Hereditary Structural Completeness in Intermediate Logics

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TACL, June 23, 2015

Outline

Introduction

Primitive Quasivarieties

Applications to Intermediate Logics

Introduction

G. Mints (1939 - 2014)

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"If for some time I am unable to find an answer to a question, first, I go on Internet and ask the interest groups. If I am not getting an answer, I am writing to Prof. Mints, and in a couple of days I am getting the answer." S. Ghilardi TACL-2013

A (finitary structural) consequence relation:

(R) $A \vdash A$;

(M) $\Gamma \vdash A$ yields $\Gamma \cup \Delta \vdash A$;

(T) if $\Gamma \vdash A$ and $\Delta \vdash B$ for all $B \in \Gamma$, then $\Delta \vdash A$.

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Given $\vdash$, $\mathsf{L} := \{A \ : \vdash A\}$ is a logic *defined* by $\vdash$. A consequence relation $\vdash$ defining a logic L is *structurally complete* ($\mathcal{SC}$) if every proper extension of $\vdash$ defines a proper extension of L. A logic L is *structurally complete* if a consequence relation defined by the axioms of L and modus ponens is structurally complete.

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Definition

A consequence relation $\vdash$ is said to be *hereditarily structurally complete* ($\mathcal{HSC}$) if $\vdash$ and all its extensions are structurally complete. Logic L is $\mathcal{HSC}$ if L and all its extensions are $\mathcal{SC}$.

Introduction

The notion of hereditary structural completeness ($\mathcal{HSC}$) for intermediate logics was introduced in [Citkin, 1978], where the criterion of $\mathcal{HSC}$ for intermediate logics had been proven.
In [Rybakov, 1995] V.V. Rybakov has established a similar criterion for normal extensions of **K4**.

More recently, the hereditarily structurally complete consequence relations have been studied by J. S. Olson, J. G. Raftery and C. J. van Alten, in [Olson et al., 2008], and by P. Cintula and G. Metcalfe in [Cintula and Metcalfe, 2009], G. Metcalfe in [Metcalfe, 2013].

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In this presentation, we will focus on $\mathcal{HSC}$ consequence relations in intermediate logics.

The hereditary structural completeness has also another meaning.

Let L be a logic. A rule r is said to be *admissible* for L if L is closed under r. A set of rules R admissible for L is called a *basis of admissible rules* if any rule admissible for L is derivable from R.

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Given a logic L, by L° we denote its *structural completion* (or admissible closure [Rybakov, 1997]): the greatest consequence relation having L as a set of theorems (that is, the consequence relation induced by all rules admissible for L).

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Theorem

Let L *be a logic and* R *be a basis of rules admissible for* L*. The structural completion* L° *is* $\mathcal{HSC}$ *if and only if the rules* R *form a basis of admissible rules for every extension of* L*, for which all rules from* R *are admissible.*

The properties of classes of $\mathcal{HSC}$ logics and $\mathcal{HSC}$ consequence relations are quite different.

Let $\mathscr{L}$ be a class of all intermediate $\mathcal{HSC}$ logics, and $\mathscr{R}$ be a class of all intermediate $\mathcal{HSC}$ consequence relations.

	Property	$\mathscr{L}$	$\mathscr{R}$
1	Has a smallest element	Yes	No
2	Is countable	Yes	No
3	All members are f. axiomatizable	Yes	No
4	All members are locally tabular	Yes	No (may not have the fmp)

The Case of Intermediate Logics

Theorem ([Iemhoff, 2001])

The Visser rules

$$V_n := (A^{(n)} \to (A_{n+1} \vee A_{n+1})) \vee C / \vee_{j=1}^{n+2} (A^n \to A_j) \vee C,$$

where $A^{(n)} = \wedge_{i=1}^{n}(A_i \to B_i)$*, form a basis of rules admissible for* **Int**.

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Corollary

$\mathbf{Int}^\circ$ *is hereditarily structurally complete.*

Properties of $\mathscr{R}$

1. $\mathbf{Int}^\circ$ is minimal in $\mathscr{R}$. Logic L_9 (of single-generated 9-element Heyting algebra) satisfies the $\mathcal{HSC}$-criterion. But Visser rules are not admissible for L_9. Hence, $\mathbf{Int}^\circ$ and L_9° are incomparable and, therefore, $\mathscr{R}$ does not have the smallest element (so, $\mathscr{R}$ is not a lattice).

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4. $\mathbf{Int}^\circ$ does not have the fmp: from [Citkin, 1977] it follows that formula

$$((p \to q) \to (p \lor r)) \to (((p \to q) \to p) \lor ((p \to q) \to r))$$

is valid in all finite models of $\mathbf{Int}^\circ$, but is not valid in $\mathbf{Int}$.

