# An algebraic approach to Probabilistic Dynamic Epistemic Logic

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# Probabilistic Dynamic Epistemic Logic (PDEL)

PDEL is a family of logics for multiagent interaction.

- **Epistemic:** Describing knowledge and belief, i.e. *qualitative* (un)certainty.
- **Probabilistic:** Describing *quantitative* (un)certainty.
- **Dynamic:** How the above are affected by epistemic events (e.g. new information, belief revision)

General aim: Formalise decision procedures in uncertain competitive environments.

# Language of PDEL

The set $\mathcal{L}$ of *PDEL-formulas* $\varphi$ and the class $\mathrm{PEM}_{\mathcal{L}}$ of *probabilistic event structures* $\mathcal{E}$ over $\mathcal{L}$ are built by simultaneous recursion as follows:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \Diamond_i \varphi \mid$$

$$\Box_i \varphi \mid \langle \mathcal{E}, e \rangle \varphi \mid [\mathcal{E}, e] \varphi \mid (\sum_{k=1}^{n} \alpha_k \mu_i(\varphi)) \geq \beta.$$

# Probabilistic DEL

A *probabilistic event structure over* $\mathcal{L}$ is a tuple

$$\mathcal{E} = (E, (\sim_i)_{i \in \mathrm{Ag}}, (P_i)_{i \in \mathrm{Ag}}, \Phi, \mathrm{pre}),$$

such that

- $E$ is a non-empty finite set,
- each $\sim_i$ is an equivalence relation on $E$,
- each $P_i : E \to [0,1]$ assigns a probability distribution on each $\sim_i$-equivalence class (i.e., $\sum\{P_i(e') : e' \sim_i e\} = 1$),
- $\Phi$ is a finite set of pairwise inconsistent $\mathcal{L}$-formulas, and
- pre assigns a probability distribution $\mathrm{pre}(\bullet|\phi)$ over $E$ for every $\phi \in \Phi$.

# Models and semantics

A *probabilistic epistemic state model (PES-model)* is a structure

$$\mathbb{M} = (S, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, [\![\cdot]\!])$$

where each $P_i : S \to [0,1]$ assigns a probability distribution on each $\sim_i$-equivalence class, i.e., $\sum\{P_i(s') : s' \sim_i s\} = 1$.
Interpretation of formulas:

$$\mathbb{M}, s \Vdash \alpha\mu_i(\varphi) \leq \beta \quad \text{iff} \quad \sum_{\mathbb{M},t\Vdash\varphi} P_i(t) \leq \beta$$

$$\mathbb{M}, s \Vdash [\mathcal{E}, e]\varphi \quad \text{iff} \quad (s,e) \in \mathbb{M}^{\mathcal{E}} \implies \mathbb{M}^{\mathcal{E}}, (s,e) \Vdash \varphi$$

$$\mathbb{M}, s \Vdash \langle\mathcal{E}, e\rangle\varphi \quad \text{iff} \quad (s,e) \in \mathbb{M}^{\mathcal{E}} \text{ and } \mathbb{M}^{\mathcal{E}}, (s,e) \Vdash \varphi$$

# The update structure

$$\mathbb{M} \hookrightarrow \coprod_{\mathcal{E}} \mathbb{M} \hookleftarrow \mathbb{M}^{\mathcal{E}}.$$

The structure $\mathbb{M}^{\mathcal{E}}$, is created in two steps:
First we take the coproduct of $E$ copies of $\mathbb{M}$.
For each $e \in E$ we define:

$$Pre(e) = \bigvee_{\mathrm{pre}(e|\varphi) \neq 0} \varphi.$$

The domain of $\mathbb{M}^{\mathcal{E}}$ is a subset of the coproduct:

$$(s, e) \in \mathbb{M}^{\mathcal{E}} \quad \text{iff} \quad \mathbb{M}, s \models Pre(e).$$

