Constructions of Pretoposes

Vassilis Aravantinos - Sotiropoulos
University of Patras
Patras, Greece

Panagis Karazeris
University of Patras
Patras, Greece

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The general answer is: When the theories, seen as categories, have equivalent completions of some kind

Focusing on theories arising in geometrical contexts: Coherent theories, comprising sentences of the form $\forall\vec{x}(\varphi(\vec{x}) \to \psi(\vec{x}))$, where φ, ψ are built from atomic formulae by $\exists$, $\wedge$, $\vee$.

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A coherent category has the same category of models as its **pretopos completion**: Adding formally finite coproducts and, roughly, quotients of equivalence relations.

Conceptual Completeness for Pretoposes (M. Makkai, G. Reyes 1976) An interpretation of theories $I\colon \mathbb{T} \to \mathbb{T}'$ induces an equivalence between the categories of models iff $P(I)\colon P(\mathbb{T}) \to P(\mathbb{T}')$ is an equivalence between the respective pretopos completions of the theories.

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Conceptual Completeness for Pretoposes (M. Makkai, G. Reyes 1976, **A. Pitts** 1986) An interpretation of theories $I \colon \mathbb{T} \to \mathbb{T}'$ induces an equivalence between the categories of models **in a sufficient class of toposes** iff $P(I) \colon P(\mathbb{T}) \to P(\mathbb{T}')$ is an equivalence between the respective pretopos completions of the theories.

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In modern terminology: The pretopos completion of a coherent category is the (**exact completion** of the (**finite coproduct completion** of it) **as a regular category**).

In particular we find in *Sketches of an Elephant*, A1.4.5:

Proposition 1: If $\mathcal{C}$ is a coherent category, then its free completion under finite coproducts, $\mathrm{fam}\mathcal{C}$, is also coherent.

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Proposition 2: If $\mathcal{C}$ is a regular category, then its free completion under finite coproducts, $\mathrm{fam}\mathcal{C}$, is also regular.

Proof: The proof of 1. relies on the existence of suprema of subobjects, while we characterize strong epimorphisms $(\alpha, f_i)\colon (C_i)_{i\in I} \to (C'_j)_{j\in J}$ in $\mathrm{fam}\mathcal{C}$, as those arrows given by a surjection $\alpha\colon I \to J$ such that each $f_i\colon C_i \to C'_{\alpha(i)}$ is a strong epimorphism. ∎

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For any exact category $\mathcal{E}$, any regular functor $F\colon \mathcal{D} \to \mathcal{E}$,

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\zeta_{\mathcal{D}}} & \mathcal{D}_{ex/reg} \\ & {\scriptstyle F}\searrow & \downarrow{\scriptstyle F^*} \\ & & \mathcal{E}, \end{array}$$

F^* regular, unique up to natural iso.

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$(\mathrm{fam}(-))_{ex/reg}\colon \mathrm{REG} \to \mathrm{PRETOP}$ provides a left biadjoint to the forgetful functor.

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Relying on the work of Pitts we can give a constructive, categorical one.

Theorem: A regular functor $I \colon \mathbb{T} \to \mathbb{T}'$ induces an equivalence between the categories of models **in a sufficient class of toposes** iff $I_{ex/reg} \colon \mathbb{T}_{ex/reg} \to \mathbb{T}'_{ex/reg}$ is an equivalence.

Proof: Uses a result of [MR], that a regular functor to an exact category, which is fully faithful and **covering**, is also essentially surjective on objects.

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Lemma 1: If $I : \mathcal{K} \to \mathcal{L}$ is a regular functor, such that the induced $(\mathrm{fam} I)_{ex/reg} : (\mathrm{fam}\mathcal{K})_{ex/reg} \to (\mathrm{fam}\mathcal{L})_{ex/reg}$ is covering, then I is covering (hence so is $I_{ex/reg} : \mathcal{K}_{ex/reg} \to \mathcal{L}_{ex/reg}$)

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Lemma 2: If $I : \mathcal{K} \to \mathcal{L}$ is a regular functor, such that the induced $(\mathrm{fam} I)_{ex/reg} : (\mathrm{fam}\mathcal{K})_{ex/reg} \to (\mathrm{fam}\mathcal{L})_{ex/reg}$ is fully faithful, then $I_{ex/reg} : \mathcal{K}_{ex/reg} \to \mathcal{L}_{ex/reg}$ is fully faithful.

