On Regular Congruences of Partially Ordered Semigroups

Boža Tasić
Ryerson University, Toronto

June 21, 2015

Partially Ordered Semigroups

- A partially ordered semigroup (po-semigroup) is an ordered triple $\mathbf{S} = \langle S, \cdot, \leq \rangle$ such that

Partially Ordered Semigroups

- A partially ordered semigroup (po-semigroup) is an ordered triple $\mathbf{S} = \langle S, \cdot, \leq \rangle$ such that
  - a) $\langle S, \cdot, \rangle$ is a semigroup,

Partially Ordered Semigroups

- A partially ordered semigroup (po-semigroup) is an ordered triple $\mathbf{S} = \langle S, \cdot, \leq \rangle$ such that
  - a) $\langle S, \cdot, \rangle$ is a semigroup,
  - b) $\langle S, \leq \rangle$ is a partially ordered set (poset),

Partially Ordered Semigroups

- A partially ordered semigroup (po-semigroup) is an ordered triple $\mathbf{S} = \langle S, \cdot, \leq \rangle$ such that
  - a) $\langle S, \cdot, \rangle$ is a semigroup,
  - b) $\langle S, \leq \rangle$ is a partially ordered set (poset),
  - c) $(x \leq y \ \&\ u \leq v) \rightarrow x \cdot u \leq y \cdot v$ for all $x, y, u, v \in S$.

Partially Ordered Semigroups

- A partially ordered semigroup (po-semigroup) is an ordered triple $\mathbf{S} = \langle S, \cdot, \leq \rangle$ such that
  - a) $\langle S, \cdot, \rangle$ is a semigroup,
  - b) $\langle S, \leq \rangle$ is a partially ordered set (poset),
  - c) $(x \leq y \;\&\; u \leq v) \rightarrow x \cdot u \leq y \cdot v$ for all $x, y, u, v \in S$.
- Condition c) is equivalent to

Partially Ordered Semigroups

- A partially ordered semigroup (po-semigroup) is an ordered triple $\mathbf{S} = \langle S, \cdot, \leq \rangle$ such that
  - a) $\langle S, \cdot, \rangle$ is a semigroup,
  - b) $\langle S, \leq \rangle$ is a partially ordered set (poset),
  - c) $(x \leq y \ \& \ u \leq v) \to x \cdot u \leq y \cdot v$ for all $x, y, u, v \in S$.
- Condition c) is equivalent to
  - c') $x \leq y \to (x \cdot z \leq y \cdot z \ \& \ z \cdot x \leq z \cdot y)$ for all $x, y, z \in S$

Congruence Relations of Semigroups

- A congruence relation of the semigroup $\mathbf{S} = \langle S, \cdot \rangle$ is an equivalence relation $\theta \subseteq S^2$ that satisfies the following compatibility property:

Congruence Relations of Semigroups

- A congruence relation of the semigroup $\mathbf{S} = \langle S, \cdot \rangle$ is an equivalence relation $\theta \subseteq S^2$ that satisfies the following compatibility property:
- $$(x \,\theta\, y \;\&\; u \,\theta\, v) \rightarrow x \cdot u \,\theta\, y \cdot v \quad \text{for all } x, y, u, v \in S. \qquad \text{(CP)}$$

Congruence Relations of Semigroups

- Given a congruence relation θ of the semigroup $\mathbf{S} = \langle S, \cdot \rangle$ we define the quotient semigroup (a.k.a the homomorphic image) of $\mathbf{S}$ by θ to be the semigroup $\mathbf{S}/\theta = \langle S/\theta, \odot \rangle$ where

Congruence Relations of Semigroups

- Given a congruence relation θ of the semigroup $\mathbf{S} = \langle S, \cdot \rangle$ we define the quotient semigroup (a.k.a the homomorphic image) of $\mathbf{S}$ by θ to be the semigroup $\mathbf{S}/\theta = \langle S/\theta, \odot \rangle$ where
- $x/\theta \odot y/\theta = (x \cdot y)/\theta \quad$ for all $x, y \in S$.

Congruence Relations of Semigroups

- Given a congruence relation θ of the semigroup $\mathbf{S} = \langle S, \cdot \rangle$ we define the quotient semigroup (a.k.a the homomorphic image) of $\mathbf{S}$ by θ to be the semigroup $\mathbf{S}/\theta = \langle S/\theta, \odot \rangle$ where
- $x/\theta \odot y/\theta = (x \cdot y)/\theta \quad$ for all $x, y \in S$.
- The set of all congruences of a semigroup $\mathbf{S} = \langle S, \cdot \rangle$ is denoted by $Con(\mathbf{S})$

Congruence Relations of Semigroups

- How does one extend the concept of quotients to po-semigroups?