# Classical PDEL, axiomatised

PDEL is axiomatized completely by the axioms and rules for the modal logic S5/K plus the following axioms, for all $i \in Ag$ and $\mathcal{E}$:

$$\langle \mathcal{E}, e \rangle p \leftrightarrow (Pre(e) \wedge p);$$

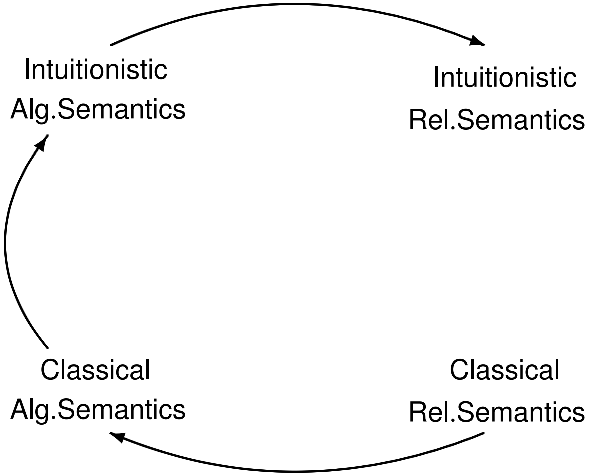
$$\langle \mathcal{E}, e \rangle \neg \phi \leftrightarrow (Pre(e) \wedge \neg \langle \mathcal{E}, e \rangle \phi);$$

$$\langle \mathcal{E}, e \rangle (\phi \vee \psi) \leftrightarrow (\langle \mathcal{E}, e \rangle \phi \vee \langle \mathcal{E}, e \rangle \psi);$$

$$\langle \mathcal{E}, e \rangle \Diamond_i \phi \leftrightarrow (Pre(e) \wedge \bigvee \{\Diamond_i \langle \mathcal{E}, e' \rangle \phi \mid e \sim_i e'\});$$

$$\langle \mathcal{E}, e \rangle \left(\sum_{k=1}^{n} \alpha_k \mu_i(\varphi_k) \geq \beta\right) \leftrightarrow$$
$$Pre(e) \wedge \Bigg( \sum_{k=1}^{n} \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} \alpha_k \cdot P_i(e') \cdot pre(e' \mid \phi) \mu_i(\phi \wedge \langle \mathcal{E}, e' \rangle \varphi_k)$$
$$+ \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} -\beta \cdot P_i(e') \cdot pre(e' \mid \phi) \mu_i(\phi) \geq 0 \Bigg)$$

# Methodology: dual characterizations



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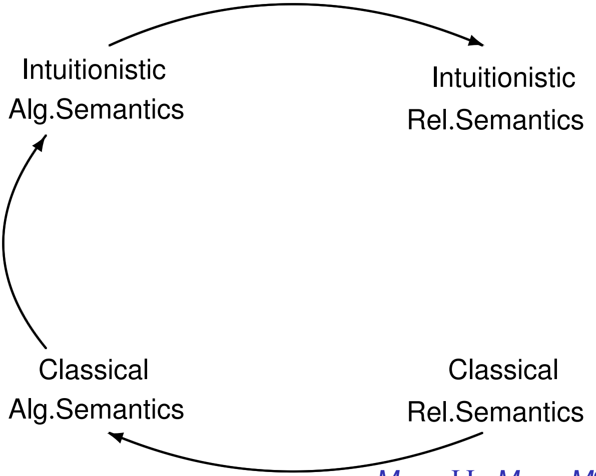
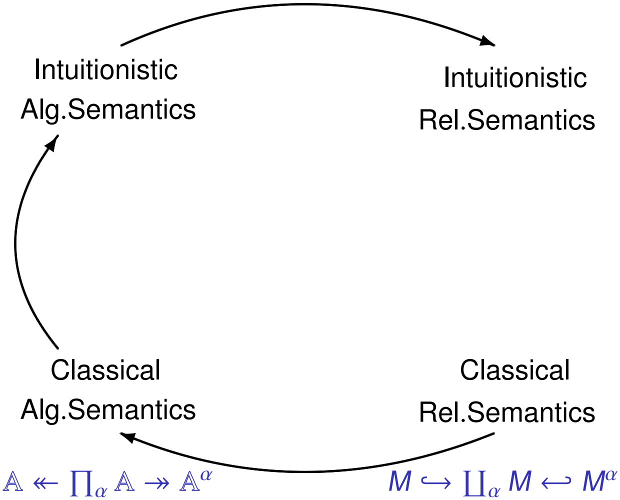



