Relational semantics via TiRS graphs

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Representation/duality results for (non-distributive) lattices

- Urquhart (1978)
- Hartung (1992)
- Allwein & Hartonas (1993)
- Ploščica (1995)
- Hartonas & Dunn (1997)
- Gehrke & van Gool (2014)
- Moshier & Jipsen (2014)

Relational semantics have provided a useful tool for the study of modal and other non-classical logics.

The aim of this work is to develop *single-sorted* relational semantics for logics whose algebraic semantics are given by non-distributive lattices. We hope to mimic as closely as possible the relational semantics for modal logic based on Kripke frames.

Relational representations and semantics

- Dunn, Gehrke, Palmigiano (2005): relational semantics for implication-fusion fragment of substructural logics, perfect posets
- Gehrke (2006): relational semantics for implication-fusion fragment of substructural logics, RS frames
- Dzik, Orlowska, van Alten (2006): relational representation of lattices with negation, (Urquhart) doubly-ordered sets
- Almeida (2009): relational representation of lattices with negation, RS frames
- Chernilovskaya, Gehrke, van Rooijen (2012): relational semantics for Lambek-Grishin calculus, RS frames
- Coumans, Gehrke, van Rooijen (2013): relational semantics for linear logics, RS frames

Maximal partial worlds

We will use a set of maximal partial worlds as the underlying set of our relational structures. In the algebraic setting, these worlds are the maximal partial homomorphisms of Ploščica (or the maximal disjoint filter-ideal pairs of Urquhart).

Following Ploščica we equip these worlds with a binary relation E. We think of this relation as saying

$$xEy \qquad \text{iff} \qquad \text{“}x \text{ trusts } y\text{”}$$

Each world x will have a set of formulas that it asserts and a (disjoint) set of formulas that it denies. However, because the worlds are *partial*, there will be some formulas about which they have no opinion.

The canonical extension

Given a bounded lattice $\mathbf{A}$ we denote its *canonical extension* by $\mathbf{A}^\delta$.

$\mathbf{A}^\delta$ is a completion of $\mathbf{A}$ such that:

- every $x \in A^\delta$ is a join of meets of elements of A;
- every $x \in A^\delta$ is a meet of joins of elements of A;
- for $S, T \subseteq A$, if $\bigwedge S \leqslant \bigvee T$ then $\bigwedge S' \leqslant \bigvee T'$ for some $S' \subseteq^{fin} S$ and $T' \subseteq^{fin} T$.

For $\mathbf{C}$ a complete lattice

(i) $j \in C$ is *completely join-irreducible* if for any $X \subseteq C$, if $j = \bigvee X$ then $j = x$ for some $x \in X$;

(ii) $m \in C$ is *completely meet-irreducible* if for any $Y \subseteq C$, if $m = \bigwedge Y$ then $m = y$ for some $y \in Y$;

NB: $\mathbf{A}^\delta$ is *perfect*, i.e. the completely join-irreducible elements, $J^\infty(\mathbf{A}^\delta)$, are join-dense in $\mathbf{A}^\delta$, and the completely meet-irreducible elements, $M^\infty(\mathbf{A}^\delta)$ are meet-dense in $\mathbf{A}^\delta$.

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TiRS graphs

Let $\mathbf{X} = (X, E)$ be a (directed) graph and define:

$$xE = \{ y \in X \mid (x, y) \in E \} \qquad \text{and} \qquad Ex = \{ y \in X \mid (y, x) \in E \}.$$

Consider the following conditions:

(S) for every $x, y \in X$, if $x \neq y$ then $xE \neq yE$ or $Ex \neq Ey$;

(R) (i) for all $x, z \in X$, if $zE \subsetneq xE$ then $(z, x) \notin E$;
(ii) for all $y, z \in X$, if $Ez \subsetneq Ey$ then $(y, z) \notin E$;

The (Ti) property

(Ti) for all $x, y \in X$, if $(x, y) \in E$, then there exists $z \in X$ such that $zE \subseteq xE$ and $Ez \subseteq Ey$.