Properties of $\mathscr{R}$

The above formula is obtained from the following admissible for **Int** not derivable rule introduced by Mints [Mints, 1976]:

$$(A \to B) \to (A \vee C)/((A \to B) \to A) \vee ((A \to B) \to C) \qquad \text{(Mints Rule)}$$

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An algebraic meaning of the Mints rule is as follows:

Theorem ([Citkin, 1977])

Let A *be a finite s.i. Heyting algebra. Then the following is equivalent*

1. A *is projective;*
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Since every finite subalgebra of F_ω is s.i., every finite subalgebra of F_ω is projective.

Properties of $\mathscr{R}$

Due to **Int** enjoying the Disjunction Property, the following rule (a generalized Mints rule) is also admissible and not derivable in **Int**:

$$((A \to B) \to (A \vee C)) \vee D / ((A \to B) \to A) \vee ((A \to B) \to C) \vee D \tag{GMR}$$

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Note: Generalized Mints rule is interderivable in **Int** with 1-st Visser rule.

Theorem

Let A *be a finite Heyting algebra. Then the following is equivalent*

1. $\mathsf{A} \in \mathbf{Q}(\mathsf{F}_\omega)$*;*
2. A *is a subdirect product of projective algebras;*
3. *Generalized Mints rule is valid in* A.

Properties of $\mathscr{R}$

Theorem

If an intermediate logic L *admits the generalized Mints rule (or the 1-st Visser rule) and* L° *enjoys the fmp, then* L *is hereditarily structurally complete.*

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Theorem

If an intermediate logic L *admits the generalized Mints rule (or the 1-st Visser rule) and* L° *enjoys the fmp, then* L *is hereditarily structurally complete.*

Recall that there are just countably many $\mathcal{HSC}$ intermediate logics, and there are continuum many extensions of $\mathbf{Int}^\circ$.

Corollary

There are continuum many $\mathcal{HSC}$ *consequence relations the structural completions of which do not enjoy the fmp.*

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Primitive Quasivarieties

A propositional logic L is algebraizable (in a sense of Blok and Pigozzi), if with L we can associate a variety of algebras. Accordingly, we can associate a quasivariety with any given algebraizable consequence relation. And a consequence relation is $\mathcal{HSC}$ if and only if the corresponding quasivariety $\mathbf{Q}$ is *primitive*, that is, any proper subquasivariety of $\mathbf{Q}$ can be defined over $\mathbf{Q}$ by identities.

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Theorem ([Gorbunov, 1976])

Any subquasivariety of a primitive quasivariety is primitive. A class of subquasivarieties of a given primitive quasivariety forms a distributive lattice.

Corollary

Every extension of a given $\mathcal{HSC}$ consequence relation is $\mathcal{HSC}$.

Weakly **Q**-Projective Algebras

Definition

Let $\mathbf{Q}$ be a quasivariety, $\mathsf{A} \in \mathbf{Q}$ be an algebra and θ be a congruence of A. Then θ is said to be a $\mathbf{Q}$-*congruence* if $\mathsf{A}/\theta \in \mathbf{Q}$. And algebra A is said to be $\mathbf{Q}$-*irreducible* if the meet of all distinct from identity $\mathbf{Q}$-congruences of A is distinct from identity congruence.

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Definition

Algebra $\mathsf{A} \in \mathbf{Q}$ is *weakly* $\mathbf{Q}$-*projective*, if A is embedded into every of its homomorphic pre-images from $\mathbf{Q}$, that is, $\mathsf{A} \in \mathbf{H}\mathsf{B}$ entails $\mathsf{A} \in \mathbf{S}\mathsf{B}$ for every $\mathsf{B} \in \mathbf{Q}$.

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Proposition

Every countable weakly $\mathbf{Q}$*-projective algebra is embedded into a free algebra of quasivariety* $\mathbf{Q}$ *of at most countable rank.*

Weakly $\mathbf{Q}$-Projective Algebras

The following theorem gives a simple sufficient condition of primitiveness.

Theorem ([Gorbunov, 1976])

If all finitely generated $\mathbf{Q}$-irreducible algebras of a quasivariety $\mathbf{Q}$ are weakly $\mathbf{Q}$-projective, then $\mathbf{Q}$ is primitive.

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If all finitely generated $\mathbf{Q}$-irreducible algebras of a quasivariety $\mathbf{Q}$ are weakly $\mathbf{Q}$-projective, then $\mathbf{Q}$ is primitive.