Congruence Relations of Semigroups

- How does one extend the concept of quotients to po-semigroups?
- For structures that have only operations, i.e., algebras, quotients are defined using congruences.

Congruence Relations of Semigroups

- How does one extend the concept of quotients to po-semigroups?
- For structures that have only operations, i.e., algebras, quotients are defined using congruences.
- Structures that have both operations and relations are studied in model theory and their quotients are defined as

$$\text{The algebraic quotient} + \text{Relations on the algebraic quotient.}$$

Quotients of Po-Semigroups

- Given a po-semigroup $\mathbf{S} = \langle S, \cdot, \leq \rangle$ and $\theta \in Con\langle S, \cdot \rangle$ we define the quotient po-semigroup (a.k.a the homomorphic image) of $\mathbf{S}$ by θ to be the structure $\mathbf{S}/\theta = \langle S/\theta, \odot, \preceq \rangle$, where

Quotients of Po-Semigroups

- Given a po-semigroup $\mathbf{S} = \langle S, \cdot, \leq \rangle$ and $\theta \in Con\langle S, \cdot \rangle$ we define the quotient po-semigroup (a.k.a the homomorphic image) of $\mathbf{S}$ by θ to be the structure $\mathbf{S}/\theta = \langle S/\theta, \odot, \preceq \rangle$, where
  - a) $\langle S/\theta, \odot \rangle$ is a quotient semigroup,

Quotients of Po-Semigroups

- Given a po-semigroup $\mathbf{S} = \langle S, \cdot, \leq \rangle$ and $\theta \in Con\langle S, \cdot \rangle$ we define the quotient po-semigroup (a.k.a the homomorphic image) of $\mathbf{S}$ by θ to be the structure $\mathbf{S}/\theta = \langle S/\theta, \odot, \preceq \rangle$, where
  - a) $\langle S/\theta, \odot \rangle$ is a quotient semigroup,
  - b) The relation $\preceq$ on S/θ is defined by

$$x/\theta \preceq y/\theta \leftrightarrow (\exists x_1 \in x/\theta)(\exists y_1 \in y/\theta)\, x_1 \leq y_1$$

Quotients of Po-Semigroups

- Quotients preserve identities satisfied by the original algebra

Quotients of Po-Semigroups

- Quotients preserve identities satisfied by the original algebra
- Quotients may not preserve all formulas involving relations satisfied by the original structure.

Quotients of Po-Semigroups

- Quotients preserve identities satisfied by the original algebra
- Quotients may not preserve all formulas involving relations satisfied by the original structure.
- A quotient of a semigroup is a semigroup,

Quotients of Po-Semigroups

- Quotients preserve identities satisfied by the original algebra
- Quotients may not preserve all formulas involving relations satisfied by the original structure.
- A quotient of a semigroup is a semigroup,
- Unfortunately, a quotient of a po-semigroup is not necessarily a po-semigroup.

Quotients of Po-Semigroups

- The quotient structure $\mathbf{S}/\theta = \langle S/\theta, \odot, \preceq \rangle$ satisfies

Quotients of Po-Semigroups

- The quotient structure $\mathbf{S}/\theta = \langle S/\theta, \odot, \preceq \rangle$ satisfies
  - $\forall x\, (x \preceq x)$,

Quotients of Po-Semigroups

- The quotient structure $\mathbf{S}/\theta = \langle S/\theta, \odot, \preceq \rangle$ satisfies
  - $\forall x\,(x \preceq x)$,
  - and $\preceq$ on S/θ is compatible with $\odot$, i.e.,

$$(x/\theta \preceq y/\theta \wedge u/\theta \preceq v/\theta) \rightarrow x/\theta \odot u/\theta \preceq y/\theta \odot v/\theta \quad \text{(CPQ)}$$

Quotients of Po-Semigroups

- The quotient structure $\mathbf{S}/\theta = \langle S/\theta, \odot, \preceq \rangle$ satisfies
  - $\forall x\,(x \preceq x)$,
  - and $\preceq$ on S/θ is compatible with $\odot$, i.e.,

$$(x/\theta \preceq y/\theta \wedge u/\theta \preceq v/\theta) \rightarrow x/\theta \odot u/\theta \preceq y/\theta \odot v/\theta \quad \text{(CPQ)}$$

- Neither anti-symmetry nor transitivity is preserved by $\preceq$ in general.

