Leibniz and Suszko filters for non-protoalgebraic logics

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Leibniz and Suszko congruences

- Given $F \subseteq A$, the *Leibniz congruence of $F$*, which shall be denoted by $\Omega^A(F)$, is the largest congruence of $A$ compatible with $F$.

- Given $F \subseteq A$, the *Suszko congruence of $F$*, which shall be denoted by $\sim\Omega^A_S(F)$, is the largest congruence of $A$ compatible with every $G \in (\mathcal{F}_S A)^F$, or, equivalently,

\[
\sim\Omega^A_S(F) := \bigcap \{ \Omega^A(G) : G \in \mathcal{F}_S A, F \subseteq G \}.
\]

Leibniz and Suszko operators

- The *Leibniz operator on $A$* is the map $\Omega^A : \mathcal{F}_S A \to \text{Co}A$ defined by $F \mapsto \Omega^A(F)$.

- The *Suszko operator on $A$* is the map $\sim\Omega^A_S : \mathcal{F}_S A \to \text{Co}A$ defined by $F \mapsto \sim\Omega^A_S(F)$. 
The classes of algebras $\text{Alg}^* S$ and $\text{Alg} S$

**Definition**

Let $S$ be a logic.

- $\text{Alg}^* S := \{ A : \text{there is } F \in \mathcal{F}_{iS} A \text{ such that } \Omega^A(F) = id_A \}$;
- $\text{Alg} S := \{ A : \text{there is } F \in \mathcal{F}_{iS} A \text{ such that } \Omega^A_S(F) = id_A \}$.

**Proposition**

$\text{Alg} S = \mathbb{P}_S(\text{Alg}^* S)$.
The Leibniz hierarchy

Figure: Fragment of the Leibniz hierarchy.
Leibniz classes and Leibniz filters

Definition

Let $S$ be a logic, $A$ be an algebra and $F \in F\mathcal{I}_{S}(A)$. The Leibniz class of $F$ is defined by

$$[F]^* := \{ G \in F\mathcal{I}_{S}A : \Omega^{A}(F) \subseteq \Omega^{A}(G) \}.$$ 

The least element of $[F]^*$ shall be denoted by $F^*$. We say that $F$ is a Leibniz filter if $F = F^*$, and we denote the set of all Leibniz filters of $A$ by $F\mathcal{I}^{*}_{S}A$. 

The new definition of Leibniz filters generalizes for arbitrary logics the existing one for protoalgebraic logics [Font and Jansana, 2001].
Leibniz classes and Leibniz filters

Definition

Let $S$ be a logic, $A$ be an algebra and $F \in \mathcal{F}i_S(A)$. The Leibniz class of $F$ is defined by

$$[F]^* := \{ G \in \mathcal{F}i_S A : \Omega^A(F) \subseteq \Omega^A(G) \}.$$  

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- The new definition of Leibniz filters generalizes for arbitrary logics the existing one for protoalgebraic logics [Font and Jansana, 2001].
**Suszko classes and Suszko filters**

**Definition**

Let \( S \) be a logic, \( A \) be an algebra and \( F \in \mathcal{F}i_{S}(A) \). The **Suszko class of** \( F \) is defined by

\[
\left[ F \right]^{Su} := \left\{ G \in \mathcal{F}i_{S}A : \tilde{\Omega}^{A}_{S}(F) \subseteq \Omega^{A}(G) \right\}.
\]

The least element of \( \left[ F \right]^{Su} \) shall be denoted by \( F^{Su} \). We say that \( F \) is a **Suszko filter** if \( F = F^{Su} \), and we denote the set of all Suszko filters of \( A \) by \( \mathcal{F}i_{S}^{Su}A \).
**Definition**

Let \( S \) be a logic, \( A \) be an algebra and \( F \in \mathcal{F}i_S(A) \). The *Suszko class of* \( F \) is defined by

\[
\left[ F \right]^{Su} := \left\{ G \in \mathcal{F}i_S A : \mathcal{N}_S^A(F) \subseteq \mathcal{O}_A(G) \right\} .
\]

The least element of \( \left[ F \right]^{Su} \) shall be denoted by \( F^{Su} \). We say that \( F \) is a *Suszko filter* if \( F = F^{Su} \), and we denote the set of all Suszko filters of \( A \) by \( \mathcal{F}i_S^{Su} A \).

- \( F^{Su} \subseteq F^* \subseteq F \).

**Lemma**

*Every Suszko filter of* \( A \) *is a Leibniz filter of* \( A \).*
[Raftery, 2006] introduces the class of truth-equational logics.
Truth-equational vs. Suszko filters

- [Raftery, 2006] introduces the class of truth-equational logics.

**Theorem** [Font, Jansana, A.]

Let $S$ be a logic. The following are equivalent:

1. $S$ is truth-equational;
2. For every $A$, every $S$-filter of $A$ is a Suszko filter;
3. For every $A \in \text{Alg}(S)$, every $S$-filter of $A$ is a Suszko filter.
Isomorphism Theorem for protoalgebraic logics
[Font, Jansana, A.]