In case of the locally finite varieties, the sufficient condition is also necessary.

Theorem ([Gorbunov, 1976])

A locally finite quasivariety $\mathbf{Q}$ is primitive if and only if every of its finite $\mathbf{Q}$-irreducible algebras is weakly $\mathbf{Q}$-projective.

Totally Non-Projective Algebras

An algebra A is *totally non-projective* if A is not weakly **Q**-projective in the quasivariety it generates.

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For varieties, the above necessary condition is also sufficient.

Theorem

A variety of Heyting algebras is primitive if and only if it does not contain any totally non-projective algebras.

Totally Non-Projective Algebras

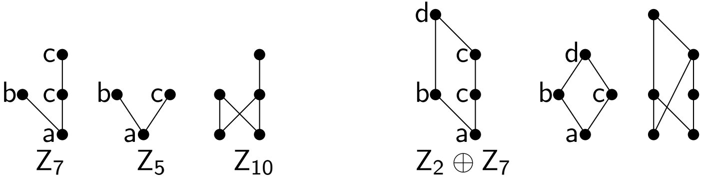
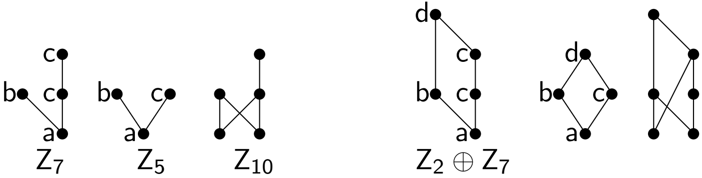


Fig. 1

$Z_5 \in \mathbf{Q}Z_7$, for $Z_5 \in \mathbf{S}Z_7$.



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$Z_5 \in \mathbf{Q}\mathrm{Z}_7$, for $Z_5 \in \mathbf{S}\mathrm{Z}_7$.

$Z_{10} \in \mathbf{Q}\mathrm{Z}_7$, for Z_{10} is a subdirect product of Z_7 and Z_5.

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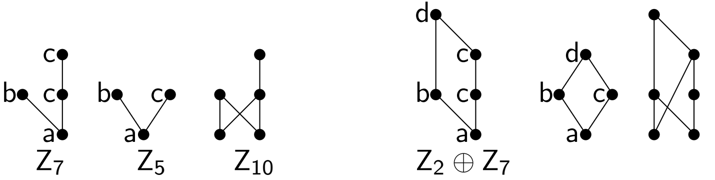

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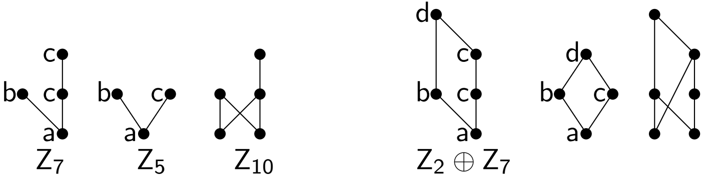
Z_7 is a homomorphic image of Z_{10}, but $\mathrm{Z}_7 \notin \mathbf{S}\mathrm{Z}_{10}$, hence Z_7 is not weakly $\mathbf{Q}\mathrm{Z}_7$-projective. Thus, Z_7 is totally non-projective.



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By a similar argument, one can show that $\mathrm{Z}_2 \oplus \mathrm{Z}_7$ is totally non-projective too.

Weakly $\mathbf{Q}$-Projective Algebras

If $\mathbf{Q}$ is a quasivariety, by $\mathbf{Q}^n, n = 1, 2, \ldots$ we denote a subquasivariety of $\mathbf{Q}$ generated by $\mathsf{F}_{\mathbf{Q}}(n)$, and by $\mathbf{Q}^0$ we denote a subquasivariety of $\mathbf{Q}$ generated by $\mathsf{F}_{\mathbf{Q}}(\omega)$.

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Note, that $\mathbf{Q}^0 = \cup_{n=1}^{\infty} \mathbf{Q}_i$.

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A quasivariety $\mathbf{Q}^0$ is primitive if and only if $\mathbf{Q}^n$ is primitive for every $n = 1, 2, \ldots$.

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First, we will give a criterion of primitiveness of $\mathbf{Q}^1$.

Primitiveness of $\mathbf{Q}(\mathrm{Z}_n)$

Denote by Z_n the single-generated Heyting algebra of cardinality n, and by Z - the countable single-generated algebra (the Rieger-Nishimura ladder).

