Leibniz and Suszko filters for non-protoalgebraic logics

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Outline

Preliminaries

Leibniz and Suszko congruences

- Given $F \subseteq A$, the *Leibniz congruence of F*, which shall be denoted by $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, is the largest congruence of $\boldsymbol{A}$ compatible with F.

- Given $F \subseteq A$, the *Suszko congruence of F*, which shall be denoted by $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$, is the largest congruence of $\boldsymbol{A}$ compatible with every $G \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F}$, or, equivalently,

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) := \bigcap \{ \boldsymbol{\Omega}^{\boldsymbol{A}}(G) : G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}, F \subseteq G \} .$$

Leibniz and Suszko operators

- The *Leibniz operator on $\boldsymbol{A}$* is the map $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Co}\boldsymbol{A}$ defined by $F \mapsto \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$.

- The *Suszko operator on $\boldsymbol{A}$* is the map $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} : \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Co}\boldsymbol{A}$ defined by $F \mapsto \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$.

The classes of algebras $\mathrm{Alg}^*\mathcal{S}$ and $\mathrm{Alg}\mathcal{S}$

Definition

Let $\mathcal{S}$ be a logic.

- $\mathrm{Alg}^*\mathcal{S} := \{\boldsymbol{A} : \text{there is } F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \boldsymbol{\Omega}^{\boldsymbol{A}}(F) = id_{\boldsymbol{A}}\}$;
- $\mathrm{Alg}\mathcal{S} := \{\boldsymbol{A} : \text{there is } F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = id_{\boldsymbol{A}}\}$.

Proposition

$\mathrm{Alg}\mathcal{S} = \mathbb{P}_{\mathrm{S}}(\mathrm{Alg}^*\mathcal{S})$.

The Leibniz hierarchy

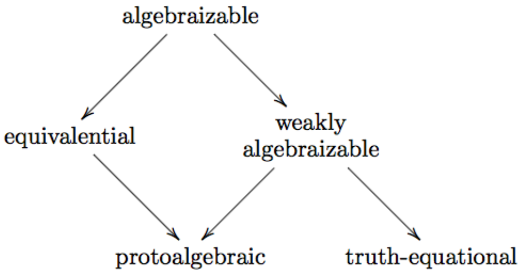

Figure: Fragment of the Leibniz hierarchy.

Leibniz classes and Leibniz filters

Definition

Let $\mathcal{S}$ be a logic, $\boldsymbol{A}$ be an algebra and $F \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A})$. The *Leibniz class of F* is defined by

$$[\![F]\!]^* := \left\{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\right\} .$$

The least element of $[\![F]\!]^*$ shall be denoted by F^*. We say that F is a *Leibniz filter* if $F = F^*$, and we denote the set of all Leibniz filters of $\boldsymbol{A}$ by $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$.

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- The new definition of Leibniz filters generalizes for arbitrary logics the existing one for protoalgebraic logics [Font and Jansana, 2001].

Suszko classes and Suszko filters

Definition

Let $\mathcal{S}$ be a logic, $\boldsymbol{A}$ be an algebra and $F \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A})$. The *Suszko class of F* is defined by

$$[\![F]\!]^{\mathrm{Su}} := \left\{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\right\}.$$

The least element of $[\![F]\!]^{\mathrm{Su}}$ shall be denoted by F^{Su}. We say that F is a *Suszko filter* if $F = F^{\mathrm{Su}}$, and we denote the set of all Suszko filters of $\boldsymbol{A}$ by $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$.

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The least element of $[\![F]\!]^{\mathrm{Su}}$ shall be denoted by F^{Su}. We say that F is a *Suszko filter* if $F = F^{\mathrm{Su}}$, and we denote the set of all Suszko filters of $\boldsymbol{A}$ by $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$.

- $F^{\mathrm{Su}} \subseteq F^{*} \subseteq F$.

Lemma

Every Suszko filter of $\boldsymbol{A}$ *is a Leibniz filter of* $\boldsymbol{A}$.

Truth-equational vs. Suszko filters

- [Raftery, 2006] introduces the class of truth-equational logics.

Truth-equational vs. Suszko filters

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Theorem [Font, Jansana, A.]