Let $S$ be a logic. The following are equivalent:

1. $S$ is protoalgebraic;
2. The Leibniz operator $\Omega^A : \mathcal{F}_{\text{Su}}^S(A) \rightarrow \text{CoAlg}^*(S)(A)$, restricted to the Suszko filters, is an order-isomorphism, for every algebra $A$. 
Isomorphism theorems in AAL

Isomorphism Theorem for protoalgebraic logics

[Font, Jansana, A.]

Let $S$ be a logic. The following are equivalent:

1. $S$ is protoalgebraic;
2. The Leibniz operator $\Omega^A : \mathcal{F}_{iS}^\text{Su}(A) \rightarrow \mathcal{C}_{\text{Alg}^*}(S)(A)$, restricted to the Suszko filters, is an order-isomorphism, for every algebra $A$.

Isomorphism Theorem for equivalential logics

[Font, Jansana, A.]

Let $S$ be a logic. The following are equivalent:

1. $S$ is equivalential;
2. The Leibniz operator $\Omega^A : \mathcal{F}_{iS}^\text{Su}(A) \rightarrow \mathcal{C}_{\text{Alg}^*}(S)(A)$, restricted to the Suszko filters, is an order-isomorphism which commutes with inverse images of homomorphisms, for every algebra $A$. 
Isomorphism Theorem for weakly algebraizable logics
[Czelakowski and Jansana, 2000]

Let $S$ be a logic. The following are equivalent:

1. $S$ is weakly algebraizable;
2. The Leibniz operator $\Omega^A : \mathcal{F}_S A \to \mathcal{C}_{O\text{Alg}^*(S)}(A)$ is an order-isomorphism, for every algebra $A$. 

Isomorphism Theorems in AAL
Isomorphism theorems in AAL

Isomorphism Theorem for weakly algebraizable logics  
[Czelakowski and Jansana, 2000]

Let $S$ be a logic. The following are equivalent:

1. $S$ is weakly algebraizable;
2. The Leibniz operator $\Omega^A : \mathcal{F}_S A \rightarrow \mathcal{C}_{\text{Alg}^*(S)}(A)$ is an order-isomorphism, for every algebra $A$.

Isomorphism Theorem for algebraizable logics  
[Blok and Pigozzi, 1989] and [Herrmann, 1997]

Let $S$ be a logic. The following are equivalent:

1. $S$ is algebraizable;
2. The Leibniz operator $\Omega^A : \mathcal{F}_S A \rightarrow \mathcal{C}_{\text{Alg}^*(S)}(A)$ is an order-isomorphism which commutes with inverse images of homomorphisms, for every algebra $A$. 
The strong version of a sententional logic

With the new definition of Leibniz filter, we can generalize the notion of strong version of a protoalgebraic logic [Font and Jansana, 2001] to arbitrary logics:

**Definition**

Let $S$ be a logic. The strong version of $S$, which we shall denote by $S^+$, is the logic induced by the class of matrices whose distinguished set is a Leibniz filter. That is,

$$S^+ = \bigcap \{\models_{A,F}: A \text{ an algebra, } F \in \mathcal{F}_S(A)\}.$$
The strong version of a sententional logic

Proposition

\[ S^+ = \bigcap\{\models_{(A,F)}: A \text{ an algebra, } F \in \mathcal{F}_{iS}^{Su}(A)\} \]
Preliminaries
Leibniz and Suszko filters
Strong version of a sententional logic
Non-protoalgebraic examples

The strong version of a sententional logic

Proposition

\[ S^+ = \bigcap \{ \models_{\langle A,F \rangle} : A \text{ an algebra, } F \in F_{iS}^{Su}(A) \} \]

Proposition

\[ S^+ = \bigcap \{ \models_{\langle A, \bigcap F_{iS} A \rangle} : A \text{ an algebra} \} \]

= \bigcap \{ \models_{\langle A, \bigcap F_{iS} A \rangle} : A \in \text{Alg}(S) \} \]
The strong version of a sentential logic

Proposition

\[ S^+ = \bigcap \{ \models_{\langle A, F \rangle} : A \text{ an algebra, } F \in F_{iS}^{Su}(A) \} . \]

Proposition

\[ S^+ = \bigcap \{ \models_{\langle A, F \rangle} : A \text{ an algebra } \} \]
\[ = \bigcap \{ \models_{\langle A, F \rangle} : A \in \text{Alg}(S) \} \]

- If \( S \) is truth-equational, then \( S^+ = S \).
The strong version of a sententional logic

Proposition

\[ S^+ = \bigcap \{ \models_{\langle A, F \rangle} \colon A \text{ an algebra, } F \in \mathcal{F}^{\text{Su}}_S(A) \} \]

Proposition

\[ S^+ = \bigcap \{ \models_{\langle A, \bigcap \mathcal{F} S A \rangle} \colon A \text{ an algebra} \} \]
\[ = \bigcap \{ \models_{\langle A, \bigcap \mathcal{F} S A \rangle} \colon A \in \text{Alg}(S) \} \]