When E is also reflexive, it is easily seen that the condition (Ti) can equivalently be written as follows:

(Ti)′ for all $x, y \in X$, if $(x, y) \in E$, then there exists z such that $(x, z) \in E$ and $(z, y) \in E$ and for every $w \in X$, $(z, w) \in E$ implies $(x, w) \in E$ and $(w, z) \in E$ implies $(w, y) \in E$.

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A TiRS graph is a reflexive graph that satisfies (R), (S) and (Ti)′.

For our TiRS graphs $\mathbf{X} = (X, E)$ we shall consider the context

$$\mathbb{K}(\mathbf{X}) := (X, X, E^{\complement})$$

where the base set X of the graph $\mathbf{X}$ stands for both objects and attributes and the relation $E^{\complement} = (X \times X) \backslash E$. We define a Galois connection via polars

$$E^{\complement}_{\triangleright} : (\wp(X), \subseteq) \to (\wp(X), \supseteq) \quad \text{and} \quad E^{\complement}_{\triangleleft} : (\wp(X), \supseteq) \to (\wp(X), \subseteq)$$

with these maps defined by

$$E^{\complement}_{\triangleright}(Y) = \{ x \in X \mid (\forall y \in Y)(y, x) \notin E \}, \; E^{\complement}_{\triangleleft}(Y) = \{ z \in X \mid (\forall y \in Y)(z, y) \notin E \}.$$

The concept lattice $\mathcal{G}(\mathbb{K}(\mathbf{X}))$ of the context $\mathbb{K}(\mathbf{X}) = (X, X, E^{\complement})$ is

$$\mathcal{G}(\mathbb{K}(\mathbf{X})) = \{ Y \subseteq X \mid (E^{\complement}_{\triangleleft} \circ E^{\complement}_{\triangleright})(Y) = Y \},$$

ordered by inclusion.

When $\mathbf{X} = (\mathcal{L}^{mp}(\mathbf{L}, \underline{\mathbf{2}}), E)$, we have $\mathcal{G}(\mathbb{K}(\mathbf{X})) \simeq \mathbf{L}^{\delta}$.

Completely join- and completely meet-irreducible elements

The completely join- and completely meet- irreducibles of the concept lattice $\mathcal{G}(\mathbb{K}(\mathbf{X}))$ are described by the following proposition.

Proposition (Craig, Gouveia, Haviar 2015)

Let $\mathbf{X} = (X, E)$ *be a TiRS graph and consider the complete lattice*

$$\mathcal{G}(\mathbb{K}(\mathbf{X})) = \{\, Y \subseteq X \mid (E_{\triangleleft}^{\complement} \circ E_{\triangleright}^{\complement})(Y) = Y \,\}.$$

Then

$$J^{\infty}(\mathcal{G}(\mathbb{K}(\mathbf{X}))) = \{\, (E_{\triangleleft}^{\complement} \circ E_{\triangleright}^{\complement})(\{x\}) \mid x \in X \,\}$$

and

$$M^{\infty}(\mathcal{G}(\mathbb{K}(\mathbf{X}))) = \{\, E_{\triangleleft}^{\complement}(\{y\}) \mid y \in X \,\}.$$

For $x \in X$ we will use the following abbreviations:

$$x_J = (E_{\triangleleft}^{\complement} \circ E_{\triangleright}^{\complement})(\{x\}) \qquad \text{and} \qquad x_M = E_{\triangleleft}^{\complement}(\{x\}).$$

Deriving relational semantics for $\Box$

In classical modal logic we have:

$$\mathfrak{M}, w \Vdash \Diamond\varphi \quad \text{iff} \quad (\exists u \in W)(wRu \,\&\, \mathfrak{M}, u \Vdash \varphi)$$
$$\mathfrak{M}, w \Vdash \Box\varphi \quad \text{iff} \quad (\forall u \in W)(wRu \Rightarrow \mathfrak{M}, u \Vdash \varphi)$$

