Representation of Partial Traces

TACL 2015, Ischia, Italy

Marc Bagnol — *University of Ottawa*

Traces in symmetric monoidal categories

Monoidal: a category with an associative bifunctor $\otimes$ and a unit object $\mathbf{1}$.

Traces in symmetric monoidal categories

Monoidal: a category with an associative bifunctor $\otimes$ and a unit object $\mathbf{1}$.

Symmetric: moreover has natural isomorphisms $\sigma_{A,B} : A \otimes B \to B \otimes A$, such that $\sigma_{A,B} \circ \sigma_{B,A} = \mathrm{Id}_{A \otimes B}$.

Traces in symmetric monoidal categories

Monoidal: a category with an associative bifunctor $\otimes$ and a unit object $\mathbf{1}$.

Symmetric: moreover has natural isomorphisms $\sigma_{A,B} : A \otimes B \to B \otimes A$, such that $\sigma_{A,B} \circ \sigma_{B,A} = \mathrm{Id}_{A \otimes B}$.

Trace (A. Joyal, R. Street, D. Verity): operation turning $f : A \otimes U \to B \otimes U$

Traces in symmetric monoidal categories

Monoidal: a category with an associative bifunctor $\otimes$ and a unit object $\mathbf{1}$.

Symmetric: moreover has natural isomorphisms $\sigma_{A,B} : A \otimes B \to B \otimes A$, such that $\sigma_{A,B} \circ \sigma_{B,A} = \mathrm{Id}_{A \otimes B}$.

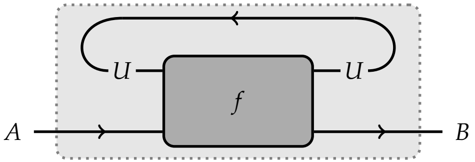
Trace (A. Joyal, R. Street, D. Verity): operation turning $f : A \otimes U \to B \otimes U$ into $\mathbf{Tr}^{U}[f] : A \to B$.

Traces in symmetric monoidal categories

Monoidal: a category with an associative bifunctor $\otimes$ and a unit object $\mathbf{1}$.

Symmetric: moreover has natural isomorphisms $\sigma_{A,B} : A \otimes B \to B \otimes A$, such that $\sigma_{A,B} \circ \sigma_{B,A} = \mathrm{Id}_{A \otimes B}$.

Trace (A. Joyal, R. Street, D. Verity): operation turning $f : A \otimes U \to B \otimes U$ into $\mathbf{Tr}^{U}[f] : A \to B$.

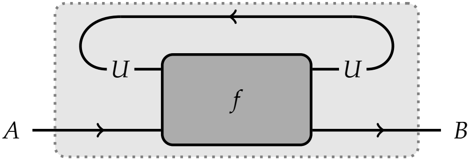


Traces in symmetric monoidal categories

Monoidal: a category with an associative bifunctor $\otimes$ and a unit object $\mathbf{1}$.

Symmetric: moreover has natural isomorphisms $\sigma_{A,B} : A \otimes B \to B \otimes A$, such that $\sigma_{A,B} \circ \sigma_{B,A} = \mathrm{Id}_{A \otimes B}$.

Trace (A. Joyal, R. Street, D. Verity): operation turning $f : A \otimes U \to B \otimes U$ into $\mathbf{Tr}^U[f] : A \to B$.



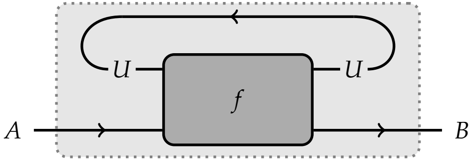
Understood as a *feedback along* U.

Traces in symmetric monoidal categories

Monoidal: a category with an associative bifunctor $\otimes$ and a unit object $\mathbf{1}$.

Symmetric: moreover has natural isomorphisms $\sigma_{A,B} : A \otimes B \to B \otimes A$, such that $\sigma_{A,B} \circ \sigma_{B,A} = \mathrm{Id}_{A \otimes B}$.

Trace (A. Joyal, R. Street, D. Verity): operation turning $f : A \otimes U \to B \otimes U$ into $\mathbf{Tr}^U[f] : A \to B$.



Understood as a *feedback along* U.