Theorem

(a) *Quasivariety* $\mathbf{Q}(\mathrm{Z}_{2k+1})$ *is primitive if and only if* $k < 5$*;*

(b) *Quasivariety* $\mathbf{Q}(\mathrm{Z}_{2k})$ *is primitive if and only if* $k < 8$*.*

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(b) *Quasivariety $\mathbf{Q}(Z_{2k})$ is primitive if and only if $k < 8$.*

For every $k \geq 5$ the quasivariety $\mathbf{Q}(Z_{2k+1})$ is not primitive, for $Z_2 \oplus Z_7$ is embedded into Z_{2k+1} and $Z_2 \oplus Z_7$ is totally non-projective.

For every $k \geq 8$ the quasivariety $\mathbf{Q}(Z_{2k})$ is not primitive, for algebra $Z_2 \oplus Z_{10}$ is $\mathbf{Q}(Z_{2k})$-irreducible, although NOT weakly $\mathbf{Q}(Z_{2k})$-projective: $Z_2 \oplus Z_{10} \in \mathbf{H}(Z_2 \oplus Z_{12})$, but $Z_2 \oplus Z_{10} \notin \mathbf{S}(Z_2 \oplus Z_{12})$.

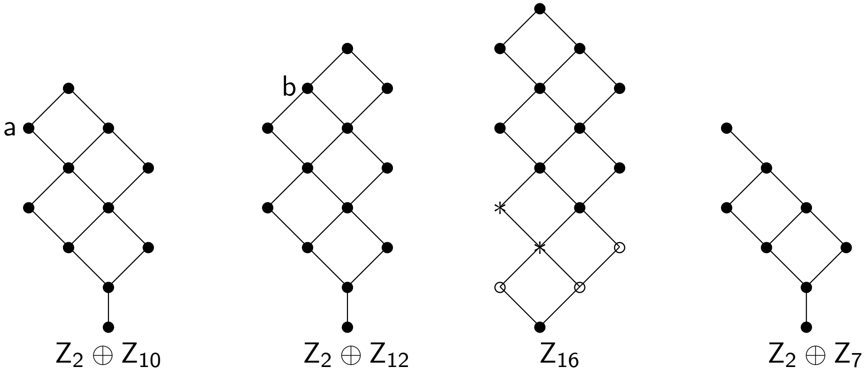

Fig. 2

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Since $\mathbf{Int}^{\circ}$ is $\mathcal{HSC}$, and every extension of an $\mathcal{HSC}$ consequence relation is $\mathcal{HSC}$, we have

Theorem

The structural completions of the following logics are hereditarily structurally complete:

(a) $\mathsf{G_n}$ - *Gödel logics*

(b) KC - *Yankov logic*

(c) LC - *Gödel - Dummett logic*

(d) P - *logic of projective algebras*

(e) RN - *logic of Rieger-Nishimura ladder*

(f) Sm - *Smetanich logic.*

Structural completions of BD_n

Theorem

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By $\mathsf{BD}_n, n = 1, 2, 3, \ldots$ we denote the logic of frames of depth at most $n + 1$ and by $\mathbf{BD}_n$ – a corresponding variety of Heyting algebras.

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By $\mathrm{BD}_n, n = 1, 2, 3, \dots$ we denote the logic of frames of depth at most $n+1$ and by $\mathbf{BD}_n$ – a corresponding variety of Heyting algebras.

Since Z_{4n} is a free cyclic algebra of $\mathbf{BD}_n$, the following holds.

Theorem

Structural completions of BD_n *for all* $n > 5$ *are not* $\mathcal{HSC}$*.*

Structural completions in intermediate logics

L	Description	L is $\mathcal{HSC}$	L° is $\mathcal{HSC}$
Int	(intuitionistic logic)	No	Yes
$\mathsf{BD_n}$	(depth at most n)	No for $n > 1$	No for $n > 4$
$\mathsf{D_n}$	(Gabbay - de Jongh)	No for $n > 2$	?
$\mathsf{G_n}$	(Gödel logics)	Yes	Yes
KC	(Yankov logic)	No	Yes
KP	(Kreisel-Putnam logic)	No	No (Jeřábek)
LC	(Gödel - Dummett logic)	Yes	Yes
$\mathsf{M_n}$	(at most n maximal nodes)	No for $n > 2$	?
ML	(Medvedev logic)	No	No
P	(logic of projective algebras)	Yes	Yes
RN	(logic of Z)	No	Yes
RN_n	(logic of Z_n)	No for $n = 7$ and $n > 9$	No for $n > 11$
Sm	(Smetanich logic)	Yes	Yes

Conclusion

G. Mints (1939 - 2014)

If we have a question that Prof. S.Ghilardi and the Internet interest groups cannot answer, who do we ask?...

Thanks

Thank you!

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