Use algebraic intuition to derive relational conditions for *assertion*:

$$\begin{aligned} x_J \leqslant v(\Box\psi) \quad & \text{iff} \quad x_J \leqslant \Box(\bigwedge\{ z_M \mid z \in X, v(\psi) \leqslant z_M \}) \\ & \text{iff} \quad x_J \leqslant \bigwedge\{ \Box(z_M) \mid z \in X, v(\psi) \leqslant z_M \} \\ & \text{iff} \quad (\forall z \in X)(v(\psi) \leqslant z_M \Rightarrow x_J \leqslant \Box z_M) \\ & \text{iff} \quad (\forall z \in X)(x_J \nleqslant \Box z_M \Rightarrow v(\psi) \nleqslant z_M) \end{aligned}$$

Now we define $R_\Box xz$ iff $x_J \nleqslant \Box z_M$. Now we can write

$$\begin{aligned} x \Vdash \Box\psi \quad & \text{iff} \quad (\forall z \in X)(R_\Box xz \Rightarrow v(\psi) \nleqslant z_M) \\ & \text{iff} \quad (\forall z \in X)(R_\Box xz \Rightarrow \neg(z \succ \psi)) \end{aligned}$$

For denial we get:

$$x \succ \Box\psi \quad \text{iff} \quad (\forall y \in X)(yEx \Rightarrow \neg(\mathfrak{M}, y \Vdash \Box\psi))$$

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Deriving relational semantics for the $\Diamond$

$$\begin{aligned} v(\Diamond\varphi) \leqslant x_M \quad & \text{iff} \quad \Diamond(\bigvee\{\, y_J \mid y \in X, y_J \leqslant v(\varphi)\,\}) \leqslant x_M \\ & \text{iff} \quad \bigvee\{\, \Diamond y_J \mid y \in X, y_J \leqslant v(\varphi)\,\} \leqslant x_M \\ & \text{iff} \quad (\forall y \in X)(y_J \leqslant v(\varphi) \Rightarrow \Diamond y_J \leqslant x_M) \\ & \text{iff} \quad (\forall y \in X)(\Diamond y_J \nleqslant x_M \Rightarrow y_J \nleqslant v(\varphi)) \end{aligned}$$

We define: $R_\Diamond xy$ iff $\Diamond y_J \nleqslant x_M$.

Hence we get $x \succ \Diamond\varphi$ iff $(\forall y \in X)(R_\Diamond xy \Rightarrow \neg(y \Vdash \varphi))$.
To derive assertion we apply the same (algebraic) ideas and get

$$x \Vdash \Diamond\varphi \quad \text{iff} \quad \forall y(xEy \Rightarrow \exists z(yR_\Diamond z \text{ and } z \Vdash \varphi))$$

Or, more simply $x \Vdash \Diamond\varphi$ iff $\forall y(xEy \Rightarrow \neg(y \succ \Diamond\varphi))$

Interaction conditions between $R_\Diamond$, $R_\Box$ and E

We denote by $[\![\varphi]\!]$ the set of worlds that assert φ and by $\langle\!\langle\varphi\rangle\!\rangle$ the set of worlds that *deny* φ.

We want to enforce the following:

(1) for every formula φ we have $[\![\varphi]\!] \in \mathcal{G}(\mathbb{K}(\mathbf{X}))$
i.e. $[\![\varphi]\!] = (E_\triangleleft^\complement \circ E_\triangleright^\complement)([\![\varphi]\!])$

(2) for every formula ψ we have $\langle\!\langle\psi\rangle\!\rangle = (E_\triangleright^\complement \circ E_\triangleleft^\complement)(\langle\!\langle\psi\rangle\!\rangle)$

We do this by imposing the following interaction conditions on $R_\Diamond$ and $R_\Box$ for all $y, z \in X$:

(a) $X \backslash R_\Diamond^{-1}[y] = (E_\triangleright^\complement \circ E_\triangleleft^\complement)(X \backslash R_\Diamond^{-1}[y])$

(b) $X \backslash R_\Box^{-1}[z] \in \mathcal{G}(\mathbb{K}(\mathbf{X}))$

If $\mathbf{X} = (X, E)$ is a TiRS graph and $R_\Diamond$ is a binary relation on X satisfying (a) and $R_\Box$ satisfies (b), then we call $\mathscr{X} = (X, E, R_\Diamond, R_\Box)$ a modal TiRS graph.

Let $\mathtt{PROP}$ be a countable set of propositional variables. Formulas in the language $\mathcal{L}$ are defined by

$$\varphi ::= \bot \mid \top \mid p \in \mathtt{PROP} \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \Diamond\varphi \mid \Box\varphi$$

A model is a pair $\mathfrak{M} = (\mathscr{X}, v)$ where $\mathscr{X}$ is a modal TiRS graph and $v : \mathtt{PROP} \to \mathcal{G}(\mathbb{K}(\mathbf{X}))$. Let $x \in X$ be a maximal partial world. Then the semantics for $\mathcal{L}$ is defined by:

$\mathfrak{M}, x \succ \top$	$\mathfrak{M}, x \Vdash \bot$		never
$\mathfrak{M}, x \Vdash \top$	$\mathfrak{M}, x \succ \bot$		always
	$\mathfrak{M}, x \Vdash p$	iff	$x \in v(p)$
	$\mathfrak{M}, x \succ p$	iff	$(\forall y \in X)(yEx \Rightarrow \neg(y \Vdash p))$
	$\mathfrak{M}, x \succ \varphi \vee \psi$	iff	$\mathfrak{M}, x \succ \varphi$ and $\mathfrak{M}, x \succ \psi$
	$\mathfrak{M}, x \Vdash \varphi \vee \psi$	iff	$(\forall y \in X)(xEy \Rightarrow \neg(\mathfrak{M}, y \succ \varphi \vee \psi))$
	$\mathfrak{M}, x \Vdash \varphi \wedge \psi$	iff	$\mathfrak{M}, x \Vdash \varphi$ and $\mathfrak{M}, x \Vdash \psi$
	$\mathfrak{M}, x \succ \varphi \wedge \psi$	iff	$(\forall y \in X)(yEx \Rightarrow \neg(\mathfrak{M}, y \Vdash \varphi \wedge \psi))$
	$\mathfrak{M}, x \succ \Diamond\varphi$	iff	$(\forall y \in X)(R_{\Diamond}xy \Rightarrow \neg(\mathfrak{M}, y \Vdash \varphi))$
	$\mathfrak{M}, x \Vdash \Diamond\varphi$	iff	$(\forall y \in X)(xEy \Rightarrow \neg(\mathfrak{M}, y \succ \Diamond\varphi))$
	$\mathfrak{M}, x \Vdash \Box\psi$	iff	$(\forall y \in X)(R_{\Box}xy \Rightarrow \neg(\mathfrak{M}, y \succ \psi))$
	$\mathfrak{M}, x \succ \Box\psi$	iff	$(\forall y \in X)(yEx \Rightarrow \neg(\mathfrak{M}, y \Vdash \Box\psi))$

Duality and completeness

Theorem

There is a duality between perfect lattices with $\Diamond$ and $\Box$ and modal TiRS graphs.*

It follows that the logic of modal lattices is sound and complete with respect to modal TiRS graphs.

Further work

In our derivation of the TiRS semantics for $\Diamond$ and $\Box$, we made use of the fact that they are, respectively, join- and meet-preserving.

We can apply these same techniques to other logics where the algebraic operations interpreting the logical connectives are meet/join preserving/reversing in each co-ordinate, e.g. fusion. TiRS graph semantics can thus be applied to other non-classical logics.