Ubiquitous structure in mathematics: linear algebra, topology, knot theory, proof theory…

Partial traces

P. Scott & E. Haghverdi: axiomatization of partially-defined trace,

Partial traces

P. Scott & E. Haghverdi: axiomatization of partially-defined trace, capturing the idea of (partially defined) categorical feedback.

Partial traces

P. Scott & E. Haghverdi: axiomatization of partially-defined trace, capturing the idea of (partially defined) categorical feedback.

One example of partial traces axiom: sliding

Partial traces

P. Scott & E. Haghverdi: axiomatization of partially-defined trace, capturing the idea of (partially defined) categorical feedback.

One example of partial traces axiom: sliding

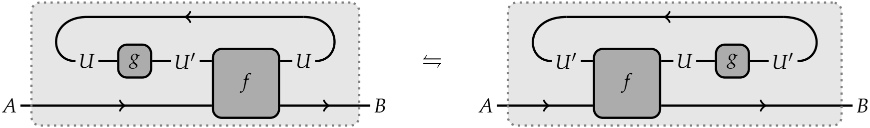
$$\mathbf{Tr}^{U}\left[f(\mathrm{Id}_A \otimes g)\right] \leftrightharpoons \mathbf{Tr}^{U'}\left[(\mathrm{Id}_B \otimes g)f\right]$$

Partial traces

P. Scott & E. Haghverdi: axiomatization of partially-defined trace, capturing the idea of (partially defined) categorical feedback.

One example of partial traces axiom: sliding

$$\mathsf{Tr}^{U}\left[f(\mathrm{Id}_A \otimes g)\right] \leftrightharpoons \mathsf{Tr}^{U'}\left[(\mathrm{Id}_B \otimes g)f\right]$$



Partial traces and sub-categories

A straightforward way to build partial traces:

Partial traces and sub-categories

A straightforward way to build partial traces:

- Consider a totally traced category $\mathcal{D}$.

Partial traces and sub-categories

A straightforward way to build partial traces:

- Consider a totally traced category $\mathcal{D}$.
- Take any sub-symmetric monoidal category $\mathcal{C} \subseteq \mathcal{D}$.

Partial traces and sub-categories

A straightforward way to build partial traces:

- Consider a totally traced category $\mathcal{D}$.
- Take any sub-symmetric monoidal category $\mathcal{C} \subseteq \mathcal{D}$.
- If $f : A \otimes U \to B \otimes U$ is in $\mathcal{C}$,

Partial traces and sub-categories

A straightforward way to build partial traces:

- Consider a totally traced category $\mathcal{D}$.
- Take any sub-symmetric monoidal category $\mathcal{C} \subseteq \mathcal{D}$.
- If $f : A \otimes U \to B \otimes U$ is in $\mathcal{C}$, it always has a trace $\mathsf{Tr}^U[f]$ in $\mathcal{D}$.

Partial traces and sub-categories

A straightforward way to build partial traces:

- Consider a totally traced category $\mathcal{D}$.
- Take any sub-symmetric monoidal category $\mathcal{C} \subseteq \mathcal{D}$.
- If $f : A \otimes U \to B \otimes U$ is in $\mathcal{C}$, it always has a trace $\mathsf{Tr}^U[f]$ in $\mathcal{D}$.
  ($\mathsf{Tr}^U[f]$ may or may not end up in $\mathcal{C}$)

Partial traces and sub-categories

A straightforward way to build partial traces:

- Consider a totally traced category $\mathcal{D}$.
- Take any sub-symmetric monoidal category $\mathcal{C} \subseteq \mathcal{D}$.
- If $f : A \otimes U \to B \otimes U$ is in $\mathcal{C}$, it always has a trace $\mathbf{Tr}^U[f]$ in $\mathcal{D}$.
  ($\mathbf{Tr}^U[f]$ may or may not end up in $\mathcal{C}$)

  Define a partial trace $\widehat{\mathbf{Tr}}$ on $\mathcal{C}$ as:

Partial traces and sub-categories

A straightforward way to build partial traces:

- Consider a totally traced category $\mathcal{D}$.
- Take any sub-symmetric monoidal category $\mathcal{C} \subseteq \mathcal{D}$.
- If $f : A \otimes U \to B \otimes U$ is in $\mathcal{C}$, it always has a trace $\mathbf{Tr}^U[f]$ in $\mathcal{D}$.
  ($\mathbf{Tr}^U[f]$ may or may not end up in $\mathcal{C}$)

