Unified Correspondence as a Proof-Theoretic Tool (Unified Correspondence VII)

Apostolos Tzimoulis

26 June 2015
Ischia, Italy

joint work with:
Giuseppe Greco, Minghui Ma,
Alessandra Palmigiano, Zhiguang Zhao
http://www.appliedlogictudelft.nl

Motivation

Main question: which axioms give rise to analytic rules?

Motivation

Main question: which axioms give rise to analytic rules?

Correspondence theory can help in answering this question!

Motivation

Main question: which axioms give rise to analytic rules?

Correspondence theory can help in answering this question!

- Formal connections between **correspondence theory** and **display calculi**.
- **Primitive formulas** [Kracht '96] for classical modal logic **K** generalised to **primitive inequalities** for general **DLE-logics**.

Display Calculi

Natural generalization of sequent calculi.
Sequents $X \vdash Y$, where X, Y are structures:
$A, \quad A ; B, \quad \ldots \quad X > Y, \quad \ldots$

structural symbols assemble and disassemble structures
operational symbols assemble formulas.

Main feature: display property

$$\frac{\frac{\frac{Y \vdash X > Z}{X ; Y \vdash Z}}{Y ; X \vdash Z}}{X \vdash Y > Z}$$

display property: **adjunction** at the structural level.
Canonical proof of cut elimination

Canonical Cut elimination

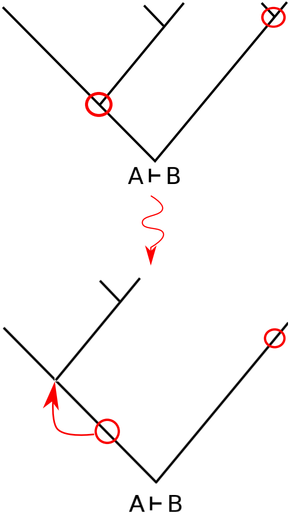
Complexity of the cut formula

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \pi_1\\ Z \vdash \circ A\end{matrix}}{Z \vdash \Box A} \qquad \dfrac{\begin{matrix}\vdots\ \pi_2\\ A \vdash Y\end{matrix}}{\Box A \vdash \circ Y}}{Z \vdash \circ Y}\ Cut$$

$$\Downarrow$$

$$\text{Display}\ \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\ \pi_1\\ Z \vdash \circ A\end{matrix}}{\bullet Z \vdash A}\ \text{Display} \qquad \begin{matrix}\vdots\ \pi_2\\ A \vdash Y\end{matrix}}{\bullet Z \vdash Y}\ Cut}{Z \vdash \circ Y}$$

Height of the cut



Proper Display Calculi

Theorem (Canonical cut elimination)

If a calculus satisfies the properties below, then it enjoys cut elimination.

- C1: structures can disappear, formulas are **forever**;
- **tree-traceable** formula-occurrences, via suitably defined congruence:
    - C2: same shape, C3: non-proliferation, C4: same position;
- C5: **principal = displayed**;
- C6, C7: rules are closed under **uniform substitution** of congruent parameters;
- C8: **reduction strategy** exists when cut formulas are both principal.

DLE-languages and expansions

$$\varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid f(\overline{\varphi}) \mid g(\overline{\varphi})$$

where $p \in \mathsf{PROP}$, $f \in \mathcal{F}$, $g \in \mathcal{G}$.

Str.	I		;		>		H		K	
Op.	$\top$	$\bot$	$\wedge$	$\vee$	$(\succ\!\!-)$	$(\rightarrow)$	f			g

Str.	H_i		K_h	
Op.		$(f_i^\sharp)$	$(g_h^\flat)$	

  for $\varepsilon_f(i) = \varepsilon_g(h) = 1$

Str.	H_i		K_h	
Op.	$(f_i^\sharp)$			$(g_h^\flat)$

  for $\varepsilon_f(i) = \varepsilon_g(h) = \partial$

Introduction rules for $f \in \mathcal{F}$ and $g \in \mathcal{G}$

$$
f_L \; \frac{H(A_1, \ldots, A_{n_f}) \vdash X}{f(A_1, \ldots, A_{n_f}) \vdash X} \qquad \frac{X \vdash K(A_1, \ldots, A_{n_g})}{X \vdash g(A_1, \ldots, A_{n_g})} \; g_R
$$

$$
f_R \; \frac{\left(X_i \vdash A_i \quad A_j \vdash X_j \quad \mid \quad \varepsilon_f(i) = 1 \quad \varepsilon_f(j) = \partial \right)}{H(X_1, \ldots, X_{n_f}) \vdash f(A_1, \ldots, A_{n_f})}
$$

$$
g_L \; \frac{\left(A_i \vdash X_i \quad X_j \vdash A_j \quad \mid \quad \varepsilon_g(i) = 1 \quad \varepsilon_g(j) = \partial \right)}{g(A_1, \ldots, A_{n_g}) \vdash K(X_1, \ldots, X_{n_g})}
$$