Let $\mathcal{S}$ be a logic. The following are equivalent:

1. $\mathcal{S}$ *is truth-equational;*
2. *For every* $\boldsymbol{A}$*, every* $\mathcal{S}$*-filter of* $\boldsymbol{A}$ *is a Suszko filter;*
3. *For every* $\boldsymbol{A} \in \mathrm{Alg}(\mathcal{S})$*, every* $\mathcal{S}$*-filter of* $\boldsymbol{A}$ *is a Suszko filter.*

Isomorphism theorems in AAL

Isomorphism Theorem for protoalgebraic logics [Font, Jansana, A.]

Let $\mathcal{S}$ be a logic. The following are equivalent:

1. *$\mathcal{S}$ is protoalgebraic;*
2. *The Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}(\boldsymbol{A}) \to \mathrm{Co}_{\mathsf{Alg}^*(\mathcal{S})}(\boldsymbol{A})$, restricted to the Suszko filters, is an order-isomorphism, for every algebra $\boldsymbol{A}$.*

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Isomorphism Theorem for equivalential logics [Font, Jansana, A.]

Let $\mathcal{S}$ be a logic. The following are equivalent:

1. *$\mathcal{S}$ is equivalential;*
2. *The Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}(\boldsymbol{A}) \to \mathrm{Co}_{\mathsf{Alg}^*(\mathcal{S})}(\boldsymbol{A})$, restricted to the Suszko filters, is an order-isomorphism which commutes with inverse images of homomorphisms, for every algebra $\boldsymbol{A}$.*

Isomorphism theorems in AAL

Isomorphism Theorem for weakly algebraizable logics [Czelakowski and Jansana, 2000]

Let $\mathcal{S}$ be a logic. The following are equivalent:

1. *$\mathcal{S}$ is weakly algebraizable;*
2. *The Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Co}_{\mathsf{Alg}^*(\mathcal{S})}(\boldsymbol{A})$ is an order-isomorphism, for every algebra $\boldsymbol{A}$.*

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Isomorphism Theorem for algebraizable logics [Blok and Pigozzi, 1989] and [Herrmann, 1997]

Let $\mathcal{S}$ be a logic. The following are equivalent:

1. *$\mathcal{S}$ is algebraizable;*
2. *The Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Co}_{\mathsf{Alg}^*(\mathcal{S})}(\boldsymbol{A})$ is an order-isomorphism which commutes with inverse images of homomorphisms, for every algebra $\boldsymbol{A}$.*

The strong version of a sentential logic

- With the new definition of Leibniz filter, we can generalize the notion of strong version of a protoalgebraic logic [Font and Jansana, 2001] to arbitrary logics:

Definition

Let $\mathcal{S}$ be a logic. The *strong version of* $\mathcal{S}$, which we shall denote by $\mathcal{S}^+$, is the logic induced by the class of matrices whose distinguished set is a Leibniz filter. That is,

$$\mathcal{S}^+ = \bigcap\{\vDash_{\langle \boldsymbol{A},F\rangle} : \boldsymbol{A} \text{ an algebra, } F \in \mathcal{F}i^*_{\mathcal{S}}(\boldsymbol{A})\} .$$

The strong version of a sentential logic

Proposition

$$\mathcal{S}^{+} = \bigcap\{\vDash_{\langle \boldsymbol{A},F\rangle} : \boldsymbol{A} \text{ an algebra, } F \in \mathcal{F}i_{\mathcal{S}}^{\mathrm{Su}}(\boldsymbol{A})\} .$$

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$$\begin{aligned}\mathcal{S}^+ &= \bigcap\{\models_{\langle \boldsymbol{A},\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle} : \boldsymbol{A} \text{ an algebra }\} \\ &= \bigcap\{\models_{\langle \boldsymbol{A},\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle} : \boldsymbol{A} \in \mathrm{Alg}(\mathcal{S})\}\end{aligned}$$

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- If $\mathcal{S}$ is truth-equational, then $\mathcal{S}^+ = \mathcal{S}$.

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- If $\mathcal{S}$ is truth-equational, then $\mathcal{S}^+ = \mathcal{S}$.
- If $\mathcal{S}$ is protoalgebraic, then we fall into the scope of [Font and Jansana, 2001].

