Trakhtenbrot theorem and first-order axiomatic extensions of MTL

Matteo Bianchi

Università degli Studi di Milano
Department of Computer Science

matteo.bianchi@unimi.it

joint work with

Franco Montagna

Outline

- In 1950, B.A. Trakhtenbrot showed that the set of first-order tautologies associated to finite models is not recursively enumerable (Π_1^0-complete).

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MTL and MTL-algebras

Syntax

- Monoidal t-norm based logic (MTL) was introduced in [EG01]: it is based over connectives $\&, \wedge, \rightarrow, \perp$ (the first three are binary, whilst the last one is 0-ary).
- The notion of formula is defined inductively starting from the fact that all propositional variables and $\perp$ are formulas.
- MTL can be axiomatized with a Hilbert style calculus, having MP as an inference rule.
- An axiomatic extension of MTL is a logic obtained by adding one or more axiom schemata to it.

Semantics

- An MTL algebra is a ▸ prelinear residuated lattice $\langle A, *, \Rightarrow, \sqcap, \sqcup, 0, 1\rangle$: MTL-algebras forms an algebraic variety, and every axiomatic extension of MTL is algebraizable in the sense of [BP89].
- An L-algebra is an MTL-algebra satisfying all the axioms of L.
- A totally ordered MTL-algebra is called MTL-chain.
- An MTL-algebra is called *standard* when its lattice reduct is $\langle [0, 1], \min, \max, 0, 1\rangle$: this happens (see [EG01, BEG99]) if and only if $*$ is a left-continuous t-norm.

First-order case

- A first-order language is a *countable* set **P** of predicate symbols, containing *at least* a binary one (i.e. we do not work with monadic fragments).
- We overlook constant, function symbols, and we work without equality.
- We have the "classical" quantifiers $\forall, \exists$.
- The notions of term (note that our terms coincide with variables), formula, closed formula, term substitutable in a formula are defined like in the classical case; the connectives are those of the propositional level.

As regards to semantics, we need to restrict to L-chains: given an L-chain $\mathcal{A}$, a finite **A**-model is a structure $\mathbf{M} = \langle M, \{r_P\}_{P \in \mathbf{P}} \rangle$, where:

- M is a *finite* non-empty set.
- for each $P \in \mathbf{P}$ of arity[1] n, $r_P : M^n \to A$.
- For each evaluation over variables $v : Var \to M$, the truth value of a formula φ ($\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v}$) is defined inductively as follows:
- $\|P(x_1, \dots, x_n)\|^{\mathcal{A}}_{\mathbf{M},v} = r_P(v(x_1), \dots, v(x_n))$.
- The truth value commutes with the connectives of L$\forall$, i.e.

$$\|\varphi \to \psi\|^{\mathcal{A}}_{\mathbf{M},v} = \|\varphi\|^{\mathcal{A}}_{\mathbf{M},v} \Rightarrow \|\psi\|^{\mathcal{A}}_{\mathbf{M},v}$$
$$\|\varphi \& \psi\|^{\mathcal{A}}_{\mathbf{M},v} = \|\varphi\|^{\mathcal{A}}_{\mathbf{M},v} * \|\psi\|^{\mathcal{A}}_{\mathbf{M},v}$$
$$\|\bot\|^{\mathcal{A}}_{\mathbf{M},v} = 0$$
$$\|\varphi \wedge \psi\|^{\mathcal{A}}_{\mathbf{M},v} = \|\varphi\|^{\mathcal{A}}_{\mathbf{M},v} \sqcap \|\psi\|^{\mathcal{A}}_{\mathbf{M},v}$$

- $\|(\forall x)\varphi\|^{\mathcal{A}}_{\mathbf{M},v} = \min\{\|\varphi\|^{\mathcal{A}}_{M,v'} : \ v' \equiv_x v$, i.e. $v'(y) = v(y)$ for all variables except for $x\}$
- $\|(\exists x)\varphi\|^{\mathcal{A}}_{\mathbf{M},v} = \max\{\|\varphi\|^{\mathcal{A}}_{M,v'} : \ v' \equiv_x v$, i.e. $v'(y) = v(y)$ for all variables except for $x\}$.

[1] If P has arity zero, then $r_P \in A$.