Regular Congruences

- Let $\mathbf{S} = \langle S, \cdot, \leq \rangle$ be a po-semigroup and $\theta \in Con\langle S, \cdot \rangle$. The congruence θ is called **regular** if there exists an order $\preceq$ on S/θ such that:

Regular Congruences

- Let $\mathbf{S} = \langle S, \cdot, \leq \rangle$ be a po-semigroup and $\theta \in Con\langle S, \cdot \rangle$. The congruence θ is called **regular** if there exists an order $\preceq$ on S/θ such that:
  - a) $\mathbf{S}/\theta = \langle S/\theta, \odot, \preceq \rangle$ is a po-semigroup,

Regular Congruences

- Let $\mathbf{S} = \langle S, \cdot, \leq \rangle$ be a po-semigroup and $\theta \in Con\langle S, \cdot \rangle$. The congruence θ is called **regular** if there exists an order $\preceq$ on S/θ such that:
  - a) $\mathbf{S}/\theta = \langle S/\theta, \odot, \preceq \rangle$ is a po-semigroup,
  - b) The mapping $\varphi : S \mapsto S/\theta$ defined by $\varphi(x) = x/\theta$ is isotone, i.e.,

$$x \leq y \to x/\theta \preceq y/\theta.$$

Regular Congruences

- Let $\mathbf{S} = \langle S, \cdot, \leq \rangle$ be a po-semigroup and $\theta \in Con\langle S, \cdot \rangle$. The congruence θ is called **regular** if there exists an order $\preceq$ on S/θ such that:
  - a) $\mathbf{S}/\theta = \langle S/\theta, \odot, \preceq \rangle$ is a po-semigroup,
  - b) The mapping $\varphi : S \mapsto S/\theta$ defined by $\varphi(x) = x/\theta$ is isotone, i.e.,
  $$x \leq y \to x/\theta \preceq y/\theta.$$
- Regularity can be expressed using a special property of θ !

Open Bracelet Modulo θ

- Let $\mathbf{S} = \langle S, \cdot, \leq \rangle$ be a po-semigroup and $\theta \in Con\langle S, \cdot \rangle$. A sequence of elements $(a_1, a_2, a_3, \ldots, a_{2n})$ in $\mathbf{S}$ is called

Open Bracelet Modulo θ

- Let $\mathbf{S} = \langle S, \cdot, \leq \rangle$ be a po-semigroup and $\theta \in Con\langle S, \cdot \rangle$. A sequence of elements $(a_1, a_2, a_3, \ldots, a_{2n})$ in $\mathbf{S}$ is called
- an **open bracelet modulo** θ if it satisfies

$$a_1 \,\theta\, a_2 \leq a_3 \,\theta\, a_4 \leq \ldots \cdots \leq a_{2n-1} \,\theta\, a_{2n}$$

Open Bracelet Modulo θ

- Let $\mathbf{S} = \langle S, \cdot, \leq \rangle$ be a po-semigroup and $\theta \in Con\langle S, \cdot \rangle$. A sequence of elements $(a_1, a_2, a_3, \ldots, a_{2n})$ in $\mathbf{S}$ is called
- an **open bracelet modulo** θ if it satisfies

$$a_1 \, \theta \, a_2 \leq a_3 \, \theta \, a_4 \leq \ldots \cdots \leq \; a_{2n-1} \, \theta \, a_{2n}$$

- a **closed bracelet modulo** θ if it satisfies

$$a_1 \, \theta \, a_2 \leq a_3 \, \theta \, a_4 \leq \ldots \cdots \leq \; a_{2n-1} \, \theta \, a_{2n} \; \leq a_1$$