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- If $\mathcal{S}$ is truth-equational, then $\mathcal{S}^+ = \mathcal{S}$.
- If $\mathcal{S}$ is protoalgebraic, then we fall into the scope of [Font and Jansana, 2001].
- We are therefore interested in examples of non-truth-equational and non-protoalgebraic logics. That is, examples of logics outside the Leibniz hierarchy.

Positive Modal Logic

- $\mathcal{L} = \{\wedge, \vee, \Box, \top, \bot\}$

Definition

A *positive modal algebra* is an $\mathcal{L}$-algebra $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \Box^{\boldsymbol{A}}, \Diamond^{\boldsymbol{A}}, 1, 0\rangle$, such that $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, 1, 0\rangle$ is a bounded distributive lattice and $\Box^{\boldsymbol{A}}, \Diamond^{\boldsymbol{A}}$ are two unary modal connectives satisfying:

1. $\Box^{\boldsymbol{A}}(a \wedge^{\boldsymbol{A}} b) = \Box^{\boldsymbol{A}} a \wedge^{\boldsymbol{A}} \Box^{\boldsymbol{A}} b$;
2. $\Diamond^{\boldsymbol{A}}(a \vee^{\boldsymbol{A}} b) = \Diamond^{\boldsymbol{A}} a \vee^{\boldsymbol{A}} \Diamond^{\boldsymbol{A}} b$;
3. $\Box^{\boldsymbol{A}} a \wedge^{\boldsymbol{A}} \Diamond^{\boldsymbol{A}} b \leq \Diamond^{\boldsymbol{A}}(a \wedge^{\boldsymbol{A}} b)$;
4. $\Box^{\boldsymbol{A}}(a \vee^{\boldsymbol{A}} b) \leq \Box^{\boldsymbol{A}} a \vee^{\boldsymbol{A}} \Diamond^{\boldsymbol{A}} b$;
5. $\Box^{\boldsymbol{A}} 1 = 1$;
6. $\Diamond^{\boldsymbol{A}} 0 = 0$.

The class of all positive modal algebras will be denoted by PMA.

Definition

PML is the logic preserving degrees of truth wrt. PMA, i.e, **PML** $= \models^{\leq}_{\mathrm{PMA}}$.

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- **PML** is not truth-equational neither protoalgebraic.
- For every $\boldsymbol{A} \in \mathrm{PMA}$, $\mathcal{F}i_{\mathbf{PML}}(\boldsymbol{A}) = \mathrm{Filt}(\boldsymbol{A})$.

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Definition

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- For every $\boldsymbol{A} \in \mathrm{PMA}$, $\mathcal{F}i_{\mathbf{PML}}(\boldsymbol{A}) = \mathrm{Filt}(\boldsymbol{A})$.
- $\mathrm{Alg}^*(\mathbf{PML}) \subsetneq \mathrm{Alg}(\mathbf{PML}) = \mathrm{PMA}$.

Positive Modal Logic

Leibniz and Suszko PML-filters

Definition

Let $\boldsymbol{A} \in \mathrm{PMA}$. A lattice filter $F \in \mathrm{Filt}(\boldsymbol{A})$ is *open*, if it is closed under $\Box$, i.e., for every $a \in A$, if $a \in F$, then $\Box^{\boldsymbol{A}} a \in F$.
The set of all open lattice filters of $\boldsymbol{A}$ will be denoted by $\mathrm{Filt}_{\Box}(\boldsymbol{A})$.

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Theorem

Let $\boldsymbol{A} \in \mathrm{PMA}$. The Leibniz and Suszko **PML***-filters of* $\boldsymbol{A}$ *coincide with the open lattice filters of* $\boldsymbol{A}$. *That is,*

$$\mathcal{F}i^{*}_{\mathbf{PML}}(\boldsymbol{A}) = \mathcal{F}i^{\mathrm{Su}}_{\mathbf{PML}}(\boldsymbol{A}) = \mathrm{Filt}_{\Box}(\boldsymbol{A}) \ .$$

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Theorem

Let $\boldsymbol{A} \in \mathrm{PMA}$. The Leibniz and Suszko **PML***-filters of $\boldsymbol{A}$ coincide with the open lattice filters of $\boldsymbol{A}$. That is,*

$$\mathcal{F}i^{*}_{\mathbf{PML}}(\boldsymbol{A}) = \mathcal{F}i^{\mathrm{Su}}_{\mathbf{PML}}(\boldsymbol{A}) = \mathrm{Filt}_{\Box}(\boldsymbol{A}) .$$

Strong version of PML

Theorem

$\mathbf{PML}^{+}$ *is the logic preserving truth wrt.* PMA, *i.e.,* $\mathbf{PML}^{+} = \vDash^{1}_{\mathrm{PMA}}$.