Theorem 1 ([Tra50, Vau60, BGG01])

Consider countable language containing only predicates, with at least a binary one, and without equality. Then the set $fTAUT^{\mathbf{2}}_{\forall}$ *is* Π^0_1*-complete.*

Trakhtenbrot theorem, previous results

Theorem 1 ([Tra50, Vau60, BGG01])

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Theorem 2 ([Háj99])

Consider countable language containing only predicates, with at least a binary one, and without equality. If $\mathcal{A} \in \{[0,1]_G, [0,1]_\Pi, [0,1]_Ł\}$ *then* $fTAUT^{\mathcal{A}}_{\forall}$ *is* Π^0_1*-complete.*

Theorem 3

(i) *For every non-trivial MTL-chain $\mathcal{A}$, $fTAUT_{\forall}^{\mathcal{A}}$ is Π_1^0-hard. More in general, for every class of non-trivial MTL-chains $\mathbb{K}$, $fTAUT_{\forall}^{\mathbb{K}}$ is Π_1^0-hard.*

(ii) *Let L be a consistent axiomatic extension of MTL. If $\mathbb{K}$ is a class of L-chains and $TAUT_L = TAUT(\mathbb{K})$, then $fTAUT_{\forall}^{\mathbb{K}}$ is Π_1^0-hard.*

(iii) *For every consistent axiomatic extension L of MTL, the set $fTAUT(L\forall)$ is Π_1^0-hard.*

Definition 4

A first-order formula is said to be *Boolean* if its connectives are among $\neg$, $\vee$ and $\wedge$. For each n-ary predicate P, let us define:

$$PREDEF(P) \stackrel{\text{def}}{=} \forall x_1 \ldots \forall x_n \neg(\neg\neg P(x_1, \ldots, x_n) \leftrightarrow \neg P(x_1, \ldots, x_n)).$$

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Lemma 5

Let $\mathcal{A}$ be any non-trivial MTL-chain, $\mathbb{K}$ be a class of non-trivial MTL-chains, and let ϕ be a Boolean first-order closed formula. The following are equivalent.

(1) $\phi \in fTAUT^{\mathbf{2}}_{\forall}$.

(2) $\neg PREDEF(\phi) \vee (\neg\phi^{\neg\neg} \to \phi^{\neg\neg}) \in fTAUT^{\mathcal{A}}_{\forall}$.

(3) $\neg PREDEF(\phi) \vee (\neg\phi^{\neg\neg} \to \phi^{\neg\neg}) \in fTAUT^{\mathbb{K}}_{\forall}$.

Theorem 6

(i) *If $\mathcal{A}$ is any non-trivial MTL-chain and $TAUT(\mathcal{A})$ is decidable, then $fTAUT_{\forall}^{\mathcal{A}}$ is in Π_1^0. More in general, if $\mathbb{K}$ is any class of non-trivial MTL-chains and $TAUT(\mathbb{K})$ is decidable, then $fTAUT_{\forall}^{\mathbb{K}}$ is in Π_1^0.*

(ii) *Let L be a consistent axiomatic extension of MTL. If L is decidable and is sound and complete with respect to a class of L-chains $\mathbb{K}$, that is, if $TAUT_L = TAUT(\mathbb{K})$, then $fTAUT_{\forall}^{\mathbb{K}}$ is in Π_1^0.*

(iii) *For every decidable axiomatic extension L of MTL, the set $fTAUT(L\forall)$ is in Π_1^0.*

A translation from first-order to propositional formulas

Let $\mathbb{N}^+ \stackrel{\text{def}}{=} \mathbb{N} \setminus \{0\}$. For every $n \in \mathbb{N}^+$, with $\mathcal{L}^n\forall$ we denote the language of MTL$\forall$ expanded with the constants $c_1, \ldots, c_n$. The set of $\mathcal{L}^n\forall$ formulas will be called FORM$_n$.

Definition 7

Let $c_n : FORM_n \to \mathbb{N}$ be a *computable* map that encodes a first-order formulas into natural numbers. Since we are working with a countable language, this can be done without any problem.

For $n \in \mathbb{N}^+$, we define by induction an interpretation $^*_n$ from the closed formulas of $\mathcal{L}^n_\forall$ into propositional formulas of MTL as follows.

- If ϕ is atomic, say, $\phi = P(a_1, \ldots, a_k)$, with $a_1, \ldots, a_k$ among $c_1, \ldots, c_n$, then $\phi^*_n = x_{c_n(P(a_1,\ldots,a_n))}$. In other terms, every closed atomic formula is mapped into a propositional variable.
- $^*_n$ commutes with all logical connectives.
- $(\forall x \phi(x))^*_n = \bigwedge_{i=1}^n (\phi(c_i))^*_n$.
- $(\exists x \phi(x))^*_n = \bigvee_{i=1}^n (\phi(c_i))^*_n$.