- If \( S \) is truth-equational, then \( S^+ = S \).
- If \( S \) is protoalgebraic, then we fall into the scope of [Font and Jansana, 2001].
The strong version of a sentential logic

Proposition

\[ S^+ = \bigcap \{ \models_{\langle A, F \rangle}: A \text{ an algebra, } F \in F_{iS}^SU(A) \} . \]

Proposition

\[ S^+ = \bigcap \{ \models_{\langle A, \bigcap F_{iS} A \rangle}: A \text{ an algebra } \} \]
\[ = \bigcap \{ \models_{\langle A, \bigcap F_{iS} A \rangle}: A \in \text{Alg}(S) \} \]

- If \( S \) is truth-equational, then \( S^+ = S \).
- If \( S \) is protoalgebraic, then we fall into the scope of [Font and Jansana, 2001].
- We are therefore interested in examples of non-truth-equational and non,protoalgebraic logics. That is, examples of logics outside the Leibniz hierarchy.
Positive Modal Logic

- $\mathcal{L} = \{\land, \lor, \Box, \top, \bot\}$

**Definition**

A *positive modal algebra* is an $\mathcal{L}$-algebra $A = \langle A, \land^A, \lor^A, \Box^A, \Diamond^A, 1, 0 \rangle$, such that $\langle A, \land^A, \lor^A, 1, 0 \rangle$ is a bounded distributive lattice and $\Box^A, \Diamond^A$ are two unary modal connectives satisfying:

1. $\Box^A(a \land^A b) = \Box^A a \land^A \Box^A b$;
2. $\Diamond^A(a \lor^A b) = \Diamond^A a \lor^A \Diamond^A b$;
3. $\Box^A a \land^A \Diamond^A b \leq \Diamond^A(a \land^A b)$;
4. $\Box^A(a \lor^A b) \leq \Box^A a \lor^A \Diamond^A b$;
5. $\Box^A 1 = 1$;
6. $\Diamond^A 0 = 0$.

The class of all positive modal algebras will be denoted by PMA.

**Definition**

$\text{PML}$ is the logic preserving degrees of truth wrt. PMA, i.e, $\text{PML} = \models_{\text{PMA}}$. 
Positive Modal Logic

$\mathcal{L} = \{\land, \lor, \Box, T, \bot\}$

**Definition**

A *positive modal algebra* is an $\mathcal{L}$-algebra $A = \langle A, \land^A, \lor^A, \Box^A, \Diamond^A, 1, 0 \rangle$, such that $\langle A, \land^A, \lor^A, 1, 0 \rangle$ is a bounded distributive lattice and $\Box^A, \Diamond^A$ are two unary modal connectives satisfying:

1. $\Box^A(a \land^A b) = \Box^Aa \land^A \Box^Ab$;
2. $\Diamond^A(a \lor^A b) = \Diamond^Aa \lor^A \Diamond^Ab$;
3. $\Box^Aa \land^A \Diamond^Ab \leq \Diamond^A(a \land^A b)$;
4. $\Box^A(a \lor^A b) \leq \Box^Aa \lor^A \Diamond^Ab$;
5. $\Box^A1 = 1$;
6. $\Diamond^A0 = 0$.

The class of all positive modal algebras will be denoted by $\text{PMA}$.

**Definition**

$\text{PML}$ is the logic preserving degrees of truth wrt. $\text{PMA}$, i.e, $\text{PML} = \models_{\text{PMA}}$.

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Positive Modal Logic

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1. $\Box^A(a \land^A b) = \Box^A a \land^A \Box^A b$;
2. $\Diamond^A(a \lor^A b) = \Diamond^A a \lor^A \Diamond^A b$;
3. $\Box^A a \land^A \Diamond^A b \leq \Diamond^A(a \land^A b)$;
4. $\Box^A(a \lor^A b) \leq \Box^A a \lor^A \Diamond^A b$;
5. $\Box^A 1 = 1$;
6. $\Diamond^A 0 = 0$.

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**Definition**

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- $\text{PML}$ is not truth-equational neither protoalgebraic.
- For every $A \in \text{PMA}$, $\mathcal{F}{\text{PML}}(A) = \text{Filt}(A)$.
Positive Modal Logic

- $\mathcal{L} = \{\land, \lor, \Box, T, \bot\}$

**Definition**

A **positive modal algebra** is an $\mathcal{L}$-algebra $A = \langle A, \land^A, \lor^A, \Box^A, \Diamond^A, 1, 0 \rangle$, such that $\langle A, \land^A, \lor^A, 1, 0 \rangle$ is a bounded distributive lattice and $\Box^A, \Diamond^A$ are two unary modal connectives satisfying:

1. $\Box^A(a \land^A b) = \Box^Aa \land^A \Box^Ab$;
2. $\Diamond^A(a \lor^A b) = \Diamond^Aa \lor^A \Diamond^Ab$;
3. $\Box^Aa \land^A \Diamond^Ab \leq \Diamond^A(a \land^A b)$;
4. $\Box^A(a \lor^A b) \leq \Box^Aa \lor^A \Diamond^Ab$;
5. $\Box^A1 = 1$;
6. $\Diamond^A0 = 0$.

