Canonicity results for mu-calculi

(Unified Correspondence III)

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TACL, Ischia (Italy) 21 - 26 June 2015

Unified correspondence

Hybrid logics
[CR15]

DLE-logics
[CP12, CPS]

Mu-calculi
[CFPS15, CGP14, CC15]

Substructural logics
[CP15]

Regular DLE-logics
Kripke frames with
impossible worlds
[PSZ15a]

Display calculi
[GMPTZ]

Jónsson-style vs
Sambin-style canonicity
[PSZ15b]

Finite lattices and
monotone ML
[FPS15]

Canonicity via
pseudo-correspondence
[CPSZ]

Fixed points on complete lattices

Theorem (Knaster-Tarski)

Let L be a complete lattice and $f : L \to L$ an order preserving map. Then the least fixed-point of f exists and is given by

$$\mathrm{LFP}x.f(x) = \bigwedge\{a \in L \mid f(a) \leq a\}.$$

Fixed points on complete lattices

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$$\mathrm{LFP}x.f(x) = \bigwedge\{a \in L \mid f(a) \leq a\}.$$

Or by ordinal unfolding:

$$\mathrm{LFP}x.f(x) = \bigvee_{0\leq\alpha} f^{\alpha}(\bot).$$

where

$$\begin{aligned} f^0(\bot) &= \bot \\ f^{\alpha+1}(\bot) &= f\big(f^{\alpha}(\bot)\big) \\ f^{\lambda}(\bot) &= \bigvee_{\alpha<\lambda} f^{\alpha}(\bot) \end{aligned}$$

Fixed points on general (incomplete) modal bi-Heyting algebras (1)

Language $\mathcal{L}$ of modal bi-Heyting algebras defined by:

$$\varphi ::= \bot \mid \top \mid p \mid X \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \varphi - \psi \mid \Diamond\varphi \mid \Box\varphi$$

where $p \in \mathrm{PROP}$ and $X \in \mathrm{FVAR}$.

$$\mathcal{L}_1 := \mathcal{L} + \mu_1 X.\varphi \mid \nu_1 X.\varphi$$

$$\mu x.t(x, a_1, \ldots, a_{n-1}) := \bigwedge \{\, a \in A \mid t(a, a_1, \ldots, a_{n-1}) \leq a \,\}$$

if this meet exists, otherwise $\mu x.t(x, a_1, \ldots, a_{n-1})$ is undefined.

$$\nu x.t(x, a_1, \ldots, a_{n-1}) := \bigvee \{\, a \in A \mid a \leq t(a, a_1, \ldots, a_{n-1}) \,\}$$

if this join exists, otherwise $\nu x.t(x, a_1, \ldots, a_{n-1})$ is undefined.

Definition [Ambler, Kwiatkowska, Measor, '95]

A modal bi-Heyting algebra $\mathbf{A}$ is of the first kind if $t^{\mathbf{A}}(a_1, \ldots, a_n)$ is defined for all $a_1, \ldots, a_n \in \mathbf{A}$ and all $t \in \mathcal{L}_1$.

Fixed points on general (incomplete) modal bi-Heyting algebras (2)

Language $\mathcal{L}$ of modal bi-Heyting algebras are defined recursively by:

For $t(x_1, \ldots, x_n) \in \mathcal{L}_2$ we then define

$$\mu_2 x.t(x, a_1, \ldots, a_{n-1}) := \bigvee_{\alpha \geq 0} t^{\alpha}(\bot, a_1, \ldots, a_{n-1})$$

and

$$\nu_2 x.t(x, a_1, \ldots, a_{n-1}) := \bigwedge_{\alpha \geq 0} t_{\alpha}(\top, a_1, \ldots, a_{n-1})$$

if this join and this meet exist, and they are undefined otherwise.

$$\mathcal{L}_2 := \mathcal{L} + \mu_2 X.\varphi \mid \nu_2 X.\varphi$$

Definition [AKM, '95]

A modal bi-Heyting algebra $\mathbf{A}$ is said to be of the second kind if $t^{\mathbf{A}}(a_1, \ldots, a_n)$ is defined for all $a_1, \ldots, a_n \in \mathbf{A}$ and all $t \in \mathcal{L}_2$.

