Dense completeness theorem for protoalgebraic logics

Petr Cintula[1] Carles Noguera[2]

[1]Institute of Computer Science
Czech Academy of Sciences

[2]Institute of Information Theory and Automation
Czech Academy of Sciences

Outline

1. A logic, a question, and a logician
2. Let us first study the problem in a controlled setting
3. Let us now look at the problem in full generality

Once upon a time, there was a logic . . .

Once upon a time, there was a logic . . .

Its name was full Lambek calculus with exchange $\mathrm{FL_e}$, and it has

Once upon a time, there was a logic ...

Its name was full Lambek calculus with exchange $\mathrm{FL_e}$, and it has

- a language $\mathcal{L} = \{\to, \&, \wedge, \vee, 0, 1\}$
- a nice Getzen and Hilbert style calculi
- a provability relation $\vdash_{\mathrm{FL_e}}$
- an algebraic semantics given by the variety $\mathbb{FL}_\mathrm{e}$ of all pointed commutative residuated lattices
- a natural notion of semantical consequence w.r.t. arbitrary $\mathbb{K} \subseteq \mathbb{FL}_\mathrm{e}$:

$$\Gamma \models_{\mathbb{K}} \varphi \text{ iff } (\forall \boldsymbol{A} \in \mathbb{K})(\forall e\colon \boldsymbol{Term}_{\mathcal{L}} \to \boldsymbol{A})[(\forall \gamma \in \Gamma)(e(\gamma) \geq 1) \Rightarrow e(\varphi) \geq 1]$$

- a completeness theorem: $\Gamma \vdash_{\mathrm{FL_e}} \varphi$ iff $\Gamma \models_{\mathbb{FL}_\mathrm{e}} \varphi$

And some people wondered . . .

And some people wondered . . .

How can we axiomatize its extension given by *completely densely linearly ordered* $\mathrm{FL_e}$*-algebras?*

Then a wise man splitted the question into three parts . . .


Franco Montagn
Italy

Then a wise man splitted the question into three parts . . .

Franco Montagna observed that:

$$\Gamma \vdash_{\mathrm{FL_e}} \varphi \quad \text{iff} \quad \Gamma \models_{\mathbb{FL}_\mathrm{e}} \varphi \;\Rightarrow\; \Gamma \models_{\mathbb{FL}_\mathrm{e}^\ell} \varphi \;\Rightarrow\; \Gamma \models_{\mathbb{FL}_\mathrm{e}^\delta} \varphi \;\Rightarrow\; \Gamma \models_{\mathbb{FL}_\mathrm{e}^{\mathrm{std}}} \varphi,$$

where:

- $\mathbb{FL}_\mathrm{e}^\ell$ are the linearly ordered $\mathrm{FL_e}$-algebras
- $\mathbb{FL}_\mathrm{e}^\delta$ are the densely linearly ordered $\mathrm{FL_e}$-algebras
- $\mathbb{FL}_\mathrm{e}^{\mathrm{std}}$ are the completely densely linearly ordered $\mathrm{FL_e}$-algebras

Then a wise man splitted the question into three parts . . .

Franco Montagna observed that:

$$\Gamma \vdash_{\mathrm{FL_e}} \varphi \quad \text{iff} \quad \Gamma \models_{\mathbb{FL}_\mathrm{e}} \varphi \overset{1}{\Rightarrow} \Gamma \models_{\mathbb{FL}_\mathrm{e}^\ell} \varphi \overset{2}{\Rightarrow} \Gamma \models_{\mathbb{FL}_\mathrm{e}^\delta} \varphi \overset{3}{\Rightarrow} \Gamma \models_{\mathbb{FL}_\mathrm{e}^{\mathrm{std}}} \varphi,$$

where:

- $\mathbb{FL}_\mathrm{e}^\ell$ are the linearly ordered $\mathrm{FL_e}$-algebras
- $\mathbb{FL}_\mathrm{e}^\delta$ are the densely linearly ordered $\mathrm{FL_e}$-algebras
- $\mathbb{FL}_\mathrm{e}^{\mathrm{std}}$ are the completely densely linearly ordered $\mathrm{FL_e}$-algebras

- He knew that there was an axiomatization for $\models_{\mathbb{FL}_\mathrm{e}^\ell}$
- He knew that the implication 3 could be reversed
- He showed that the implication 2 could be also reversed

In this talk we study the last step in general

How can we tell that a logic of chains is also a logic of dense chains?