The class of all positive modal algebras will be denoted by $\text{PMA}$.

**Definition**

$\text{PML}$ is the logic preserving degrees of truth wrt. $\text{PMA}$, i.e, $\text{PML} = \models_{\text{PMA}}$.

- $\text{PML}$ is not truth-equational neither protoalgebraic.
- For every $A \in \text{PMA}$, $\mathcal{F}_{\text{PML}}(A) = \text{Filt}(A)$.
- $\text{Alg}^*(\text{PML}) \subsetneq \text{Alg}(\text{PML}) = \text{PMA}$. 
Positive Modal Logic

Leibniz and Suszko PML-filters

**Definition**

Let $A \in \text{PMA}$. A lattice filter $F \in \text{Filt}(A)$ is *open*, if it is closed under $\Box$, i.e., for every $a \in A$, if $a \in F$, then $\Box^A a \in F$.

The set of all open lattice filters of $A$ will be denoted by $\text{Filt}\Box(A)$. 
Positive Modal Logic

Leibniz and Suszko PML-filters

Definition

Let $A \in \text{PMA}$. A lattice filter $F \in \text{Filt}(A)$ is open, if it is closed under $\Box$, i.e., for every $a \in A$, if $a \in F$, then $\Box^A a \in F$.

The set of all open lattice filters of $A$ will be denoted by $\text{Filt}^\Box(A)$.

Theorem

Let $A \in \text{PMA}$. The Leibniz and Suszko PML-filters of $A$ coincide with the open lattice filters of $A$. That is,

$$\mathcal{F}_{i^*}^{\text{PML}}(A) = \mathcal{F}_{i}^{\text{Su}}(A) = \text{Filt}^\Box(A).$$
Positive Modal Logic

Leibniz and Suszko PML-filters

Definition

Let $A \in \text{PMA}$. A lattice filter $F \in \text{Filt}(A)$ is open, if it is closed under $\square$, i.e., for every $a \in A$, if $a \in F$, then $\square^A a \in F$.

The set of all open lattice filters of $A$ will be denoted by $\text{Filt} \square(A)$.

Theorem

Let $A \in \text{PMA}$. The Leibniz and Suszko PML-filters of $A$ coincide with the open lattice filters of $A$. That is,

$$\mathcal{F}^*_{\text{PML}}(A) = \mathcal{F}^\text{Su}_{\text{PML}}(A) = \text{Filt} \square(A).$$

Strong version of PML

Theorem

$\text{PML}^+$ is the logic preserving truth wrt. PMA, i.e., $\text{PML}^+ = \models_1^{\text{PMA}}$. 

Dunn-Belnap’s Logic

\[ \mathcal{L} = \langle \land, \lor, \neg, \top, \bot \rangle \]

**Definition**

A *De Morgan algebra* is an \( \mathcal{L} \)-algebra \( \mathbf{A} = \langle A, \land^A, \lor^A, \neg^A, 0, 1 \rangle \) such that:

(i) The reduct \( \langle A, \land^A, \lor^A \rangle \) is a distributive lattice;

(ii) The De Morgan laws hold, that is, \( \neg^A (a \lor^A b) = (\neg^A a \land^A \neg^A b) \) and \( \neg^A (a \land^A b) = (\neg^A a \lor^A \neg^A b) \);

(iii) The unary operation \( \neg^A \) is idempotent, that is, \( a = \neg^A \neg^A a \);

(iv) 1 and 0 are the top and bottom elements, respectively, that is, \( x \lor^A 1 = 1 \) and \( x \land^A 0 = 0 \).

The class of all De Morgan algebras will be denoted by DMA.

**Definition**

\( \mathcal{B} \) is the logic preserving degrees of truth wrt. DMA, i.e, \( \mathcal{B} = \models_{\text{DMA}} \).
**Dunn-Belnap’s Logic**

- $\mathcal{L} = \langle \land, \lor, \neg, \top, \bot \rangle$

**Definition**

A *De Morgan algebra* is an $\mathcal{L}$-algebra $\mathbf{A} = \langle A, \land^A, \lor^A, \neg^A, 0, 1 \rangle$ such that:

1. The reduct $\langle A, \land^A, \lor^A \rangle$ is a distributive lattice;
2. The De Morgan laws hold, that is, $\neg^A(a \lor^A b) = (\neg^A a \land^A \neg^A b)$ and $\neg^A(a \land^A b) = (\neg^A a \lor^A \neg^A b)$;
3. The unary operation $\neg^A$ is idempotent, that is, $a = \neg^A \neg^A a$;
4. $1$ and $0$ are the top and bottom elements, respectively, that is, $x \lor^A 1 = 1$ and $x \land^A 0 = 0$.

The class of all De Morgan algebras will be denoted by DMA.