Axiomatization and Π^0_1-completeness

By the previous results we have that the decidability of a consistent axiomatic extension L of MTL is a sufficient condition for the Π^0_1-completess of fTAUT(L$\forall$). Is it also necessary?

By the previous results we have that the decidability of a consistent axiomatic extension L of MTL is a sufficient condition for the Π_1^0-completess of fTAUT(L∀). Is it also necessary?

Theorem 8

Let L be a recursively axiomatizable consistent propositional logic extending MTL. The following are equivalent.

(1) *L is decidable.*

(2) *fTAUT(L∀) is in* Π_1^0.

(3) *fTAUT(L∀) is* Π_1^0*-complete.*

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Lemma 9

If a logic L (thought of as a set of formulas closed under deduction and under substitution) is not in Π^0_1, then fTAUT(L∀) is not in Π^0_1.

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Lemma 9

If a logic L (thought of as a set of formulas closed under deduction and under substitution) is not in Π_1^0, then fTAUT(L∀) is not in Π_1^0.

We have a negative answer to the question if we expand the language of L with constants, and L is not recursively axiomatizable.

Axiomatic extensions of MTL with Baaz operator Δ

- The Baaz operator Δ was firstly introduced in [Baa96].
- For every axiomatic extension L of MTL, we denote with L_Δ its expansion with an operator Δ satisfying the following axioms:

(Δ1)	$\Delta(\varphi) \vee \neg\Delta(\varphi)$.
(Δ2)	$\Delta(\varphi \vee \psi) \to ((\Delta(\varphi) \vee \Delta(\psi)))$.
(Δ3)	$\Delta(\varphi) \to \varphi$.
(Δ4)	$\Delta(\varphi) \to \Delta(\Delta(\varphi))$.
(Δ5)	$\Delta(\varphi \to \psi) \to (\Delta(\varphi) \to \Delta(\psi))$,

  and the following additional inference rule: $\frac{\varphi}{\Delta\varphi}$.
- An MTL_Δ-chain is an MTL-chain expanded with an operation δ such that, for every element x,

$$\delta(x) = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Last result

Theorem 10

(i) *If $\mathcal{A}$ is any non-trivial MTL_Δ-chain and $TAUT(\mathcal{A})$ is decidable, then $fTAUT_\forall^{\mathcal{A}}$ is Π_1^0-complete. More in general, if $\mathbb{K}$ is any class of non-trival MTL_Δ-chains and $TAUT(\mathbb{K})$ is decidable, then $fTAUT_\forall^{\mathbb{K}}$ is Π_1^0-complete.*

(ii) *Let L be a consistent axiomatic extension of MTL_Δ. If L is decidable and is sound and complete with respect to a class of L-chains $\mathbb{K}$, that is, if $TAUT_L = TAUT(\mathbb{K})$, then $fTAUT_\forall^{\mathbb{K}}$ is Π_1^0-complete.*

(iii) *For every consistent and decidable axiomatic extension L of MTL_Δ, the set $fTAUT(L\forall)$ is Π_1^0-complete.*

Proof

(i) Let $\mathbb{K}$ be a class of non-trivial MTL_{Δ}-chains. For every Boolean formula ϕ, if ϕ^{Δ} denotes the formula obtained by replacing each atomic subformula γ by $\Delta(\gamma)$, then an easy check shows that $\phi \in fTAUT_{\forall}^{\mathbf{2}}$ iff $\phi^{\Delta} \in fTAUT_{\forall}^{\mathbb{K}}$. Hence $fTAUT_{\forall}^{\mathbb{K}}$ is Π_1^0-hard.
Finally, imitating the proof of Theorem 6, we can prove that $fTAUT_{\forall}^{\mathbb{K}}$ is in Π_1^0.

(ii) Immediate from (i).

(iii) Immediate from (ii).

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APPENDIX

MTL-algebras

An MTL algebra is an algebra $\langle A, *, \Rightarrow, \sqcap, \sqcup, 0, 1\rangle$. such that:

1. $\langle A, \sqcap, \sqcup, 0, 1\rangle$ is a bounded lattice with minimum 0 and maximum 1.
2. $\langle A, *, 1\rangle$ is a commutative monoid.
3. $\langle *, \Rightarrow\rangle$ forms a *residuated pair*: $z * x \leq y$ iff $z \leq x \Rightarrow y$ for all $x, y, z \in A$.
4. The following axiom holds, for all $x, y \in A$:

(Prelinearity) $$(x \Rightarrow y) \sqcup (y \Rightarrow x) = 1$$

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