Languages for canonical extension

$$\mathcal{L}_* := \mathcal{L} \quad + \quad \mu^* X.\varphi \mid \nu^* X.\varphi$$

In the canonical extensions $\mathbf{A}^\delta$ of modal bi-Heyting algebras $\mathbf{A}$:

$$\mu^* x_1.t(x_1, a_1, \ldots, a_{n-1}) := \bigwedge \{ a \in A \mid t(a, a_1, \ldots, a_{n-1}) \leq a \}$$

and

$$\nu^* x_1.t(x_1, a_2, \ldots, a_n) := \bigvee \{ a \in A \mid a \leq t(a, a_1, \ldots, a_{n-1}) \}.$$

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Extended language $\mathcal{L}_\rho^+$, for $\mathcal{L}_\rho$, with $\rho \in \{1, 2, *\}$:

$$\mathcal{L}_1 := \mathcal{L} \quad + \quad \mathbf{j} \mid \mathbf{m} \mid \blacksquare\varphi \mid \blacklozenge\varphi$$

Two types of canonicity

$\varphi \leq \psi$ is canonical:

$$\mathbf{A} \models \varphi \leq \psi \quad \text{iff} \quad \mathbf{A}^\delta \models \varphi \leq \psi$$

$\varphi \leq \psi$ is tame canonical [Bezhanishvili, Hodkinson, 2012]:

$$\mathbf{A} \models \varphi \leq \psi \quad \text{iff} \quad \mathbf{A}^\delta \models \varphi^* \leq \psi^*$$

Canonicty via correspondence

$$\begin{array}{ccc}
\mathbf{A} \models \alpha \leq \beta & & \mathbf{A}^\delta \models \alpha \leq \beta \\
\Updownarrow & & \\
\mathbf{A}^\delta \models_{\mathbf{A}} \alpha \leq \beta & & \Big\Updownarrow \\
\Updownarrow & & \\
\mathbf{A}^\delta \models_{\mathbf{A}} \mathtt{pure}(\alpha \leq \beta) & \Longleftrightarrow & \mathbf{A}^\delta \models \mathtt{pure}(\alpha \leq \beta)
\end{array}$$

μ^*-ALBA rules

First approximation rule

$$\frac{\varphi \leq \psi}{\forall \mathbf{j} \forall \mathbf{m}[(\mathbf{j} \leq \varphi \ \& \ \psi \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}]} \text{ (FA)}$$

Residuation rules, e.g.,

$$\frac{\chi - \psi \leq \varphi}{\chi \leq \psi \vee \varphi} \text{ (}-\text{LR)}$$

Adjunction rules, e.g.,

$$\frac{\Diamond \varphi \leq \psi}{\varphi \leq \blacksquare \psi} \text{ (}\Diamond\text{LA)}$$

Approximation rule, e.g.,

$$\frac{\chi \rightarrow \varphi \leq \mathbf{m}}{\exists \mathbf{j} \exists \mathbf{n}(\mathbf{j} \rightarrow \mathbf{n} \leq \mathbf{m} \ \& \ \mathbf{j} \leq \chi \ \& \ \varphi \leq \mathbf{n})} \text{ (}\rightarrow\text{Appr)}$$

Ackerman rules.

Approximation rule for μ^*

$$\frac{\mathbf{i} \leq \mu^* X.\psi(\overline{\varphi}/\overline{x}, X)}{\bigotimes_{i=1}^{n}(\exists \mathbf{j}^{\tau_i}[\mathbf{i} \leq \mu^* X.\psi(\bar{\mathbf{j}}_i^{\tau}/\overline{x}, X) \;\&\; \mathbf{j}^{\tau_i} \leq^{\tau_i} \varphi_i])} \;(\mu^\tau\text{-A-R})$$

where

1. $\psi, \overline{\varphi} \in \mathcal{L}_*$;
2. $\overline{x} \in$ PHVAR do not occur in $\overline{\varphi}$;
3. all propositional variables and free fixed point variables in $\psi(\overline{x}, X)$ and $\varphi(\overline{x}, X)$ are among $\overline{x}$ and X.
4. $\psi(\overline{x}, X) : (\mathbf{C}^\delta)^\tau \times \mathbf{C}^\delta \to \mathbf{C}^\delta$ preserves $(\mathbf{C}^\tau \times \mathbf{C})$-targeted joins for all modal bi-Heyting algebras $\mathbf{C}$ of the second kind;
5. $\psi(\overline{x}, X)$ must be τ-positive in $\overline{x}$;

(ν^τ-A-R) is defined similarly.