Display postulates for $f \in \mathcal{F}$ and $g \in \mathcal{G}$

- If $\varepsilon_f(i) = \varepsilon_g(h) = 1$

$$\frac{H(X_1, \ldots, X_i, \ldots, X_{n_f}) \vdash Y}{X_i \vdash H_i(X_1, \ldots, Y, \ldots, X_{n_f})} \qquad \frac{Y \vdash K(X_1 \ldots, X_h, \ldots X_{n_g})}{K_h(X_1, \ldots, Y, \ldots, X_{n_g}) \vdash X_h}$$

- If $\varepsilon_f(i) = \varepsilon_g(h) = \partial$

$$\frac{H(X_1, \ldots, X_i, \ldots, X_{n_f}) \vdash Y}{H_i(X_1, \ldots, Y, \ldots, X_{n_f}) \vdash X_i} \qquad \frac{Y \vdash K(X_1, \ldots, X_h, \ldots, X_{n_g})}{X_h \vdash K_h(X_1, \ldots, Y, \ldots, X_{n_g})}$$

Unified correspondence

Hybrid logics
[CR15]

DLE-logics
[CP12, CPS]

Substructural logics
[CP15]

Mu-calculi
[CFPS15, CGP14, CC15]

Display calculi
[GMPTZ]

Regular DLE-logics
Kripke frames with
impossible worlds
[PSZ15a]

Jónsson-style vs
Sambin-style canonicity
[PSZ15b]

Finite lattices and
monotone ML
[FPS15]

Canonicity via
pseudo-correspondence
[CPSZ]

Algorithmic correspondence for DLE

Ackermann Lemma Based Algorithm

- engined by the Ackermann lemma.
- Reduction rules leading to the Ackermann elimination step.
- Residuation and approximation rules.
- Soundness on **perfect DLEs**:
  - approximation: both $\bigvee$-generated by the c. $\vee$-primes and $\bigwedge$-generated by the c. $\wedge$-primes;
  - residuation: all the operations are either right or left adjoints or residuals.

Perfect DLEs: the natural semantic environment both for ALBA and for display calculi for DLE.

Primitive inequalities

Primitive formulas: [Kracht 1996]

Left-primitive	$\varphi := p \mid \top \mid \vee \mid \wedge \mid f(\vec{\varphi}/\vec{p}, \vec{\psi}/\vec{q})$
Right-primitive	$\psi := p \mid \bot \mid \wedge \mid \vee \mid g(\vec{\psi}/\vec{p}, \vec{\varphi}/\vec{q})$

Primitive inequalities:

Left-primitive	$\varphi_1 \leq \varphi_2$ with φ_1 scattered
Right-primitive	$\psi_1 \leq \psi_2$ with ψ_2 scattered

Example:

$$\Diamond q \to \Box p \leq \Box(q \to p) \quad \rightsquigarrow \quad \frac{x \vdash \Diamond q \to \Box p}{x \vdash \Box(q \to p)} \quad \rightsquigarrow \quad \frac{X \vdash \circ Z > \circ Y}{X \vdash \circ(Z > Y)}.$$

Strategy

Crucial observation: **same** structural connectives for the **basic** and for the **expanded** DLE.