  Define a partial trace $\widehat{\mathbf{Tr}}$ on $\mathcal{C}$ as:

  if $\mathbf{Tr}^U[f] \in \mathcal{C}$

Partial traces and sub-categories

A straightforward way to build partial traces:

- Consider a totally traced category $\mathcal{D}$.
- Take any sub-symmetric monoidal category $\mathcal{C} \subseteq \mathcal{D}$.
- If $f : A \otimes U \to B \otimes U$ is in $\mathcal{C}$, it always has a trace $\mathbf{Tr}^U[f]$ in $\mathcal{D}$.
  ($\mathbf{Tr}^U[f]$ may or may not end up in $\mathcal{C}$)

  Define a partial trace $\widehat{\mathbf{Tr}}$ on $\mathcal{C}$ as:

$$\text{if } \mathbf{Tr}^U[f] \in \mathcal{C} \text{ then } \widehat{\mathbf{Tr}}^U[f] = \mathbf{Tr}^U[f]$$

Partial traces and sub-categories

A straightforward way to build partial traces:

- Consider a totally traced category $\mathcal{D}$.
- Take any sub-symmetric monoidal category $\mathcal{C} \subseteq \mathcal{D}$.
- If $f : A \otimes U \to B \otimes U$ is in $\mathcal{C}$, it always has a trace $\mathbf{Tr}^U[f]$ in $\mathcal{D}$.
  ($\mathbf{Tr}^U[f]$ may or may not end up in $\mathcal{C}$)

  Define a partial trace $\widehat{\mathbf{Tr}}$ on $\mathcal{C}$ as:

$$\text{if } \mathbf{Tr}^U[f] \in \mathcal{C} \text{ then } \widehat{\mathbf{Tr}}^U[f] = \mathbf{Tr}^U[f], \text{ undefined otherwise}$$

Partial traces and sub-categories

A straightforward way to build partial traces:

- Consider a totally traced category $\mathcal{D}$.
- Take any sub-symmetric monoidal category $\mathcal{C} \subseteq \mathcal{D}$.
- If $f : A \otimes U \to B \otimes U$ is in $\mathcal{C}$, it always has a trace $\mathbf{Tr}^U[f]$ in $\mathcal{D}$.
  ($\mathbf{Tr}^U[f]$ may or may not end up in $\mathcal{C}$)

  Define a partial trace $\widehat{\mathbf{Tr}}$ on $\mathcal{C}$ as:

$$\text{if } \mathbf{Tr}^U[f] \in \mathcal{C} \text{ then } \widehat{\mathbf{Tr}}^U[f] = \mathbf{Tr}^U[f]\text{, undefined otherwise}$$

Does any partial trace arise this way?

Partial traces and sub-categories

A straightforward way to build partial traces:

- Consider a totally traced category $\mathcal{D}$.
- Take any sub-symmetric monoidal category $\mathcal{C} \subseteq \mathcal{D}$.
- If $f : A \otimes U \to B \otimes U$ is in $\mathcal{C}$, it always has a trace $\mathsf{Tr}^U[f]$ in $\mathcal{D}$.
  ($\mathsf{Tr}^U[f]$ *may or may not end up in* $\mathcal{C}$)

  Define a partial trace $\widehat{\mathsf{Tr}}$ on $\mathcal{C}$ as:

$$\text{if } \mathsf{Tr}^U[f] \in \mathcal{C} \text{ then } \widehat{\mathsf{Tr}}^U[f] = \mathsf{Tr}^U[f], \text{ undefined otherwise}$$

Does any partial trace arise this way?

O. Malherbe, P. Scott, P. Selinger: representation theorem.

The representation theorem

More precisely: any partially traced category embeds in a totally traced one.

The representation theorem

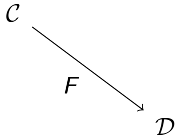
More precisely: any partially traced category embeds in a totally traced one. We also have a universal property:

$$\mathcal{C} \xrightarrow{E_{\mathcal{C}}} \mathbf{T}(\mathcal{C})$$

(where $\mathcal{C}$ is partially traced, $\mathbf{T}(\mathcal{C})$ is the totally traced category in which it embeds, $\mathcal{D}$ is any other totally traced category, with F a traced functor from $\mathcal{C}$ to $\mathcal{D}$)

The representation theorem

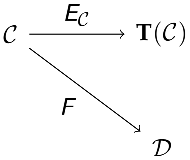
More precisely: any partially traced category embeds in a totally traced one. We also have a universal property:



(where $\mathcal{C}$ is partially traced, $\mathbf{T}(\mathcal{C})$ is the totally traced category in which it embeds, $\mathcal{D}$ is any other totally traced category, with F a traced functor from $\mathcal{C}$ to $\mathcal{D}$)

The representation theorem

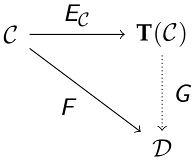
More precisely: any partially traced category embeds in a totally traced one. We also have a universal property:



(where $\mathcal{C}$ is partially traced, $\mathbf{T}(\mathcal{C})$ is the totally traced category in which it embeds, $\mathcal{D}$ is any other totally traced category, with F a traced functor from $\mathcal{C}$ to $\mathcal{D}$)

The representation theorem

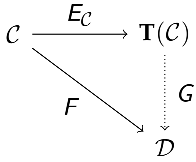
More precisely: any partially traced category embeds in a totally traced one. We also have a universal property:



(where $\mathcal{C}$ is partially traced, $\mathbf{T}(\mathcal{C})$ is the totally traced category in which it embeds, $\mathcal{D}$ is any other totally traced category, with F a traced functor from $\mathcal{C}$ to $\mathcal{D}$)

The representation theorem

More precisely: any partially traced category embeds in a totally traced one. We also have a universal property:

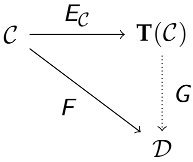


(where $\mathcal{C}$ is partially traced, $\mathbf{T}(\mathcal{C})$ is the totally traced category in which it embeds, $\mathcal{D}$ is any other totally traced category, with F a traced functor from $\mathcal{C}$ to $\mathcal{D}$)

Original proof: intermediate partial version of the $\mathbf{Int}(\cdot)$ construction and "paracategories".

The representation theorem

More precisely: any partially traced category embeds in a totally traced one. We also have a universal property:



(where $\mathcal{C}$ is partially traced, $\mathbf{T}(\mathcal{C})$ is the totally traced category in which it embeds, $\mathcal{D}$ is any other totally traced category, with F a traced functor from $\mathcal{C}$ to $\mathcal{D}$)

Original proof: intermediate partial version of the $\mathbf{Int}(\cdot)$ construction and "paracategories".

Contribution: a more direct and simplified proof.

The proof (I): the dialect construction

A generic construction $\mathbf{D}(\mathcal{C})$ on any monoidal category $\mathcal{C}$.

The proof (I): the dialect construction

A generic construction $\mathbf{D}(\mathcal{C})$ on any monoidal category $\mathcal{C}$.

Basic idea: add a "state space" to morphisms.

The proof (I): the dialect construction

A generic construction $\mathbf{D}(\mathcal{C})$ on any monoidal category $\mathcal{C}$.

Basic idea: add a "state space" to morphisms.

A morphism from A to B in $\mathbf{D}(\mathcal{C})$ is a pair (f, U) with

The proof (I): the dialect construction

A generic construction $\mathbf{D}(\mathcal{C})$ on any monoidal category $\mathcal{C}$.

Basic idea: add a "state space" to morphisms.

A morphism from A to B in $\mathbf{D}(\mathcal{C})$ is a pair (f, U) with

- U an object of $\mathcal{C}$.

The proof (I): the dialect construction

A generic construction $\mathbf{D}(\mathcal{C})$ on any monoidal category $\mathcal{C}$.

Basic idea: add a "state space" to morphisms.

A morphism from A to B in $\mathbf{D}(\mathcal{C})$ is a pair (f, U) with

- U an object of $\mathcal{C}$.
- $f : A \otimes U \to B \otimes U$ a morphism of $\mathcal{C}$.

The proof (I): the dialect construction

A generic construction $\mathbf{D}(\mathcal{C})$ on any monoidal category $\mathcal{C}$.

Basic idea: add a "state space" to morphisms.

A morphism from A to B in $\mathbf{D}(\mathcal{C})$ is a pair (f, U) with

- U an object of $\mathcal{C}$.
- $f : A \otimes U \to B \otimes U$ a morphism of $\mathcal{C}$.

When composing (f, U) and (g, V) the state spaces do not interact.