**Definition**

$\mathcal{B}$ is the logic preserving degrees of truth wrt. DMA, i.e., $\mathcal{B} = \models_{\text{DMA}}$. 

- $\mathcal{B}$ is not truth-equational neither protoalgebraic.
Dunn-Belnap’s Logic

- $\mathcal{L} = \langle \land, \lor, \neg, \top, \bot \rangle$

**Definition**

A *De Morgan algebra* is an $\mathcal{L}$-algebra $\mathbf{A} = \langle A, \land^A, \lor^A, \neg^A, 0, 1 \rangle$ such that:

(i) The reduct $\langle A, \land^A, \lor^A \rangle$ is a distributive lattice;

(ii) The De Morgan laws hold, that is, $\neg^A(a \lor^A b) = (\neg^A a \land^A \neg^A b)$ and $\neg^A(a \land^A b) = (\neg^A a \lor^A \neg^A b)$;

(iii) The unary operation $\neg^A$ is idempotent, that is, $a = \neg^A \neg^A a$;

(iv) 1 and 0 are the top and bottom elements, respectively, that is, $x \lor^A 1 = 1$ and $x \land^A 0 = 0$.

The class of all De Morgan algebras will be denoted by DMA.

**Definition**

$\mathcal{B}$ is the logic preserving degrees of truth wrt. DMA, i.e, $\mathcal{B} = \models^\leq_{\text{DMA}}$.

- $\mathcal{B}$ is not truth-equational neither protoalgebraic.
- For every $\mathbf{A} \in \text{DMA}$, $\mathcal{F}i_\mathcal{B}(\mathbf{A}) = \text{Filt}(\mathbf{A})$. 
Dunn-Belnap’s Logic

- $\mathcal{L} = \langle \land, \lor, \neg, \top, \bot \rangle$

**Definition**

A *De Morgan algebra* is an $\mathcal{L}$-algebra $\mathbf{A} = \langle A, \land^A, \lor^A, \neg^A, 0, 1 \rangle$ such that:

(i) The reduct $\langle A, \land^A, \lor^A \rangle$ is a distributive lattice;

(ii) The De Morgan laws hold, that is, $\neg^A(a \lor^A b) = (\neg^A a \land^A \neg^A b)$ and $\neg^A(a \land^A b) = (\neg^A a \lor^A \neg^A b)$;

(iii) The unary operation $\neg^A$ is idempotent, that is, $a = \neg^A \neg^A a$;

(iv) 1 and 0 are the top and bottom elements, respectively, that is, $x \lor^A 1 = 1$ and $x \land^A 0 = 0$.

The class of all De Morgan algebras will be denoted by $\mathsf{DMA}$.

**Definition**

$\mathcal{B}$ is the logic preserving degrees of truth wrt. $\mathsf{DMA}$, i.e., $\mathcal{B} = \models \leq_{\mathsf{DMA}}$.

- $\mathcal{B}$ is not truth-equational neither protoalgebraic.
- For every $\mathbf{A} \in \mathsf{DMA}$, $\mathcal{F}_{i\mathcal{B}}(\mathbf{A}) = \text{Filt}(\mathbf{A})$.
- $\mathsf{Alg}^*(\mathcal{B}) \subsetneq \mathsf{Alg}(\mathcal{B}) = \mathsf{DMA}$. 
Dunn-Belnap’s Logic

Leibniz and Suszko $B$-filters

**Definition**

Let $A \in \text{DMA}$. A lattice filter $F \in \text{Filt}(A)$ is *implicative*, if it is closed under *modus ponens*, i.e., for every $a, b \in A$, if $a, a \rightarrow^A b \in F$, then $b \in F$.

The set of all implicative filters of $A$ will be denoted by $\text{Filt} \rightarrow^A (A)$. 
Dunn-Belnap’s Logic

Leibniz and Suszko $B$-filters

**Definition**

Let $A \in DMA$. A lattice filter $F \in \text{Filt}(A)$ is *implicative*, if it is closed under *modus ponens*, i.e., for every $a, b \in A$, if $a, a \rightarrow^A b \in F$, then $b \in F$.

The set of all implicative filters of $A$ will be denoted by $\text{Filt}_{\rightarrow}(A)$.

**Theorem**

Let $A \in DMA$. *The Leibniz $B$-filters of $A$ coincide with the implicative lattice filters of $A$. That is,*

$$\mathcal{F}i^*_B(A) = \text{Filt}_{\rightarrow}(A).$$

- The Suszko $B$-filters are a strict subset of the Leibniz $B$-filters.
Dunn-Belnap’s Logic

Leibniz and Suszko $B$-filters

Definition

Let $A \in DMA$. A lattice filter $F \in \text{Filt}(A)$ is implicative, if it is closed under modus ponens, i.e., for every $a, b \in A$, if $a, a \rightarrow^A b \in F$, then $b \in F$.

The set of all implicative filters of $A$ will be denoted by $\text{Filt}_{\rightarrow}(A)$.