In this talk we study the last step in general

How can we tell that a logic of chains is also a logic of dense chains?

Originally this kind of problems were solved *purely algebraically* by embedding any countable chain into a dense one

Franco (with Sandor Jenei) did it e.g. for the logics $\mathrm{FL_{ew}}$ or $\mathrm{FL_w}$

This approach however failed in $\mathrm{FL_e}$.

In this talk we study the last step in general

How can we tell that a logic of chains is also a logic of dense chains?

So again Franco (with George Metcalfe, building on the work by Takeuti and Titani) proposed a *proof-theoretic* approach:

- Extend a suitable calculus of the logic of chains by a “density rule”
- Show that the logic of this calculus is the logic of dense chains
- Eliminate the density rule

In this talk we study the last step in general

How can we tell that a logic of chains is also a logic of dense chains?

So again Franco (with George Metcalfe, building on the work by Takeuti and Titani) proposed a *proof-theoretic* approach:

- Extend a suitable calculus of the logic of chains by a “density rule”
- Show that the logic of this calculus is the logic of dense chains
- Eliminate the density rule

The last step was studied in general by Ciabattoni and Metcalfe

We focus on the second step in the setting of abstract algebraic logic

Outline

What is a logic? (for now)

Let us recall a few known facts:

- For any quasivariety $\mathbb{Q} \subseteq \mathbb{FL}_e$ there is a consequence relation $\mathrm{L}_{\mathbb{Q}}$ axiomatized by adding axioms and finitary rules to FL_e s.t. $\vdash_{\mathrm{L}_{\mathbb{Q}}} = \models_{\mathbb{Q}}$
- For any consequence relation L axiomatized by adding axioms and finitary rules to FL_e there is a quasivariety $\mathbb{Q}_{\mathrm{L}} \subseteq \mathbb{FL}_e$ s.t. $\vdash_{\mathrm{L}} = \models_{\mathbb{Q}_{\mathrm{L}}}$
- $\mathbb{Q} = \mathbb{Q}_{\mathrm{L}_{\mathbb{Q}}}$ and $\mathrm{L} = \mathrm{L}_{\mathbb{Q}_{\mathrm{L}}}$

Logic L: an arbitrary extension of FL_e by axioms and finitary rules or, equivalently, the logic of a quasivariety of FL_e-algebras

A prerequisite: How can we recognize a logic of chains?

Notation: by $\mathbb{Q}^{\ell}_{\mathrm{L}}$ we denote the set of linearly ordered algebras in $\mathbb{Q}_{\mathrm{L}}$

Theorem (Cintula 2005)

Let L *be a logic. TFAE:*

1. L *is semilinear, i.e.,* $\vdash_{\mathrm{L}} = \models_{\mathbb{Q}^{\ell}_{\mathrm{L}}}$
2. L *has the semilinearity property, i.e., for each* $\Gamma \cup \{\varphi, \psi, \chi\} \subseteq Fm_{\mathcal{L}}$*:*

$$\frac{\Gamma, \varphi \to \psi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \to \varphi \vdash_{\mathrm{L}} \chi}{\Gamma \vdash_{\mathrm{L}} \chi}$$

Main theorem (in the present, very restricted, setting)

Notation: by $\mathbb{Q}^{\delta}_{\mathrm{L}}$ we denote the set of densely ordered algebras in $\mathbb{Q}^{\ell}_{\mathrm{L}}$