Open Bracelet Modulo θ

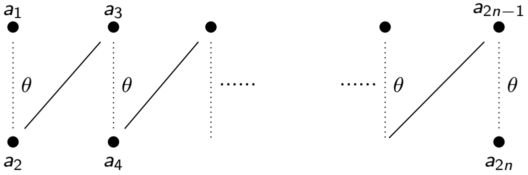


Figure : Open bracelet modulo θ

Closed Bracelet Modulo θ

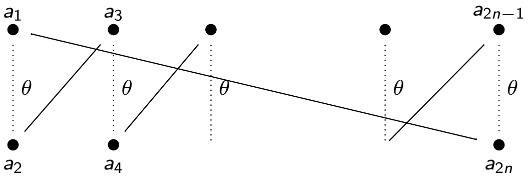


Figure : Closed bracelet modulo θ

Regular Congruences

Theorem

Let $\mathbf{S} = \langle S, \cdot, \leq \rangle$ be a po-semigroup. A congruence $\theta \in Con\langle S, \cdot \rangle$ is regular if and only if it satisfies the property: for every $n \in \mathbb{N}$ and every $a_1, \ldots, a_{2n} \in S$

$$a_1 \,\theta\, a_2 \leq a_3 \,\theta\, a_4 \leq \ldots \ldots a_{2n-1} \,\theta\, a_{2n} \leq a_1 \;\rightarrow\; \{a_1, \ldots, a_{2n}\}^2 \subseteq \theta.$$

Regular Congruences

- A congruence $\theta \in Con\langle S, \cdot\rangle$ is regular if and only if every closed bracelet modulo θ is contained in a single equivalence class modulo θ.

Regular Congruences

- A congruence $\theta \in Con\langle S, \cdot\rangle$ is regular if and only if every closed bracelet modulo θ is contained in a single equivalence class modulo θ.
- If $\theta \in Con\langle S, \cdot\rangle$ is regular then θ-classes are convex subsets of $\langle S, \leq\rangle$.

Lattice of Regular Congruences

Theorem

For a given po-semigroup $\mathbf{S} = \langle S, \cdot, \leq \rangle$*, the set of all regular congruences is a complete lattice with respect to the inclusion as the ordering. We denote it by* $\mathbf{RCon}\,\mathbf{S} = \langle RConS, \subseteq \rangle$*.*

Lattice of Regular Congruences

- The lattice **RCon S** is not a sublattice of the lattice **Con S**.

Lattice of Regular Congruences

- The lattice **RCon S** is not a sublattice of the lattice **Con S**.
- The intersection of two regular congruences is a regular congruence.

Lattice of Regular Congruences

- The lattice **RCon S** is not a sublattice of the lattice **Con S**.
- The intersection of two regular congruences is a regular congruence.
- The join of two regular congruences is not a regular congruence in general.

Lattice of Regular Congruences

Theorem (Tasic)

For a po-semigroup $\mathbf{S} = \langle S, \cdot, \leq \rangle$*, there is an algebraic closure operator* Θ_R *on* $S \times S$ *such that the closed subsets of* $S \times S$ *are precisely the regular congruences on* $\mathbf{S}$*. Hence* $\mathbf{RCon}\,\mathbf{S} = \langle RCon(S), \subseteq \rangle$ *is an algebraic lattice.*

Lattice of Regular Congruences

Proof.

Consider the algebra

$$\mathbf{S}^* = \langle S \times S,\ \odot,\ (a,a)_{a \in S},\ s,\ t,\ (e^n_{ij})_{n \geq 2,\, 1 \leq i < j \leq 2n} \rangle$$

where $(a,a)_{a \in S}$ are constants (nullary operations), s is a unary operation, $\odot$ and t are binary operations, and for each $n \geq 2$ and $1 \leq i < j \leq 2n$, e^n_{ij} is an n-ary operation. □

Lattice of Regular Congruences

Proof.

The operations are defined by

- $$(a, b) \odot (c, d) = (a \cdot c,\ b \cdot d)$$

□

Lattice of Regular Congruences

Proof.

The operations are defined by

- $$(a, b) \odot (c, d) = (a \cdot c,\ b \cdot d)$$

- $$s((a, b)) = (b, a)$$

□

Lattice of Regular Congruences

Proof.

The operations are defined by

- $$(a, b) \odot (c, d) = (a \cdot c,\ b \cdot d)$$

- $$s((a, b)) = (b, a)$$

- $$t((a, b)\,(c, d)) = \begin{cases} (a, d), & \text{if } b = c \\ (a, b), & \text{otherwise} \end{cases}$$

□

Lattice of Regular Congruences

Proof.

- $$e^n_{ij}((a_1, a_2), (a_3, a_4), \ldots, (a_{2n-1}, a_{2n})) = \begin{cases} (a_i, a_j), & \text{if for all k=1,...,n-} \\ (a_1, a_2), & \text{otherwise.} \end{cases}$$

□

Lattice of Regular Congruences

Proof.