Tame run No application of (μ^τ-A-R) or (ν^τ-A-R).

Proper run All occurrences of μ^* and ν^* handled by (μ^τ-A-R) and (ν^τ-A-R).

Syntactic classes

Outer Skeleton (P_3)	Inner Skeleton (P_2)	PIA (P_1)
Δ-adjoints	Binders	Binders
$+ \quad \vee \quad \wedge$	$+ \quad \mu$	$+ \quad \nu$
$- \quad \wedge \quad \vee$	$- \quad \nu$	$- \quad \mu$
SLR	SLA	SRA
$+ \quad \Diamond \quad \lhd \quad \circ \quad -$	$+ \quad \Diamond \quad \lhd \quad \vee$	$+ \quad \Box \quad \rhd \quad \wedge$
$- \quad \Box \quad \rhd \quad \star \quad \rightarrow$	$- \quad \Box \quad \rhd \quad \wedge$	$- \quad \Diamond \quad \lhd \quad \vee$
	SLR	SRR
	$+ \quad \wedge \quad \circ \quad -$	$+ \quad \vee \quad \star \quad \rightarrow$
	$- \quad \vee \quad \star \quad \rightarrow$	$- \quad \wedge \quad \circ \quad -$

Table : Skeleton and PIA nodes.

Conditions on branches of generation trees (1)

ϵ and order type on $p_1, \ldots, p_n$ and Ω a partial order on $p_1, \ldots, p_n$.

(GB1) The formula corresponding to the uppermost node on P_1 is a mu-sentence.

(GB2) For every SRR-node in P_1 of the form $\gamma \odot \beta$ or $\beta \odot \gamma$, where β is the side where the branch lies, γ is a mu-sentence and $\epsilon^\partial(\gamma) \prec *\varphi$.

(GB3) For every SLR-node in P_2 of the form $\gamma \odot \beta$ or $\beta \odot \gamma$, where β is the side where the branch lies, γ is a mu-sentence and $\epsilon^\partial(\gamma) \prec *\varphi$ (see above for this notation).

(NB-PIA) P_1 contains no fixed point binders.

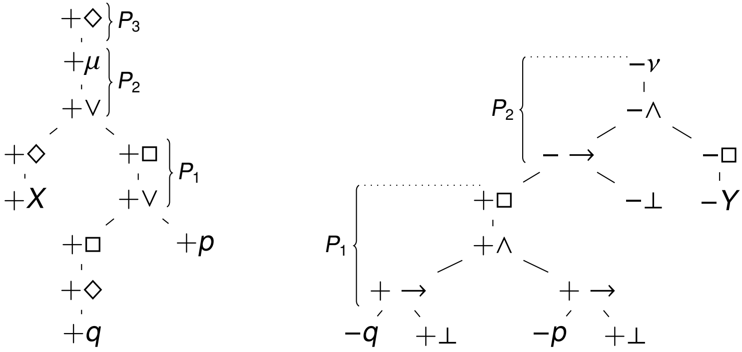
(NL) For every SLR-node in P_2 of the form $\gamma \odot \beta$ or $\beta \odot \gamma$, where β is the side where the branch lies, the signed generation tree of γ contains no live branches.

(Ω-CONF) For every SRR-node in P_1 of the form $\gamma \odot \beta$ or $\beta \odot \gamma$, where β is the side where the branch lies: $p_j <_\Omega p_i$ for every p_j occurring in γ, where p_i is the propositional variable labelling the leaf of the branch.

Conditions on branches of generation trees (2)

1. ϵ-recursive if every ϵ-critical branch is ϵ-good.
2. (Ω, ϵ)-inductive it is ϵ-recursive and every ϵ-critical branch satisfies (Ω-CONF).
3. restricted (Ω, ϵ)-inductive if it is (Ω, ϵ)-inductive and
   3.1 every ϵ-critical branch satisfies (NB-PIA) and (NL),
   3.2 every occurrence of a binder is on an ϵ-critical branch.
4. tame (Ω, ϵ)-inductive if it is (Ω, ϵ)-inductive and
   4.1 $\Omega = \varnothing$,
   4.2 no binder occurs on any ϵ-critical branch,
   4.3 the only nodes involving binders which are allowed to occur are $+\nu$ and $-\mu$.