Theorem

Let L *be a semilinear logic. TFAE:*

1. L *is dense complete, i.e.,* $\vdash_{\mathrm{L}} = \models_{\mathbb{Q}^{\delta}_{\mathrm{L}}}$
2. *Every countable chain in* $\mathbb{Q}^{\ell}_{\mathrm{L}}$ *can be embedded into a dense one*
3. L *has the density property, i.e., for each* $\Gamma \cup \{\varphi, \psi, \chi\} \subseteq Fm_{\mathcal{L}}$ *and any variable* p *not occurring in* $\Gamma \cup \{\varphi, \psi, \chi\}$*:*

$$\frac{\Gamma \vdash_{\mathrm{L}} (\varphi \to p) \vee (p \to \psi) \vee \chi}{\Gamma \vdash_{\mathrm{L}} (\varphi \to \psi) \vee \chi}$$

Main theorem (in the present, very restricted, setting)

Notation: by $\mathbb{Q}^{\delta}_{\mathrm{L}}$ we denote the set of densely ordered algebras in $\mathbb{Q}^{\ell}_{\mathrm{L}}$

Theorem

Let L *be a semilinear logic. TFAE:*

1. L *is dense complete, i.e.,* $\vdash_{\mathrm{L}} = \models_{\mathbb{Q}^{\delta}_{\mathrm{L}}}$
2. *Every countable chain in* $\mathbb{Q}^{\ell}_{\mathrm{L}}$ *can be embedded into a dense one*
3. L *has the density property, i.e., for each* $\Gamma \cup \{\varphi, \psi, \chi\} \subseteq Fm_{\mathcal{L}}$ *and any variable* p *not occurring in* $\Gamma \cup \{\varphi, \psi, \chi\}$*:*

$$\frac{\Gamma \vdash_{\mathrm{L}} (\varphi \to p) \vee (p \to \psi) \vee \chi}{\Gamma \vdash_{\mathrm{L}} (\varphi \to \psi) \vee \chi}$$

Corollary (Recovering Metcalfe–Montagna's method)

A semilinear logic enjoys the dense completeness iff it equals the intersection of all its (finitary) extensions satisfying the density property.

The first ingredient of the proof: implication

Recall: In any $\mathrm{FL_e}$-algebra: $x \leq y$ iff $x \to y \geq 1$

The first ingredient of the proof: implication

Recall: In any $\mathrm{FL_e}$-algebra: $x \leq y$ iff $x \to y \geq 1$

By $\mathrm{Th}(\mathrm{L})$ we denote the set of theories (deductively closed sets) of L

A theory T is linear if for any pair φ, ψ we have: $\varphi \to \psi \in T$ or $\psi \to \varphi \in T$

Fact: T is linear iff its free (Lindenbaum–Tarski) algebra is linear

Theorem (Cintula 2005)

Let L be a logic. TFAE:

1. L *is semilinear, i.e.,* $\vdash_{\mathrm{L}} = \models_{\mathbb{Q}^{\ell}_{\mathrm{L}}}$
2. *Linear theories form a basis of* $\mathrm{Th}(\mathrm{L})$, *i.e., any theory* $T \nvdash_{\mathrm{L}} \varphi$ *can be extended into a linear theory* $T' \nvdash_{\mathrm{L}} \varphi$

The first ingredient of the proof: implication

A linear theory T is dense if for any pair φ, ψ, if $\varphi \to \psi \notin T$ there is χ s.t. $\chi \to \psi \notin T$ and $\varphi \to \chi \notin T$

Non-theorem based on a naive generalization

Let L *be a semilinear logic. TFAE:*

1. L *is dense complete, i.e.,* $\vdash_{\mathrm{L}} = \models_{\mathbb{Q}^{\delta}_{\mathrm{L}}}$
2. *Dense theories form a basis of* $\mathrm{Th}(\mathrm{L})$*, i.e., any theory* $T \nvdash_{\mathrm{L}} \varphi$ *can be extended into a dense theory* $T' \nvdash_{\mathrm{L}} \varphi$

The first ingredient of the proof: implication

A linear theory T is dense if for any pair φ, ψ, if $\varphi \to \psi \notin T$ there is χ s.t. $\chi \to \psi \notin T$ and $\varphi \to \chi \notin T$