Main strategy: transform **non-primitive** DLE inequalities into (conjunctions of) **primitive** DLE inequalities in the **expanded** language:

$$s(\vec{p},\vec{q}) \leq s'(\vec{p},\vec{q}) \qquad \& \left\{\varphi_i^*(\vec{p},\vec{q}) \leq \varphi'^*_i(\vec{p},\vec{q}) \mid i \in I\right\}$$

$\Updownarrow$ ALBA $\qquad$ $\Updownarrow$ ALBA on primitives

$$\& \left\{\varphi_i^*(\vec{\mathbf{i}},\vec{\mathbf{m}}) \leq \varphi_i'^*(\vec{\mathbf{i}},\vec{\mathbf{m}}) \mid i \in I\right\} \quad = \quad \& \left\{\varphi_i^*(\vec{\mathbf{i}},\vec{\mathbf{m}}) \leq \varphi_i'^*(\vec{\mathbf{i}},\vec{\mathbf{m}}) \mid i \in I\right\}$$

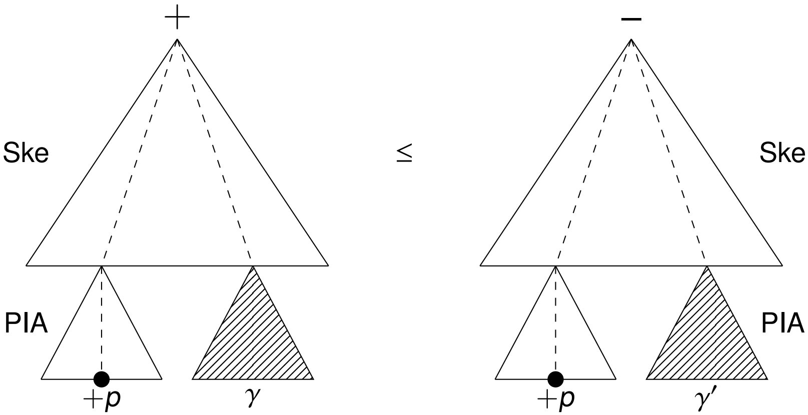
Inductive but not analytic

$$\forall[\Diamond p \leq \Diamond\Box p]$$

Inductive but not analytic

$$
\begin{array}{ll}
 & \forall[\Diamond p \leq \Diamond\Box p] \\
\text{iff} & \forall[(\mathbf{i} \leq \Diamond p \;\&\; \Diamond\Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{i} \leq \Diamond \mathbf{j} \;\&\; \mathbf{j} \leq p \;\&\; \Diamond\Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{i} \leq \Diamond \mathbf{j} \;\&\; \Diamond\Box \mathbf{j} \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[\mathbf{i} \leq \Diamond \mathbf{j} \Rightarrow \forall \mathbf{m}[\Diamond\Box \mathbf{j} \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}]] \\
\text{iff} & \forall[\mathbf{i} \leq \Diamond \mathbf{j} \Rightarrow \mathbf{i} \leq \Diamond\Box \mathbf{j}] \\
\text{iff} & \forall[\Diamond \mathbf{j} \leq \Diamond\Box \mathbf{j}]
\end{array}
$$

Analytic inductive inequalities



Example 1: The Church-Rosser inequality

Let $\mathcal{F} = \{\Diamond\}$ and $\mathcal{G} = \{\Box\}$.

$$\forall[\Diamond\Box p \leq \Box\Diamond p]$$

Example 1: The Church-Rosser inequality

Let $\mathcal{F} = \{\Diamond\}$ and $\mathcal{G} = \{\Box\}$.

$$
\begin{array}{ll}
 & \forall[\Diamond\Box p \leq \Box\Diamond p] \\
\text{iff} & \forall[\blacklozenge\Diamond\Box p \leq \Diamond p] \\
\text{iff} & \forall[\mathbf{i} \leq \blacklozenge\Diamond\Box p \ \&\ \Diamond p \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[\mathbf{i} \leq \blacklozenge\Diamond\mathbf{j} \ \&\ \mathbf{j} \leq \Box p \ \&\ \Diamond p \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[\mathbf{i} \leq \blacklozenge\Diamond\mathbf{j} \ \&\ \blacklozenge\mathbf{j} \leq p \ \&\ \Diamond p \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[\mathbf{i} \leq \blacklozenge\Diamond\mathbf{j} \ \&\ \Diamond\blacklozenge\mathbf{j} \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[\blacklozenge\Diamond\mathbf{j} \leq \Diamond\blacklozenge\mathbf{j}] \\
\hline
\text{iff} & \forall[\blacklozenge\Diamond p \leq \Diamond\blacklozenge p] \text{ (ALBA for primitive)}
\end{array}
$$

$$
\cdots \quad \rightsquigarrow \quad \frac{\Diamond\blacklozenge p \vdash z}{\blacklozenge\Diamond p \vdash z} \quad \rightsquigarrow \quad \frac{\circ \bullet X \vdash Z}{\bullet \circ X \vdash Z}
$$