Example: a restricted inductive inequality



The restricted (Ω, ϵ)-inductive inequality

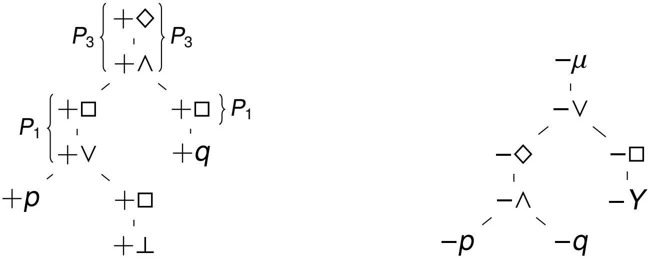
$$\Diamond\mu X.(\Diamond X \vee \Box(\Box\Diamond q \vee p)) \leq \nu Y.([\Box((q \rightarrow \bot) \wedge (p \rightarrow \bot)) \rightarrow \bot] \wedge \Box Y)$$

with $\epsilon_p = 1$ and $\epsilon_q = \partial$ and Ω such that $q <_\Omega p$.

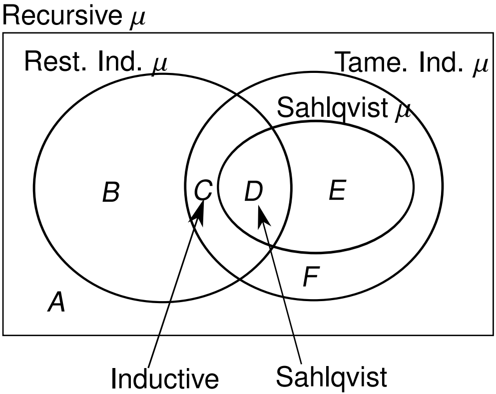
Example: a tame inductive inequality

$$\Diamond(\Box\bot \vee p) \wedge \Box q \leq \mu Y.(\Diamond(p \wedge q) \wedge \Box Y)$$

is tame (Ω, ϵ)-inductive with $\epsilon_p = 1 = \epsilon_q$ (and $\Omega = \varnothing$).



Relationships between the syntactic classes



Tame Canonicity Results (1)

Theorem (Tame Canonicity for μ^*-ALBA)

All $\mathcal{L}_1$-inequalities on which a tame run of μ^*-ALBA succeeds are tame canonical.

Proof

$$
\begin{aligned}
& \mathbf{A} \models \varphi \leq \psi && (1)\\
\iff\quad & \mathbf{A}^\delta \models_{\mathbf{A}} \varphi^* \leq \psi^* && (2)\\
\iff\quad & \mathbf{A}^\delta \models_{\mathbf{A}} \mathtt{pure}(\varphi^* \leq \psi^*) && (3)\\
\iff\quad & \mathbf{A}^\delta \models \mathtt{pure}(\varphi^* \leq \psi^*) && (4)\\
\iff\quad & \mathbf{A}^\delta \models \varphi^* \leq \psi^* && (5)
\end{aligned}
$$

Tame Canonicity Results (2)

Theorem

μ^*-ALBA successfully purifies all tame inductive mu-inequalities by means of tame runs.

Corollary

All tame inductive mu-inequalities are tame canonical.

Canonicity Results (1)

Theorem (Canonicity for μ^*-ALBA)

Let **A** be a mu-algebra of the second kind and let $\varphi \leq \psi$ be an $\mathcal{L}_1$-inequality on which a proper run of μ^*-ALBA succeeds.

Proof

$$
\begin{aligned}
& \mathbf{A} \models \varphi \leq \psi & (6)\\
\Longleftrightarrow \quad & \mathbf{A}^\delta \models_{\mathbf{A}} \varphi^* \leq \psi^* & (7)\\
\Longleftrightarrow \quad & \mathbf{A}^\delta \models_{\mathbf{A}} \mathtt{pure}(\varphi^* \leq \psi^*) & (8)\\
\Longleftrightarrow \quad & \mathbf{A}^\delta \models \mathtt{pure}(\varphi^* \leq \psi^*) & (9)\\
\Longrightarrow \quad & \mathbf{A}^\delta \models \varphi \leq \psi & (10)
\end{aligned}
$$

Canonicity Results (2)

Theorem

μ^*-ALBA successfully purifies all restrictive inductive mu-inequalities by means of proper runs.

Corollary

All restricted inductive mu-inequalities are canonical.

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