Non-theorem based on a naive generalization

Let L *be a semilinear logic. TFAE:*

1. L *is dense complete, i.e.,* $\vdash_{\mathrm{L}} \; = \; \models_{\mathbb{Q}^{\delta}_{\mathrm{L}}}$
2. *Dense theories form a basis of* $\mathrm{Th}(\mathrm{L})$*, i.e., any theory* $T \nvdash_{\mathrm{L}} \varphi$ *can be extended into a dense theory* $T' \nvdash_{\mathrm{L}} \varphi$

Failure of the naive idea: consider the Łukasiewicz infinitely-valued logic $\mathrm{Ł}$, which is well-know to enjoy the dense completeness

Consider $T = \{\varphi \mid Var \vdash_{\mathrm{Ł}} \varphi\}$ and note that for each φ, either $T \vdash_{\mathrm{Ł}} \varphi$ or $T \vdash_{\mathrm{Ł}} \neg\varphi$

Thus T is maximally consistent but not dense (we have $1 \to 0 \notin T$)

Thus T *cannot* be a extended into any dense theory

The first ingredient of the proof: implication

A linear theory T is dense if for any pair φ, ψ, if $\varphi \to \psi \notin T$ there is χ s.t. $\chi \to \psi \notin T$ and $\varphi \to \chi \notin T$

Theorem (Implicit in Metcalfe–Montagna 2007)

Let L be a semilinear logic. TFAE:

1. L *is dense complete, i.e.,* $\vdash_{\mathrm{L}} \; = \; \models_{\mathbb{Q}^{\delta}_{\mathrm{L}}}$
2. *Any set of formulae* $\Gamma \nvdash_{\mathrm{L}} \varphi$ *with infinitely many unused variables can be extended into a dense theory* $T' \nvdash_{\mathrm{L}} \varphi$

The second ingredient of the proof: disjunction

Recall the density property:

$$\frac{\Gamma \vdash_{\mathrm{L}} (\varphi \to p) \vee (p \to \psi) \vee \chi}{\Gamma \vdash_{\mathrm{L}} (\varphi \to \psi) \vee \chi}$$

The second ingredient of the proof: disjunction

Recall the density property:

$$\frac{\Gamma \vdash_{\mathrm{L}} (\varphi \to p) \vee (p \to \psi) \vee \chi}{\Gamma \vdash_{\mathrm{L}} (\varphi \to \psi) \vee \chi}$$

For the proof to work we need the disjunction to behave well:

Theorem

Let L *be logic. TFAE*

1. L *has the proof by cases property, i.e. for each* $\Gamma \cup \{\varphi, \psi, \chi\} \subseteq Fm_{\mathcal{L}}$*:*

$$\frac{\Gamma, \varphi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \vdash_{\mathrm{L}} \chi}{\Gamma, \varphi \vee \psi \vdash_{\mathrm{L}} \chi}$$

2. *for each* $\Gamma \cup \{\varphi, \chi\} \subseteq Fm_{\mathcal{L}}$

$$\frac{\Gamma \vdash_{\mathrm{L}} \varphi}{\{\psi \vee \chi \mid \psi \in \Gamma\} \vdash_{\mathrm{L}} \varphi \vee \chi}$$

Any logic extending $\models_{\mathbb{FL}_e^\ell}$ *is semilinear iff it has the proof by cases property.*

Outline

What is a logic *really*?

What is a logic *really*? (as a mathematical object)

Var: an infinite set of propositional variables

$\mathcal{L}$: an arbitrary type

$\boldsymbol{Fm}_{\mathcal{L}}$: the absolutely free $\mathcal{L}$-algebra with generators Var

elements of $\mathit{Fm}_{\mathcal{L}}$ are called *$\mathcal{L}$-formulae*

A logic L is a relation between sets of $\mathcal{L}$-formulae and $\mathcal{L}$-formulae s.t.:

we write '$\Gamma \vdash_{\mathrm{L}} \varphi$' instead of '$\langle \Gamma, \varphi \rangle \in \mathrm{L}$'

- If $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathrm{L}} \varphi$. (Reflexivity)
- If $\Gamma \vdash_{\mathrm{L}} \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_{\mathrm{L}} \varphi$. (Monotonicity)
- If $\Delta \vdash_{\mathrm{L}} \Gamma$ and $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\Delta \vdash_{\mathrm{L}} \varphi$. (Cut)
- If $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\varphi)$ for each substitution σ. (Structurality)

A logic L is finitary if $\Gamma \vdash_{\mathrm{L}} \varphi$, then there is a finite $\Gamma' \subseteq \Gamma$ st. $\Gamma' \vdash_{\mathrm{L}} \varphi$

What is a protoalgebraic logic?