Theorem

Let $A \in DMA$. The Leibniz $B$-filters of $A$ coincide with the implicative lattice filters of $A$. That is,

$$\mathcal{F}i_B^*(A) = \text{Filt}_{\rightarrow}(A).$$

- The Suszko $B$-filters are a strict subset of the Leibniz $B$-filters.

Strong version of $B$

Theorem

$B^+$ is the logic preserving truth wrt. DMA, i.e., $B^+ = \models_1^{DMA}$. 
Lukasiewicz infinite-valued logic preserving degrees of truth

\[ \mathcal{L} = \langle \land, \lor, \rightarrow, \odot, \top, \bot \rangle \]

**Definition**

An *MV-algebra* is an \( \mathcal{L} \)-algebra \( A = \langle A, \land^A, \lor^A, \rightarrow^A, \odot^A, 1, 0 \rangle \) such that:

(i) \( A \) is an integral commutative residuated lattice;

(ii) \( a \land^A b = a \odot^A (a \rightarrow^A b) \);

(iii) \( (a \rightarrow^A b) \lor^A (b \rightarrow^A a) = 1 \);

(iv) \( ((a \rightarrow^A 0) \rightarrow^A 0) = a \).

The class of all MV-algebras algebras will be denoted by MV.

**Definition**

\( L_\leq_\infty \) is the logic preserving degrees of truth wrt. MV, i.e, \( L_\leq_\infty = \models_{\leq}^{\text{MV}} \).
Lukasiewicz infinite-valued logic preserving degrees of truth

- $\mathcal{L} = \langle \wedge, \vee, \rightarrow, \odot, \top, \bot \rangle$

**Definition**

An MV-algebra is an $\mathcal{L}$-algebra $A = \langle A, \wedge^A, \vee^A, \rightarrow^A, \odot^A, 1, 0 \rangle$ such that:

(i) $A$ is an integral commutative residuated lattice;
(ii) $a \wedge^A b = a \odot^A (a \rightarrow^A b)$;
(iii) $(a \rightarrow^A b) \vee^A (b \rightarrow^A a) = 1$;
(iv) $((a \rightarrow^A 0) \rightarrow^A 0) = a$.

The class of all MV-algebras algebras will be denoted by MV.

**Definition**

$L_\leq_\infty$ is the logic preserving degrees of truth wrt. MV, i.e, $L_\leq_\infty = \vdash_\leq_{MV}$.

- $L_\leq_\infty$ is not truth-equational neither protoalgebraic.
Lukasiewicz infinite-valued logic preserving degrees of truth

- $\mathcal{L} = \langle \land, \lor, \to, \odot, \top, \bot \rangle$

**Definition**

An **MV-algebra** is an $\mathcal{L}$-algebra $A = \langle A, \land^A, \lor^A, \to^A, \odot^A, 1, 0 \rangle$ such that:

(i) $A$ is an integral commutative residuated lattice;
(ii) $a \land^A b = a \odot^A (a \to^A b)$;
(iii) $(a \to^A b) \lor^A (b \to^A a) = 1$;
(iv) $((a \to^A 0) \to^A 0) = a$.

The class of all MV-algebras algebras will be denoted by $\text{MV}$.

**Definition**

$L_{\leq \infty}$ is the logic preserving degrees of truth wrt. $\text{MV}$, i.e, $L_{\leq \infty} = \models_{\text{MV}}$.

- $L_{\leq \infty}$ is not truth-equational neither protoalgebraic.
- For every $A \in \text{MV}$, $\mathcal{F}_{L_{\leq \infty}}(A) = \text{Filt}(A)$. 

$\mathcal{L}_{\leq \infty}$ is not truth-equational neither protoalgebraic.

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Lukasiewicz infinite-valued logic preserving degrees of truth

- \( \mathcal{L} = \langle \land, \lor, \to, \odot, \top, \bot \rangle \)

**Definition**

An **MV-algebra** is an \( \mathcal{L} \)-algebra \( A = \langle A, \land^A, \lor^A, \to^A, \odot^A, 1, 0 \rangle \) such that:

(i) \( A \) is an integral commutative residuated lattice;

(ii) \( a \land^A b = a \odot^A (a \to^A b) \);

(iii) \( (a \to^A b) \lor^A (b \to^A a) = 1 \);

(iv) \( ((a \to^A 0) \to^A 0) = a \).

The class of all MV-algebras algebras will be denoted by \( \text{MV} \).

**Definition**

\( \mathcal{L}^{\leq} \) is the logic preserving degrees of truth wrt. \( \text{MV} \), i.e, \( \mathcal{L}^{\leq} = \vDash_{\text{MV}}^{\leq} \).

- \( \mathcal{L}^{\leq} \) is not truth-equational neither protoalgebraic.
- For every \( A \in \text{MV} \), \( \mathcal{F}_{\mathcal{L}^{\leq}}(A) = \text{Filt}(A) \).
- \( \text{Alg}^*(\mathcal{L}^{\leq}) = \text{Alg}(\mathcal{L}^{\leq}) = \text{MV} \).
Lukasiewicz infinite-valued logic preserving degrees of truth

Leibniz and Suszko $L_{\infty}^\leq$-filters

**Definition**

Let $A \in \text{MV}$. A lattice filter $F \in \text{Filt}(A)$ is *implicative*, if it is closed under *modus ponens*, i.e., for every $a, b \in A$, if $a, a \rightarrow^A b \in F$, then $b \in F$.