- $$e^n_{ij}((a_1, a_2), (a_3, a_4), \ldots, (a_{2n-1}, a_{2n})) = \begin{cases} (a_i, a_j), \text{ if for all k=1,...,n-} \\ (a_1, a_2), \quad \text{otherwise.} \end{cases}$$
- (a_i, a_j), if for all k=1,...,n-1 $a_{2k} \leq a_{2k+1}$ and $a_{2n} \leq a_1$

□

Lattice of Regular Congruences

- Let Θ_R be the Sg closure operator on $S \times S$ for the algebra $\mathbf{S}^*$.

Lattice of Regular Congruences

- Let Θ_R be the Sg closure operator on $S \times S$ for the algebra $\mathbf{S}^*$.
- The compact members of $\mathbf{RCon}\,\mathbf{S}$ are the finitely generated members $\Theta_R((a_1, b_1), (a_2, b_2), \ldots (a_n, b_n))$ of $\mathbf{RCon}\,\mathbf{S}$.

Regular Congruences Generated by a Set

For $\mathbf{S} = \langle S, \cdot, \leq \rangle$ a po-semigroup and $a_1, a_2, \ldots a_n \in A$ let

- $\Theta_R(a_1, a_2, \ldots a_n)$ denote the regular congruence generated by $\{(a_i, a_j) \mid 1 \leq i, j \leq n\}$

Regular Congruences Generated by a Set

For $\mathbf{S} = \langle S, \cdot, \leq \rangle$ a po-semigroup and $a_1, a_2, \ldots a_n \in A$ let

- $\Theta_R(a_1, a_2, \ldots a_n)$ denote the regular congruence generated by $\{(a_i, a_j) \mid 1 \leq i, j \leq n\}$
- The congruence $\Theta_R(a_1, a_2)$ is called a principal regular congruence.

Regular Congruences Generated by a Set

For $\mathbf{S} = \langle S, \cdot, \leq \rangle$ a po-semigroup and $a_1, a_2, \ldots a_n \in A$ let

- $\Theta_R(a_1, a_2, \ldots a_n)$ denote the regular congruence generated by $\{(a_i, a_j) \mid 1 \leq i, j \leq n\}$
- The congruence $\Theta_R(a_1, a_2)$ is called a principal regular congruence.
- For an arbitrary set $X \subseteq S$, let $\Theta_R(X)$ denote the regular congruence generated by $X \times X$.

Regular Congruences Generated by a Set

Lemma (Tasic)

Let $\mathbf{S} = \langle S, \cdot, \leq \rangle$ *be a po-semigroup, and suppose that* $a_1, b_1, \ldots a_n, b_n \in S$ *and* $\theta \in \mathbf{RCon}\,\mathbf{S}$*. Then*

Regular Congruences Generated by a Set

Lemma (Tasic)

Let $\mathbf{S} = \langle S, \cdot, \leq \rangle$ *be a po-semigroup, and suppose that* $a_1, b_1, \ldots a_n, b_n \in S$ *and* $\theta \in \mathbf{RCon}\,\mathbf{S}$. *Then*

- *a)* $\Theta_R(a_1, b_1) = \Theta_R(b_1, a_1)$

Regular Congruences Generated by a Set

Lemma (Tasic)

Let $\mathbf{S} = \langle S, \cdot, \leq \rangle$ *be a po-semigroup, and suppose that* $a_1, b_1, \ldots a_n, b_n \in S$ *and* $\theta \in \mathbf{RCon}\,\mathbf{S}$. *Then*

- *a)* $\Theta_R(a_1, b_1) = \Theta_R(b_1, a_1)$
- *b)* $\Theta_R((a_1, b_1), \ldots (a_n, b_n)) = \Theta_R(a_1, b_1) \vee \cdots \vee \Theta_R(a_n, b_n)$

Regular Congruences Generated by a Set

Lemma (Tasic)

Let $\mathbf{S} = \langle S, \cdot, \leq \rangle$ *be a po-semigroup, and suppose that* $a_1, b_1, \ldots a_n, b_n \in S$ *and* $\theta \in \mathbf{RCon}\,\mathbf{S}$*. Then*

- *a)* $\Theta_R(a_1, b_1) = \Theta_R(b_1, a_1)$
- *b)* $\Theta_R((a_1, b_1), \ldots (a_n, b_n)) = \Theta_R(a_1, b_1) \vee \cdots \vee \Theta_R(a_n, b_n)$
- *c)* $\theta = \bigcup\{\Theta_R(a, b) \mid (a, b) \in \theta\} = \bigvee\{\Theta_R(a, b) \mid (a, b) \in \theta\}$.