$$M \hookrightarrow \coprod_{\alpha} M \hookleftarrow M^{\alpha}$$

# Methodology: dual characterizations



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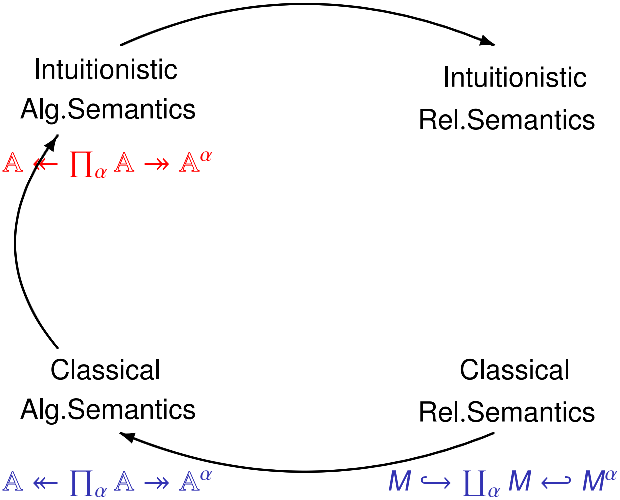

# Methodology: dual characterizations

Intuitionistic
Alg.Semantics

$\mathbb{A} \twoheadleftarrow \prod_\alpha \mathbb{A} \twoheadrightarrow \mathbb{A}^\alpha$

Intuitionistic
Rel.Semantics

$M \hookrightarrow \coprod_\alpha M \hookleftarrow M^\alpha$

Classical
Alg.Semantics

$\mathbb{A} \twoheadleftarrow \prod_\alpha \mathbb{A} \twoheadrightarrow \mathbb{A}^\alpha$

Classical
Rel.Semantics

$M \hookrightarrow \coprod_\alpha M \hookleftarrow M^\alpha$

# Interaction axioms, algebraically

$$\langle \mathcal{E}, e\rangle\left(\textstyle\sum_{k=1}^{n} \alpha_k \mu_i(\varphi_k) \geq \beta\right) \leftrightarrow$$

$$Pre(e) \wedge \Big( \textstyle\sum_{k=1}^{n} \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} \alpha_k \cdot P_i(e') \cdot pre(e' \mid \phi)\mu_i(\phi \wedge \langle \mathcal{E}, e'\rangle \varphi_k)$$

$$+ \textstyle\sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} -\beta \cdot P_i(e') \cdot pre(e' \mid \phi)\mu_i(\phi) \geq 0\Big)$$

$$[\mathcal{E}, e]\left(\textstyle\sum_{k=1}^{n} \alpha_k \mu_i(\varphi_k) \geq \beta\right) \leftrightarrow$$

$$Pre(e) \rightarrow \Big( \textstyle\sum_{k=1}^{n} \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} \alpha_k \cdot P_i(e') \cdot pre(e' \mid \phi)\mu_i(\phi \wedge [\mathcal{E}, e'] \varphi_k)$$

$$+ \textstyle\sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} -\beta \cdot P_i(e') \cdot pre(e' \mid \phi)\mu_i(\phi) \geq 0\Big)$$

# Intuitionistic probabilities

Given a Heyting algebra $\mathbb{A}$, a map $\mu : \mathbb{A} \to \mathbb{R}^+$ is a probability measure if:

- $\mu$ is order-preserving,
- $\mu(\bot) = 0$,
- $\mu(b \vee c) = \mu(b) + \mu(c) - \mu(b \wedge c)$.
- $\mu(\top) = 1$

Probability measures can be dualised with Francesco Marigo's methodology, as certain probability distributions on posets (work in progress).

# Further directions

Modelling reasoning dynamics of groups of agents. Utilities are simultaneously derived from:

- Guessing right
- Being in the majority

**Examples:**

- Stock markets
- Art collectors
- Dynamics of social acceptance (e.g. fashion)

Non-classical logics are more appropriate for these settings.

Thank you!