The proof (II): hiding and congruences

Hiding: given a partially traced $\mathcal{C}$ we can look at $\mathbf{D}(\mathcal{C})$ and define a *hiding* operation

The proof (II): hiding and congruences

Hiding: given a partially traced $\mathcal{C}$ we can look at $\mathbf{D}(\mathcal{C})$ and define a *hiding* operation turning $(f, V) : A \otimes U \to B \otimes U$ into

The proof (II): hiding and congruences

Hiding: given a partially traced $\mathcal{C}$ we can look at $\mathbf{D}(\mathcal{C})$ and define a *hiding* operation turning $(f, V) : A \otimes U \to B \otimes U$ into

$$\mathsf{H}^{U}[f, V] = (f, U \otimes V) : A \to B$$

The proof (II): hiding and congruences

Hiding: given a partially traced $\mathcal{C}$ we can look at $\mathbf{D}(\mathcal{C})$ and define a *hiding* operation turning $(f, V) : A \otimes U \to B \otimes U$ into

$$\mathsf{H}^{U}[f, V] = (f, U \otimes V) : A \to B$$

$\mathsf{H}[\cdot]$ behaves a lot like a (total) trace.

The proof (II): hiding and congruences

Hiding: given a partially traced $\mathcal{C}$ we can look at $\mathbf{D}(\mathcal{C})$ and define a *hiding* operation turning $(f, V) : A \otimes U \to B \otimes U$ into

$$\mathsf{H}^{U}[f, V] = (f, U \otimes V) : A \to B$$

$\mathsf{H}[\cdot]$ behaves a lot like a (total) trace.

Congruences: consider the equivalence relation on morphisms generated by some required equations, including

The proof (II): hiding and congruences

Hiding: given a partially traced $\mathcal{C}$ we can look at $\mathbf{D}(\mathcal{C})$ and define a *hiding* operation turning $(f, V) : A \otimes U \to B \otimes U$ into

$$\mathsf{H}^U[f, V] = (f, U \otimes V) : A \to B$$

$\mathsf{H}[\cdot]$ behaves a lot like a (total) trace.

Congruences: consider the equivalence relation on morphisms generated by some required equations, including

$$(f, U \otimes V) \approx (\mathsf{Tr}^V[f], U) \text{ when } \mathsf{Tr}^V[f] \text{ is defined.}$$

The proof (II): hiding and congruences

Hiding: given a partially traced $\mathcal{C}$ we can look at $\mathbf{D}(\mathcal{C})$ and define a *hiding* operation turning $(f, V) : A \otimes U \to B \otimes U$ into

$$\mathsf{H}^U[f, V] = (f, U \otimes V) : A \to B$$

$\mathsf{H}[\cdot]$ behaves a lot like a (total) trace.

Congruences: consider the equivalence relation on morphisms generated by some required equations, including

$$(f, U \otimes V) \approx (\mathsf{Tr}^V[f], U) \text{ when } \mathsf{Tr}^V[f] \text{ is defined.}$$

Then we can set $\mathbf{T}(\mathcal{C}) = \mathbf{D}(\mathcal{C})/\approx$

The proof (II): hiding and congruences

Hiding: given a partially traced $\mathcal{C}$ we can look at $\mathbf{D}(\mathcal{C})$ and define a *hiding* operation turning $(f, V) : A \otimes U \to B \otimes U$ into

$$\mathsf{H}^{U}[f, V] = (f, U \otimes V) : A \to B$$

$\mathsf{H}[\cdot]$ behaves a lot like a (total) trace.

Congruences: consider the equivalence relation on morphisms generated by some required equations, including

$$(f, U \otimes V) \approx (\mathsf{Tr}^{V}[f], U) \text{ when } \mathsf{Tr}^{V}[f] \text{ is defined.}$$

Then we can set $\mathbf{T}(\mathcal{C}) = \mathbf{D}(\mathcal{C})/\approx$ in which $\mathsf{H}[\cdot]$ induces a total trace, encompassing the original partial trace of $\mathcal{C}$.

The proof (III): a sketch

We can embed $\mathcal{C}$ in $\mathbf{T}(\mathcal{C})$ by setting $E_{\mathcal{C}}(f) = (f, \mathbf{1})$.

The proof (III): a sketch

We can embed $\mathcal{C}$ in $\mathbf{T}(\mathcal{C})$ by setting $E_{\mathcal{C}}(f) = (f, \mathbf{1})$.