The set of all implicative filters of $A$ will be denoted by $\text{Filt}_\rightarrow(A)$. 
Leibniz and Suszko $\mathfrak{L}_{\infty}^{\leq}$-filters

**Definition**

Let $A \in \text{MV}$. A lattice filter $F \in \text{Filt}(A)$ is **implicative**, if it is closed under *modus ponens*, i.e., for every $a, b \in A$, if $a, a \to^A b \in F$, then $b \in F$.

The set of all implicative filters of $A$ will be denoted by $\text{Filt}_{\to}(A)$.

**Theorem**

Let $A \in \text{MV}$. The Leibniz $\mathfrak{L}_{\infty}^{\leq}$-filters of $A$ coincide with the implicative lattice filters of $A$. That is,

$$\mathcal{F}i^*_\mathfrak{L}_{\leq} (A) = \text{Filt}_{\to}(A).$$

- The Suszko $\mathfrak{L}_{\infty}^{\leq}$-filters are a strict subset of the Leibniz $\mathfrak{L}_{\infty}^{\leq}$-filters.
**Lukasiewicz infinite-valued logic preserving degrees of truth**

**Leibniz and Suszko $\mathcal{L}_{\leq \infty}$-filters**

**Definition**

Let $A \in \text{MV}$. A lattice filter $F \in \text{Filt}(A)$ is implicative, if it is closed under *modus ponens*, i.e., for every $a, b \in A$, if $a, a \rightarrow^A b \in F$, then $b \in F$.

The set of all implicative filters of $A$ will be denoted by $\text{Filt}_\rightarrow(A)$.

**Theorem**

Let $A \in \text{MV}$. The Leibniz $\mathcal{L}_{\leq \infty}$-filters of $A$ coincide with the implicative lattice filters of $A$. That is,

$$\mathcal{F}i^*_L(A) = \text{Filt}_\rightarrow(A).$$

- The Suszko $\mathcal{L}_{\leq \infty}$-filters are a strict subset of the Leibniz $\mathcal{L}_{\leq \infty}$-filters.

**The strong version of $\mathcal{L}_{\leq \infty}$**

**Theorem**

$(\mathcal{L}_{\leq \infty})^+$ is the Lukasiewicz infinite valued logic, i.e., $(\mathcal{L}_{\leq \infty})^+ = \mathcal{L}_\infty$. 

Logics preserving degrees of truth from varieties of residuated lattices

\[ \mathcal{L} = \langle \land, \lor, \to, \circ, T \rangle \]

**Definition**

An (integral commutative) *residuated lattice* is an \( \mathcal{L} \)-algebra \( \mathbf{A} = \langle A, \land^A, \lor^A, \to^A, \circ^A, 1 \rangle \) such that:

(i) The reduct \( \langle A, \land^A, \lor^A \rangle \) is a lattice;

(ii) \( \langle A, \circ^A, 1 \rangle \) is a commutative monoid;

(iii) \( \to^A \) is the residuum of \( \circ^A \), that is, for every \( a, b \in A \), \( a \circ^A c \leq b \) iff \( c \leq a \to^A b \);

(iv) 1 is the top element of \( A \), that is, for every \( a \in A \), \( a \lor^A 1 = 1 \).

The class of all (integral commutative) residuated lattices will be denoted by \( \mathsf{RL} \).
Logics preserving degrees of truth from varieties of residuated lattices

- \( \mathcal{L} = \langle \wedge, \vee, \to, \odot, T \rangle \)

**Definition**

An (integral commutative) *residuated lattice* is an \( \mathcal{L} \)-algebra \( A = \langle A, \wedge^A, \vee^A, \to^A, \odot^A, 1 \rangle \) such that:

(i) The reduct \( \langle A, \wedge^A, \vee^A \rangle \) is a lattice;

(ii) \( \langle A, \odot^A, 1 \rangle \) is a commutative monoid;

(iii) \( \to^A \) is the residuum of \( \odot^A \), that is, for every \( a, b \in A \), \( a \odot^A c \leq b \) iff \( c \leq a \to^A b \);

(iv) \( 1 \) is the top element of \( A \), that is, for every \( a \in A \), \( a \vee^A 1 = 1 \).

The class of all (integral commutative) residuated lattices will be denoted by \( \text{RL} \).