Let $\overrightarrow{r}$ be a sequence of atoms and $\Rightarrow(p, q, \overrightarrow{r}) \subseteq Fm_{\mathcal{L}}$.

Convention: given formulae φ and ψ, we set

$$\varphi \Rightarrow \psi \quad = \quad \bigcup\{\Rightarrow(\varphi, \psi, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in \mathrm{Fm}_{\mathcal{L}}^{<\omega}\}$$

What is a protoalgebraic logic?

Let $\overrightarrow{r}$ be a sequence of atoms and $\Rightarrow(p, q, \overrightarrow{r}) \subseteq Fm_{\mathcal{L}}$.

Convention: given formulae φ and ψ, we set

$$\varphi \Rightarrow \psi \quad = \quad \bigcup \{\Rightarrow(\varphi, \psi, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in \mathrm{Fm}_{\mathcal{L}}^{<\omega}\}$$

A logic is protoalgebraic if it has a weak p-implication, i.e., a set $\Rightarrow$ s.t.:

(R) $\vdash_{\mathrm{L}} \varphi \Rightarrow \varphi$

(MP) $\varphi, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \psi$

(T) $\varphi \Rightarrow \psi, \psi \Rightarrow \chi \vdash_{\mathrm{L}} \varphi \Rightarrow \chi$

(sCng) $\varphi \Rightarrow \psi, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} c(\chi_1, \dots, \varphi, \dots, \chi_n) \Rightarrow c(\chi_1, \dots, \psi, \dots, \chi_n)$
for each $\langle c, n\rangle \in \mathcal{L}$ and $i \leq n$.

What is a protoalgebraic logic?

Let $\overrightarrow{r}$ be a sequence of atoms and $\Rightarrow(p, q, \overrightarrow{r}) \subseteq Fm_{\mathcal{L}}$.

Convention: given formulae φ and ψ, we set

$$\varphi \Rightarrow \psi \quad = \quad \bigcup \{\Rightarrow(\varphi, \psi, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in \mathrm{Fm}_{\mathcal{L}}^{<\omega}\}$$

A logic is protoalgebraic if it has a weak p-implication, i.e., a set $\Rightarrow$ s.t.:

(R) $\vdash_{\mathrm{L}} \varphi \Rightarrow \varphi$

(MP) $\varphi, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \psi$

(T) $\varphi \Rightarrow \psi, \psi \Rightarrow \chi \vdash_{\mathrm{L}} \varphi \Rightarrow \chi$

(sCng) $\varphi \Rightarrow \psi, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} c(\chi_1, \dots, \varphi, \dots, \chi_n) \Rightarrow c(\chi_1, \dots, \psi, \dots, \chi_n)$
for each $\langle c, n\rangle \in \mathcal{L}$ and $i \leq n$.

Let us fix a protoalgebraic logic L with a weak p-implication $\Rightarrow$

Logical matrices, semantical consequence, order

$\mathcal{L}$-matrix: a pair $\mathbf{A} = \langle A, F\rangle$ where A is an $\mathcal{L}$-algebra and $F \subseteq A$.

Definition

A formula φ is a logical consequence of a set of formulae Γ w.r.t. a class $\mathbb{K}$ of $\mathcal{L}$-matrices, $\Gamma \models_{\mathbb{K}} \varphi$, if for every $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and every homomorphism $e\colon \boldsymbol{Fm}_{\mathcal{L}} \to \boldsymbol{A}$:

$$\text{if } e(\gamma) \in F \text{ for every } \gamma \in \Gamma, \text{ then } e(\varphi) \in F.$$

A matrix $\mathbf{A}$ s.t. $\vdash_{\mathrm{L}} \subseteq \models_{\mathbf{A}}$ is called a model of L, $\mathbf{A} \in \mathbf{MOD}(\mathrm{L})$ in symbols

Consider $\mathbf{A} = \langle A, F\rangle \in \mathbf{MOD}(\mathrm{L})$; then the relation $a \leq^{\Rightarrow}_{\mathbf{A}} b$ is a preorder:

$$a \leq^{\Rightarrow}_{\mathbf{A}} b \qquad \text{iff} \qquad a \Rightarrow^{A} b \subseteq F$$

The first ingredient, implication, seems to be OK. What about the second one?