Lattice of Regular Congruences

Given a nonempty set of regular congruences θ_i for $i \in I$ we have

- $\bigwedge_R \{\theta_i \mid i \in I\} = \bigcap_{i \in I} \theta_i$

Lattice of Regular Congruences

Given a nonempty set of regular congruences θ_i for $i \in I$ we have

- $\bigwedge_R \{\theta_i \mid i \in I\} = \bigcap_{i \in I} \theta_i$
- What is the join $\bigvee_R$ in $\mathbf{RCon\,S}$?

Lattice of Regular Congruences

Given a nonempty set of regular congruences θ_i for $i \in I$ we have

- $\bigwedge_R \{\theta_i \mid i \in I\} = \bigcap_{i \in I} \theta_i$
- What is the join $\bigvee_R$ in $\mathbf{RCon\,S}$?
- In general, the join of two regular congruences is not a regular congruence

Lattice of Regular Congruences

- Given $\theta \in Con\,S$, we define the regular closure of θ, denoted by θ^c, to be the smallest regular congruence containing θ.

Lattice of Regular Congruences

- Given $\theta \in Con\,S$, we define the regular closure of θ, denoted by θ^c, to be the smallest regular congruence containing θ.
- We have $\theta^c = \bigcap\{\sigma \mid \sigma \in RConS \text{ and } \theta \subseteq \sigma\}$.

Lattice of Regular Congruences

Given a nonempty set of regular congruences θ_i for $i \in I$ we have

- $\bigvee_R \{\theta_i \mid i \in I\} = \left(\bigvee_{i \in I} \theta_i\right)^c$

Lattice of Regular Congruences

Theorem

Let $\mathbf{S} = \langle S, \cdot, \leq \rangle$ *be a po-semigroup and let* $\theta \in \mathbf{Con}\,\mathbf{S}$*. The regular closure* θ^c *of* θ *is given by*

$$x\,\theta^c\,y \;\leftrightarrow\; \text{there is a closed bracelet modulo } \theta \text{ containing } x, y.$$

Lattice of Regular Congruences

Theorem

Let $\mathbf{S} = \langle S, \cdot, \leq \rangle$ *be a po-semigroup and let* $\theta_i \in \mathbf{Con}\,\mathbf{S}$ *for* $i \in I$. *The join* $\bigvee_R$ *in* $\mathbf{RCon}\,\mathbf{S}$ *is given by* $x \bigvee_R \{\theta_i \mid i \in I\}\ y \leftrightarrow$ *there is a closed bracelet modulo* $\bigvee_{i \in I} \theta_i$ *containing* x, y.

Example

- Let $\mathbf{S} = \langle \{1, a, b, c\} \cdot, \leq \rangle$ be a po-semigroup defined by the following multiplication table and the ordering.

Example

- Let $\mathbf{S} = \langle \{1, a, b, c\} \cdot, \leq \rangle$ be a po-semigroup defined by the following multiplication table and the ordering.
-

·	1	a	b	c
1	1	a	b	c
a	a	a	c	c
b	b	c	b	c
c	c	c	c	c

Example

- The po-semigroup $\mathbf{S}$ has seven congruences determined by the following partitions:

Example

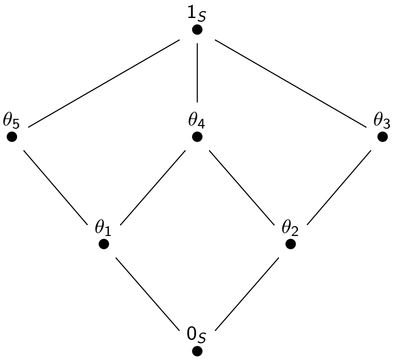
- The po-semigroup **S** has seven congruences determined by the following partitions:
-
  - $0_S = \{\{1\}, \{a\}, \{b\}, \{c\}\}$
  - $\theta_1 = \{\{1\}, \{a\}, \{b, c\}\}$
  - $\theta_2 = \{\{1\}, \{a, c\}, \{b\}\}$
  - $\theta_3 = \{\{1, b\}, \{a, c\}\}$
  - $\theta_4 = \{\{1\}, \{a, b, c\}\}$
  - $\theta_5 = \{\{1, a\}, \{b, c\}\}$
  - $1_S = \{\{1, a, b, c\}\}$