Example 2: The Prelinearity Axiom

Let $\mathcal{F} = \varnothing$, $\mathcal{G} = \{\rightharpoonup\}$ where $\rightharpoonup$ is binary and of order-type $(\partial, 1)$.

$$\forall[\top \leq (p \rightharpoonup q) \vee (q \rightharpoonup p)]$$

Example 2: The Prelinearity Axiom

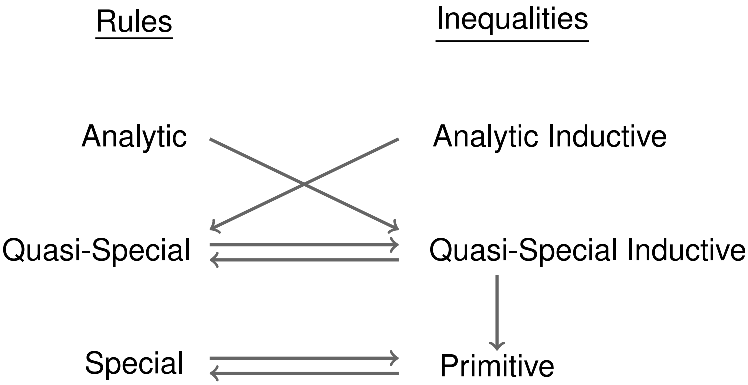
Let $\mathcal{F} = \varnothing$, $\mathcal{G} = \{\rightharpoonup\}$ where $\rightharpoonup$ is binary and of order-type $(\partial, 1)$.

$$\begin{array}{ll} & \forall[\top \leq (p \rightharpoonup q) \vee (q \rightharpoonup p)] \\ \text{iff} & \forall[(r_1 \leq p \;\&\; q \leq r_2 \;\&\; r_3 \leq q \;\&\; p \leq r_4) \Rightarrow \top \leq (r_1 \rightharpoonup r_2) \vee (r_3 \rightharpoonup r_4)] \\ \text{iff} & \forall[(r_1 \leq r_4 \;\&\; q \leq r_2 \;\&\; r_3 \leq q) \Rightarrow \top \leq (r_1 \rightharpoonup r_2) \vee (r_3 \rightharpoonup r_4)] \\ \text{iff} & \forall[(r_1 \leq r_4 \;\&\; r_3 \leq r_2) \Rightarrow \top \leq (r_1 \rightharpoonup r_2) \vee (r_3 \rightharpoonup r_4)] \end{array}$$

The last quasi-inequality above expresses the validity of the following quasi-special structural rule on perfect DLEs:

$$\frac{X \vdash W \qquad Z \vdash Y}{I \vdash (X \succ Y) \,;\, (Z \succ W)}$$

Overview of main results



[Conradie Craig] Canonicity results for mu-calculi: an algorithmic approach, *JLC*, to appear, 2015.
[Conradie Fomatati Palmigiano Sourabh] Correspondence theory for intuitionistic modal mu-calculus, *TCS*, 564:30-62 (2015).
[Conradie Ghilardi Palmigiano] Unified Correspondence, in *Johan van Benthem on Logic and Information Dynamics*, Springer, 2014.
[Conradie Palmigiano 2012] Algorithmic Correspondence and Canonicity for Distributive Modal Logic, *APAL*, 163:338-376.
[Conradie Palmigiano 2015] Algorithmic correspondence and canonicity for non-distributive logics, *JLC*, to appear.
[Conradie Palmigiano Sourabh] Algebraic modal correspondence: Sahlqvist and beyond, submitted, 2014.
[Conradie Palmigiano Sourabh Zhao] Canonicity and relativized canonicity via pseudo-correspondence, submitted, 2014.
[Conradie Robinson 2015] On Sahlqvist Theory for Hybrid Logics, *JLC*, to appear.
[Frittella Palmigiano Santocanale] Dual characterizations for finite lattices via correspondence theory for monotone modal logic, *JLC*, to appear.
[Greco Ma Palmigiano Tzimoulis Zhao] Unified correspondence as a proof-theoretic tool, submitted, 2015.
[Palmigiano Sourabh Zhao/a] Sahlqvist theory for impossible worlds, *JLC*, 2015.
[Palmigiano Sourabh Zhao/b] Jónsson-style canonicity for ALBA inequalities, *JLC*, 2015.