Dunn-Belnap's Logic

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Definition

A *De Morgan algebra* is an $\mathcal{L}$-algebra $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \neg^{\boldsymbol{A}}, 0, 1 \rangle$ such that:

(i) The reduct $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}} \rangle$ is a distributive lattice;

(ii) The De Morgan laws hold, that is, $\neg^{\boldsymbol{A}}(a \vee^{\boldsymbol{A}} b) = (\neg^{\boldsymbol{A}} a \wedge^{\boldsymbol{A}} \neg b)$ and $\neg^{\boldsymbol{A}}(a \wedge^{\boldsymbol{A}} b) = (\neg^{\boldsymbol{A}} a \vee^{\boldsymbol{A}} \neg^{\boldsymbol{A}} b)$;

(iii) The unary operation $\neg^{\boldsymbol{A}}$ is idempotent, that is, $a = \neg^{\boldsymbol{A}} \neg^{\boldsymbol{A}} a$;

(iv) 1 and 0 are the top and bottom elements, respectively, that is, $x \vee^{\boldsymbol{A}} 1 = 1$ and $x \wedge^{\boldsymbol{A}} 0 = 0$.

The class of all De Morgan algebras will be denoted by DMA.

Definition

$\mathcal{B}$ is the logic preserving degrees of truth wrt. DMA, i.e, $\mathcal{B} = \models^{\leq}_{\mathrm{DMA}}$.

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Dunn-Belnap's Logic

Leibniz and Suszko $\mathcal{B}$-filters

Definition

Let $\boldsymbol{A} \in \mathrm{DMA}$. A lattice filter $F \in \mathrm{Filt}(A)$ is *implicative*, if it is closed under *modus ponens*, i.e., for every $a, b \in A$, if $a, a \to^{\boldsymbol{A}} b \in F$, then $b \in F$.
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Strong version of $\mathcal{B}$

Theorem

$\mathcal{B}^{+}$ *is the logic preserving truth wrt.* DMA, *i.e.*, $\mathcal{B}^{+} = \vDash^{1}_{\mathrm{DMA}}$.

Lukasiewicz infinite-valued logic preserving degrees of truth

- $\mathcal{L} = \langle \wedge, \vee, \rightarrow, \odot, \top, \bot \rangle$

Definition

An *MV-algebra* is an $\mathcal{L}$-algebra $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \rightarrow^{\boldsymbol{A}}, \odot^{\boldsymbol{A}}, 1, 0 \rangle$ such that:

(i) $\boldsymbol{A}$ is an integral commutative residuated lattice;

(ii) $a \wedge^{\boldsymbol{A}} b = a \odot^{\boldsymbol{A}} (a \rightarrow^{\boldsymbol{A}} b)$;

(iii) $(a \rightarrow^{\boldsymbol{A}} b) \vee^{\boldsymbol{A}} (b \rightarrow^{\boldsymbol{A}} a) = 1$;

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The class of all MV-algebras algebras will be denoted by MV.

Definition

$Ł_\infty^{\leq}$ is the logic preserving degrees of truth wrt. MV, i.e, $Ł_\infty^{\leq} = \models_{\mathrm{MV}}^{\leq}$.

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Lukasiewicz infinite-valued logic preserving degrees of truth

Leibniz and Suszko $Ł_\infty^\leq$-filters

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The strong version of $Ł_\infty^\leq$

Theorem

$(Ł_\infty^\leq)^+$ *is the Lukasiewicz infinite valued logic, i.e.,* $(Ł_\infty^\leq)^+ = Ł_\infty$.