\( \models^\leq_{\text{RL}} \) is not truth-equational neither protoalgebraic.
Logics preserving degrees of truth from varieties of residuated lattices

\[ \mathcal{L} = \langle \land, \lor, \rightarrow, \odot, T \rangle \]

**Definition**

An (integral commutative) *residuated lattice* is an \( \mathcal{L} \)-algebra \( \mathcal{A} = \langle A, \land^A, \lor^A, \rightarrow^A, \odot^A, 1 \rangle \) such that:

(i) The reduct \( \langle A, \land^A, \lor^A \rangle \) is a lattice;

(ii) \( \langle A, \odot^A, 1 \rangle \) is a commutative monoid;

(iii) \( \rightarrow^A \) is the residuum of \( \odot^A \), that is, for every \( a, b \in A \), \( a \odot^A c \leq b \) iff \( c \leq a \rightarrow^A b \);

(iv) \( 1 \) is the top element of \( A \), that is, for every \( a \in A \), \( a \lor^A 1 = 1 \).

The class of all (integral commutative) residuated lattices will be denoted by \( \mathbf{RL} \).

\( \models_{\mathbf{RL}} \) is not truth-equational neither protoalgebraic.

For every \( \mathcal{A} \in \mathbf{RL} \), \( \mathcal{F}i_{\models_{\mathbf{RL}}} (\mathcal{A}) = \text{Filt}(\mathcal{A}) \).
Logics preserving degrees of truth from varieties of residuated lattices

- $\mathcal{L} = \langle \land, \lor, \to, \odot, \top \rangle$

**Definition**

An (integral commutative) *residuated lattice* is an $\mathcal{L}$-algebra $A = \langle A, \land^A, \lor^A, \to^A, \odot^A, 1 \rangle$ such that:

(i) The reduct $\langle A, \land^A, \lor^A \rangle$ is a lattice;

(ii) $\langle A, \odot^A, 1 \rangle$ is a commutative monoid;

(iii) $\to^A$ is the residuum of $\odot^A$, that is, for every $a, b \in A$, $a \odot^A c \leq b$ iff $c \leq a \to^A b$;

(iv) $1$ is the top element of $A$, that is, for every $a \in A$, $a \lor^A 1 = 1$.

The class of all (integral commutative) residuated lattices will be denoted by $RL$.

- $\models_{RL}$ is not truth-equational neither protoalgebraic.
- For every $A \in RL$, $\mathcal{F}i_{\models_{RL}}(A) = \text{Filt}(A)$.
- $\text{Alg}^*(\models_{RL}) = \text{Alg}(\models_{RL}) = RL$. 
**Logics preserving degrees of truth from varieties of residuated lattices**

**Leibniz and Suszko $\equiv_{RL}$-filters**

**Definition**

Let $A \in RL$. A lattice filter $F \in \text{Filt}(A)$ is *implicative*, if it is closed under *modus ponens*, i.e., for every $a, b \in A$, if $a, a \rightarrow^A b \in F$, then $b \in F$.

The set of all implicative filters of $A$ will be denoted by $\text{Filt}_{\rightarrow}(A)$.
Leibniz and Suszko $\models_{RL} \leq$-filters

**Definition**

Let $A \in RL$. A lattice filter $F \in \text{Filt}(A)$ is *implicative*, if it is closed under *modus ponens*, i.e., for every $a, b \in A$, if $a, a \rightarrow^A b \in F$, then $b \in F$.

The set of all implicative filters of $A$ will be denoted by $\text{Filt}_{\rightarrow}(A)$.

**Theorem**

*Let $A \in RL$. The Leibniz $\models_{RL} \leq$-filters of $A$ coincide with the implicative lattice filters of $A$. That is,*

$$\mathcal{F}i^*(A) = \text{Filt}_{\rightarrow}(A).$$

- The Suszko $\models_{RL} \leq$-filters are a strict subset of the Leibniz $\models_{RL} \leq$-filters.
**Logics preserving degrees of truth from varieties of residuated lattices**

**Leibniz and Suszko $\leq_{RL}$-filters**

**Definition**

Let $A \in RL$. A lattice filter $F \in \text{Filt}(A)$ is *implicative*, if it is closed under *modus ponens*, i.e., for every $a, b \in A$, if $a, a \rightarrow^A b \in F$, then $b \in F$.

The set of all implicative filters of $A$ will be denoted by $\text{Filt}_{\rightarrow}(A)$.

**Theorem**

*Let $A \in RL$. The Leibniz $\leq_{RL}$-filters of $A$ coincide with the implicative lattice filters of $A$. That is,*

$$\mathcal{F}i^* \leq_{RL}(A) = \text{Filt}_{\rightarrow}(A).$$

- The Suszko $\leq_{RL}$-filters are a strict subset of the Leibniz $\leq_{RL}$-filters.

**The strong version of $\leq_{RL}$**

**Theorem**

$(\leq_{RL})^+$ is the logic preserving truth wrt. $RL$, i.e., $(\leq_{RL})^+ = \leq^1_{RL}$. 
Thank you!
References I


Let $S$ be a protoalgebraic logic and $A$ an algebra. An $S$-filter $F \in \mathcal{F}_S A$ is a **Leibniz $S$-filter of** $A$, if it is the least element of

$$[F] = \{ G \in \mathcal{F}_S A : \Omega^A(F) = \Omega^A(G) \}.$$