Example

- The lattice of congruences of **S** is

Example

- The lattice of congruences of **S** is
-



Example

- We have

$$c\,\theta_i\,a \;\leq\; b\,\theta_i\,b \;\leq\; c\,\theta_i\,c \quad \text{but} \quad \{a,\ b,\ c\}^2 \not\subseteq \theta_i \quad \text{for} \quad i = 2, 3$$

Example

- We have

$$c\,\theta_i\,a \;\le\; b\,\theta_i\,b \;\le\; c\,\theta_i\,c \quad \text{but} \quad \{a,\ b,\ c\}^2 \not\subseteq \theta_i \quad \text{for} \quad i = 2, 3$$

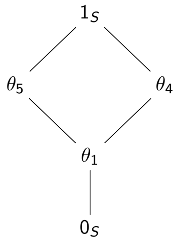
- So, θ_2 and θ_3 are not regular congruences.

Example

- We have five regular congruences of **S** and the corresponding lattice of regular congruences is

Example

- We have five regular congruences of **S** and the corresponding lattice of regular congruences is
-
    - $0_S = \{\{1\}, \{a\}, \{b\}, \{c\}\}$
    - $\theta_1 = \{\{1\}, \{a\}, \{b, c\}\}$
    - $\theta_4 = \{\{1\}, \{a, b, c\}\}$
    - $\theta_5 = \{\{1, a\}, \{b, c\}\}$
    - $1_S = \{\{1, a, b, c\}\}$



Lattice of Regular Congruences

Lemma (Tasic)

If $\mathbf{S} = \langle S, \cdot, \leq \rangle$ *is a po-semigroup, then for any* $a, b \in S$ *we have*

$$\Theta_R(a, b) = (\Theta(a, b))^c$$

Translations of a semigroup

Given a po-semigroup $\mathbf{S} = \langle S, \cdot, \leq \rangle$

- 0-translation of $\mathbf{S}$ is a function $g : S \to S$ such that $g(x) = x$ or $g(x) = a$ for every $x \in S$ and a fixed element $a \in S$.

Translations of a semigroup

Given a po-semigroup $\mathbf{S} = \langle S, \cdot, \leq \rangle$

- 0-translation of $\mathbf{S}$ is a function $g : S \to S$ such that $g(x) = x$ or $g(x) = a$ for every $x \in S$ and a fixed element $a \in S$.
- 1-translation of $\mathbf{S}$ is a 0-translation or a function $g : S \to S$ such that $g(x) = a \cdot x$ or $g(x) = x \cdot a$ for every $x \in S$ and a fixed element $a \in S$.

Translations of a semigroup

Given a po-semigroup $\mathbf{S} = \langle S, \cdot, \leq \rangle$

- 0-translation of $\mathbf{S}$ is a function $g : S \to S$ such that $g(x) = x$ or $g(x) = a$ for every $x \in S$ and a fixed element $a \in S$.
- 1-translation of $\mathbf{S}$ is a 0-translation or a function $g : S \to S$ such that $g(x) = a \cdot x$ or $g(x) = x \cdot a$ for every $x \in S$ and a fixed element $a \in S$.
- A composition of k 1-translations of $\mathbf{S}$ is called a k-translation of $\mathbf{S}$.

Translations of a semigroup

Given a po-semigroup $\mathbf{S} = \langle S, \cdot, \leq \rangle$

- 0-translation of **S** is a function $g : S \to S$ such that $g(x) = x$ or $g(x) = a$ for every $x \in S$ and a fixed element $a \in S$.
- 1-translation of **S** is a 0-translation or a function $g : S \to S$ such that $g(x) = a \cdot x$ or $g(x) = x \cdot a$ for every $x \in S$ and a fixed element $a \in S$.
- A composition of k 1-translations of **S** is called a k-translation of **S**.
- A function $g : S \to S$ is called a translation of **S** if g is a k-translation of **S** for some k.