Is it really an embedding?

The proof (III): a sketch

We can embed $\mathcal{C}$ in $\mathbf{T}(\mathcal{C})$ by setting $E_{\mathcal{C}}(f) = (f, \mathbf{1})$.

Is it really an embedding? We check that $(f, \mathbf{1}) \approx (g, \mathbf{1})$ implies $f = g$.

The proof (III): a sketch

We can embed $\mathcal{C}$ in $\mathbf{T}(\mathcal{C})$ by setting $E_{\mathcal{C}}(f) = (f, \mathbf{1})$.

Is it really an embedding? We check that $(f, \mathbf{1}) \approx (g, \mathbf{1})$ implies $f = g$.

Because $\approx$ is freely generated, we can do it by induction on chains of elementary equivalences.

The proof (III): a sketch

We can embed $\mathcal{C}$ in $\mathbf{T}(\mathcal{C})$ by setting $E_{\mathcal{C}}(f) = (f, \mathbf{1})$.

Is it really an embedding? We check that $(f, \mathbf{1}) \approx (g, \mathbf{1})$ implies $f = g$.

Because $\approx$ is freely generated, we can do it by induction on chains of elementary equivalences.

Universal property:

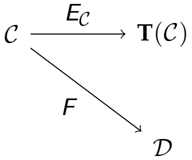
The proof (III): a sketch

We can embed $\mathcal{C}$ in $\mathbf{T}(\mathcal{C})$ by setting $E_{\mathcal{C}}(f) = (f, \mathbf{1})$.

Is it really an embedding? We check that $(f, \mathbf{1}) \approx (g, \mathbf{1})$ implies $f = g$.

Because $\approx$ is freely generated, we can do it by induction on chains of elementary equivalences.

Universal property: we can close the diagram



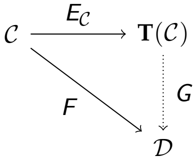
The proof (III): a sketch

We can embed $\mathcal{C}$ in $\mathbf{T}(\mathcal{C})$ by setting $E_{\mathcal{C}}(f) = (f, \mathbf{1})$.

Is it really an embedding? We check that $(f, \mathbf{1}) \approx (g, \mathbf{1})$ implies $f = g$.

Because $\approx$ is freely generated, we can do it by induction on chains of elementary equivalences.

Universal property: we can close the diagram



by setting $G(f, U) = \mathsf{Tr}^{FU}[Ff]$.

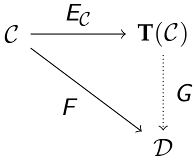
The proof (III): a sketch

We can embed $\mathcal{C}$ in $\mathbf{T}(\mathcal{C})$ by setting $E_{\mathcal{C}}(f) = (f, \mathbf{1})$.

Is it really an embedding? We check that $(f, \mathbf{1}) \approx (g, \mathbf{1})$ implies $f = g$.

Because $\approx$ is freely generated, we can do it by induction on chains of elementary equivalences.

Universal property: we can close the diagram



by setting $G(f, U) = \mathbf{Tr}^{FU}[Ff]$.

(well defined because $(f, U) \approx (g, V)$ *implies* $\mathbf{Tr}^{FU}(Ff) = \mathbf{Tr}^{FV}(Fg)$*)*

Conclusion

Easier proof of an already known result: the representation theorem for partially traced categories.

Conclusion

Easier proof of an already known result: the representation theorem for partially traced categories.

Allows intuitive diagrammatic reasoning also in the partially-defined case.

Conclusion

Easier proof of an already known result: the representation theorem for partially traced categories.

Allows intuitive diagrammatic reasoning also in the partially-defined case.

A question: how do the constructions of both proofs relate?

Conclusion

Easier proof of an already known result: the representation theorem for partially traced categories.

Allows intuitive diagrammatic reasoning also in the partially-defined case.

A question: how do the constructions of both proofs relate?

Another: can the setting be tweaked to account for partial traces of infinite-dimensional Hilbert spaces and $\mathbb{C}^*$-algebras?

Conclusion

Easier proof of an already known result: the representation theorem for partially traced categories.

Allows intuitive diagrammatic reasoning also in the partially-defined case.

A question: how do the constructions of both proofs relate?

Another: can the setting be tweaked to account for partial traces of infinite-dimensional Hilbert spaces and $\mathbb{C}^*$-algebras?

. . . Thank you for your attention