Logics preserving degrees of truth from varieties of residuated lattices

- $\mathcal{L} = \langle \wedge, \vee, \to, \odot, \top \rangle$

Definition

An (integral commutative) *residuated lattice* is an $\mathcal{L}$-algebra $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, \odot^{\boldsymbol{A}}, 1 \rangle$ such that:

(i) The reduct $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}} \rangle$ is a lattice;

(ii) $\langle A, \odot^{\boldsymbol{A}}, 1 \rangle$ is a commutative monoid;

(iii) $\to^{\boldsymbol{A}}$ is the residuum of $\odot^{\boldsymbol{A}}$, that is, for every $a, b \in A$, $a \odot^{\boldsymbol{A}} c \leq b$ iff $c \leq a \to^{\boldsymbol{A}} b$;

(iv) 1 is the top element of $\boldsymbol{A}$, that is, for every $a \in A$, $a \vee^{\boldsymbol{A}} 1 = 1$.

The class of all (integral commutative) residuated lattices will be denoted by RL.

Logics preserving degrees of truth from varieties of residuated lattices

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- $\models^{\leq}_{\mathrm{RL}}$ is not truth-equational neither protoalgebraic.
- For every $\boldsymbol{A} \in \mathrm{RL}$, $\mathcal{F}i_{\models^{\leq}_{\mathrm{RL}}}(\boldsymbol{A}) = \mathrm{Filt}(\boldsymbol{A})$.

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Logics preserving degrees of truth from varieties of residuated lattices

Leibniz and Suszko $\models^{\leq}_{\mathrm{RL}}$-filters

Definition

Let $\boldsymbol{A} \in \mathrm{RL}$. A lattice filter $F \in \mathrm{Filt}(A)$ is *implicative*, if it is closed under *modus ponens*, i.e., for every $a, b \in A$, if $a, a \rightarrow^{\boldsymbol{A}} b \in F$, then $b \in F$.
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The set of all implicative filters of $\boldsymbol{A}$ will be denoted by $\mathrm{Filt}_{\rightarrow}(\boldsymbol{A})$.

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Let $\boldsymbol{A} \in \mathrm{RL}$. The Leibniz $\models^{\leq}_{\mathrm{RL}}$-filters of $\boldsymbol{A}$ coincide with the implicative lattice filters of $\boldsymbol{A}$. That is,

$$\mathcal{F}i^{*}_{\models^{\leq}_{\mathrm{RL}}}(\boldsymbol{A}) = \mathrm{Filt}_{\rightarrow}(\boldsymbol{A}) \ .$$

- The Suszko $\models^{\leq}_{\mathrm{RL}}$-filters are a strict subset of the Leibniz $\models^{\leq}_{\mathrm{RL}}$-filters.

Logics preserving degrees of truth from varieties of residuated lattices

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The strong version of $\models^{\leq}_{\mathrm{RL}}$

Theorem

$(\models^{\leq}_{\mathrm{RL}})^{+}$ *is the logic preserving truth wrt.* RL, *i.e,* $(\models^{\leq}_{\mathrm{RL}})^{+} = \models^{1}_{\mathrm{RL}}$.

Thank you!

References I

- Albuquerque, H., Font, J. M., and Jansana, R. (201x).
  Compatibility operators in abstract algebraic logic.
  The Journal of Symbolic Logic.
  To appear.
- Blok, W. J. and Pigozzi, D. (1989).
  Algebraizable logics.
  Mem. Amer. Math. Soc., 77(396):vi+78.
- Czelakowski, J. and Jansana, R. (2000).
  Weakly algebraizable logics.
  The Journal of Symbolic Logic, 65(2):641–668.
- Font, J. M. and Jansana, R. (2001).
  Leibniz filters and the strong version of a protoalgebraic logic.
  Archive for Mathematical Logic, 40:437–465.
- Herrmann, B. (1997).
  Characterizing equivalential and algebraizable logics by the Leibniz operator.
  Studia Logica, 58(2):305–323.
- Raftery, J. G. (2006).
  The equational definability of truth predicates.
  Rep. Math. Logic, (41):95–149.

Original definition of Leibniz filters for protoalgebraic logics

Definition

Let $\mathcal{S}$ be a protoalgebraic logic and $\boldsymbol{A}$ an algebra. An $\mathcal{S}$-filter $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is a *Leibniz* *$\mathcal{S}$-filter of $\boldsymbol{A}$*, if it is the least element of

$$[F] = \{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\} \, .$$

Back to Leibniz filters.