Congruence Generation Theorem

Theorem

If $\mathbf{S} = \langle S, \cdot, \leq \rangle$ *is a po-semigroup, then for any* $a, b \in S$ *we have* $(c, d) \in \Theta(a, b)$ *iff there exist translations* $g_0, g_1, \ldots, g_n$ *of* $\mathbf{S}$ *such that* $g_1(a) = g_0(b), \ldots, g_n(a) = g_{n-1}(b)$ *and* $c = g_0(a)$, $d = g_n(b)$ *or* $d = g_0(a)$, $c = g_n(b)$.

Open bracelet modulo $\theta = \Theta(a, b)$

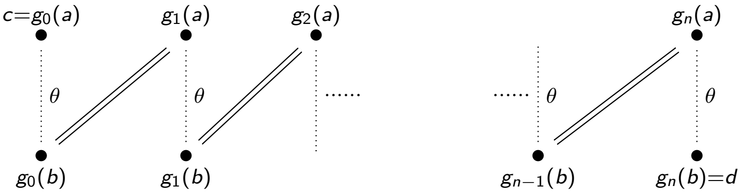


Figure : Open bracelet modulo $\theta = \Theta(a, b)$

Regular Congruence Generation Theorem

Theorem (Tasic)

If $\mathbf{S} = \langle S, \cdot, \leq \rangle$ *is a po-semigroup, then for any* $a, b, c, d \in S$ *we have* $(c, d) \in \Theta_R(a, b)$ *iff there exist* $n \in \mathbf{N}$, $e_1, e_2, \ldots, e_{2n} \in S$ *and translations* g_j^{2i-1} *for* $i = 1, 2, \ldots, n$ *and* $j = 0, 1, \ldots, m_i$ *such that*

- $c, d \in \{e_1, e_2, \ldots, e_{2n}\}$

Regular Congruence Generation Theorem

Theorem (Tasic)

If $\mathbf{S} = \langle S, \cdot, \leq \rangle$ *is a po-semigroup, then for any* $a, b, c, d \in S$ *we have* $(c, d) \in \Theta_R(a, b)$ *iff there exist* $n \in \mathbf{N}$, $e_1, e_2, \ldots, e_{2n} \in S$ *and translations* g_j^{2i-1} *for* $i = 1, 2, \ldots, n$ *and* $j = 0, 1, \ldots, m_i$ *such that*

- $c, d \in \{e_1, e_2, \ldots, e_{2n}\}$
- *For every* $i = 1, 2, \ldots, n$ *and* $j = 0, 1, \ldots, m_i$

$$\begin{aligned} e_{2i-1} &= g_0^{2i-1}(a) \\ g_j^{2i-1}(b) &= g_{j-1}^{2i-1}(a) \\ g_{m_i}^{2i-1}(b) &= e_{2i} \end{aligned} \qquad \text{or} \qquad \begin{aligned} e_{2i} &= g_0^{2i-1}(a) \\ g_j^{2i-1}(b) &= g_{j-1}^{2i-1}(a) \\ g_{m_i}^{2i-1}(b) &= e_{2i-1} \end{aligned}$$

Regular Congruence Generation Theorem

Theorem (Tasic)

If $\mathbf{S} = \langle S, \cdot, \leq \rangle$ *is a po-semigroup, then for any* $a, b, c, d \in S$ *we have* $(c, d) \in \Theta_R(a, b)$ *iff there exist* $n \in \mathbf{N}$, $e_1, e_2, \ldots, e_{2n} \in S$ *and translations* g_j^{2i-1} *for* $i = 1, 2, \ldots, n$ *and* $j = 0, 1, \ldots, m_i$ *such that*

- $c, d \in \{e_1, e_2, \ldots, e_{2n}\}$
- *For every* $i = 1, 2, \ldots, n$ *and* $j = 0, 1, \ldots, m_i$

$$\begin{aligned} e_{2i-1} &= g_0^{2i-1}(a) \\ g_j^{2i-1}(b) &= g_{j-1}^{2i-1}(a) \\ g_{m_i}^{2i-1}(b) &= e_{2i} \end{aligned} \qquad \text{or} \qquad \begin{aligned} e_{2i} &= g_0^{2i-1}(a) \\ g_j^{2i-1}(b) &= g_{j-1}^{2i-1}(a) \\ g_{m_i}^{2i-1}(b) &= e_{2i-1} \end{aligned}$$

- *For every* $i = 1, 2, \ldots, n-1$, $e_{2i} \leq e_{2i+1}$ *and* $e_{2n} \leq e_1$.