or* $\varphi_0(a) \geqslant \varphi_1(b)$*, for some* $a, b \in W$*, then* $\varphi_0(W) \cap \varphi_1(W) \neq \varnothing$*.*



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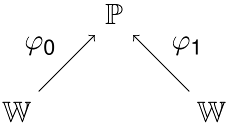

Once this result is proved, it is straightforward to prove our necessary and our sufficient conditions.

Necessary/sufficient internal conditions

Theorem 1: Necessary Conditions.

Let **A** *be a quasi-primal Cornish algebra of type F, say* $\mathbf{A} = K(\mathbb{W})$*, with* $\mathbb{W} = \langle W; F^{\mathbb{W}}, \leqslant \rangle$.

(1) $\mathbb{W}$ *has no proper non-empty substructures.*

(2) F_{-} *is nonempty (i.e., at least one of the unary maps in* $F^{\mathbb{W}}$ *must be coloured red).*

Theorem 2: Sufficient Conditions.

Let **A** *be a finite Cornish algebra, say* $\mathbf{A} = K(\mathbb{W})$*, with* $\mathbb{W} = \langle W; F^{\mathbb{W}}, \leqslant \rangle$. *Assume that* $\mathbb{W}$ *has the following properties:*

(a) $\mathbb{W}$ *has no proper non-empty substructures,*

(b) *there exists a unary term* $t \in T_{-}$ *such that, for all* $a \in W$*, the orbit of* a *under* $t^{\mathbb{W}}$ *eventually reaches an odd cycle.*

Then **A** *is quasi-primal.*

A proof

Theorem (Part of Theorem 1)

Let $\mathbf{A}$ *be a Cornish algebra of type* F*, say* $\mathbf{A} = K(\mathbb{W})$*, with* $\mathbb{W} = \langle W; F^{\mathbb{W}}, \leqslant\rangle$*. If* $F = F_+$*, then* $\mathbf{A}$ *is not quasi-primal.*

Proof.

Assume that $F = F_+$. Let $\mathbb{P} = \mathbb{W} \times \mathbb{2}$, where

$$f^{\mathbb{P}}\big((a,0)\big) := \big(f^{\mathbb{W}}(a), 0\big) \quad \text{and} \quad f^{\mathbb{P}}\big((a,1)\big) := \big(f^{\mathbb{W}}(a), 1\big),$$

for each $f \in F = F_+$. Then each $f^{\mathbb{P}}$ is order-preserving and the maps $\varphi_i \colon \mathbb{W} \to \mathbb{P}$, given by $\varphi_i(a) := (a, i)$, for $i \in \{0, 1\}$, are jointly surjective morphisms.

Let $a \in W$. Since

$$\varphi_0(a) = (a, 0) \leqslant (a, 1) = \varphi_1(a)$$

and $\varphi_0(W) \cap \varphi_1(W) = \varnothing$, it follows from Theorem 3 that $\mathbf{A} = K(\mathbb{W})$ is not quasi-primal. □