A connective $\vee$ is a lattice-disjunction if for each $\Gamma \cup \{\varphi, \psi, \chi\} \subseteq Fm_{\mathcal{L}}$:

$$\vdash_{\mathrm{L}} \varphi \Rightarrow \varphi \vee \psi \qquad \vdash_{\mathrm{L}} \psi \Rightarrow \varphi \vee \psi \qquad \varphi \Rightarrow \chi, \psi \Rightarrow \chi \vdash_{\mathrm{L}} \varphi \vee \psi \Rightarrow \chi$$

$$\frac{\Gamma, \varphi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \vdash_{\mathrm{L}} \chi}{\Gamma, \varphi \vee \psi \vdash_{\mathrm{L}} \chi}$$

For logics of chains, everything still works

A matrix $\mathbf{A}$ is linear, $\mathbf{A} \in \mathbf{MOD}^{\ell}(\mathrm{L})$, if $\leq_{\mathbf{A}}^{\Rightarrow}$ is a linear order.

By $\mathrm{Th}(\mathrm{L})$ we denote the set of theories (deductively closed sets) of L

A theory T is linear if for any pair φ, ψ we have: $\varphi \Rightarrow \psi \in T$ or $\psi \Rightarrow \varphi \in T$

Theorem (Cintula–Noguera 2010)

Let L be a protoalgebraic logic. TFAE:

1. L *is semilinear, i.e.,* $\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^{\ell}(\mathrm{L})}$
2. *Linear theories form a basis of* $\mathrm{Th}(\mathrm{L})$

If L is finitary we can add

3. L *has the semilinearity property, i.e., for each* $\Gamma \cup \{\varphi, \psi, \chi\} \subseteq Fm_{\mathcal{L}}$:

$$\frac{\Gamma, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} \chi}{\Gamma \vdash_{\mathrm{L}} \chi}$$

And now, finally, the final theorem

A matrix $\mathbf{A}$ is densely linear, $\mathbf{A} \in \mathbf{MOD}^{\delta}(\mathrm{L})$, if $\leq_{\mathbf{A}}^{\Rightarrow}$ is a dense linear order.

A linear theory T is dense if for any pair φ, ψ if $\psi \Rightarrow \varphi \notin T$ there is χ s.t. $\chi \Rightarrow \varphi \notin T$ and $\psi \Rightarrow \chi \notin T$

Theorem

Let L *be a protoalgebraic logic. TFAE:*

1. L *is dense complete, i.e.,* $\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^{\delta}(\mathrm{L})}$
2. *Any set of formulae* $\Gamma \nvdash_{\mathrm{L}} \varphi$ *with infinitely many unused variables can be extended into a dense theory* $T' \nvdash_{\mathrm{L}} \varphi$

If L *is finitary,* $\Rightarrow$ *finite and parameter-free,* $\vee$ *a lattice-disjunction, we can add*

3. *Countable chains in* $\mathbf{MOD}^{\ell}(\mathrm{L})$ *can be embedded into dense ones*
4. L *has the density property, i.e., for each* $\Gamma \cup \{\varphi, \psi, \chi\} \subseteq Fm_{\mathcal{L}}$ *and any variable* p *not occurring in* $\Gamma \cup \{\varphi, \psi, \chi\}$*:*

$$\frac{\Gamma \vdash_{\mathrm{L}} (\varphi \Rightarrow p) \vee (p \Rightarrow \psi) \vee \chi}{\Gamma \vdash_{\mathrm{L}} (\varphi \Rightarrow \psi) \vee \chi}$$