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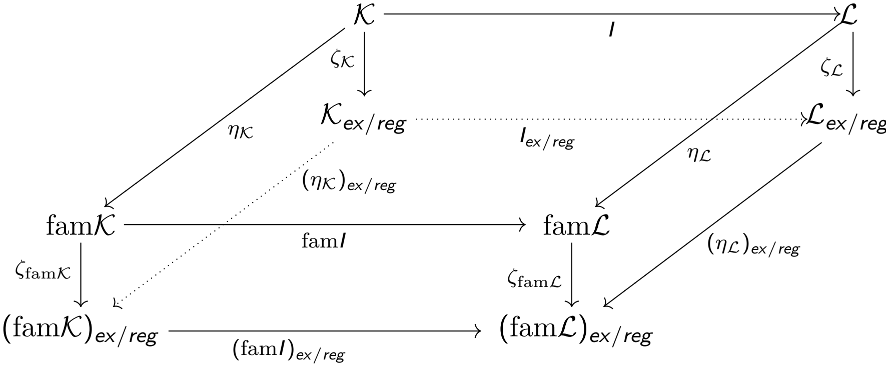
The proof of Lemma 2 is obtained by chasing around bijections of hom-sets in the following diagram


$\mathcal{K}$
$\mathcal{L}$
I
$\zeta_{\mathcal{K}}$
$\zeta_{\mathcal{L}}$
$\mathcal{K}_{ex/reg}$
$\mathcal{L}_{ex/reg}$
$I_{ex/reg}$
$\eta_{\mathcal{K}}$
$\eta_{\mathcal{L}}$
$(\eta_{\mathcal{K}})_{ex/reg}$
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$\mathrm{fam}I$
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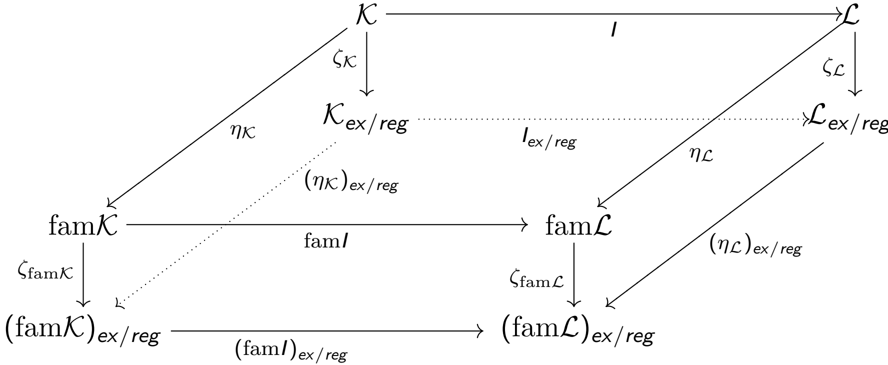

(commutative up to natural isomorphism), provided $(\eta_{\mathcal{K}})_{ex/reg}$, $(\eta_{\mathcal{L}})_{ex/reg}$ are fully faithful.

The latter is obtained by the following lemma, that seems to be of independent interest.

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Main Lemma: If $F\colon \mathcal{E} \to \mathcal{F}$ is fully faithful regular functor then $F^* = F_{ex/reg}\colon \mathcal{E}_{ex/reg} \to \mathcal{F}_{ex/reg}$ is fully faithful.

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Main Lemma: If $F\colon \mathcal{E} \to \mathcal{F}$ is fully faithful regular functor then $F^* = F_{ex/reg}\colon \mathcal{E}_{ex/reg} \to \mathcal{F}_{ex/reg}$ is fully faithful.

Proof: In particular let us examine why is $F_{ex/reg}$ faithful. That it is full follows similarly (and more easily). Recall how $F_{ex/reg}$ functions. Considering presentations of the objects of $\mathcal{E}_{ex/reg}$ as coequalizers of equivalence relations in $\mathcal{E}$ we have

$$
\begin{array}{ccccc}
\zeta_{\mathcal{E}} E' & \overset{s_1}{\underset{s_2}{\rightrightarrows}} & \zeta_{\mathcal{E}} E & \xrightarrow{q_X} & X \\
 & & & & \downarrow f \\
\zeta_{\mathcal{E}} H' & \overset{r_1}{\underset{r_2}{\rightrightarrows}} & \zeta_{\mathcal{E}} H & \xrightarrow{q_Y} & Y
\end{array}
$$

$$
F_{ex/reg}\bar{\downarrow}
$$

$$
\begin{array}{ccccc}
\zeta_{\mathcal{F}} FE' & \overset{Fs_1}{\underset{Fs_2}{\rightrightarrows}} & \zeta_{\mathcal{F}} FE & \xrightarrow{q_{F^*X}} & F^*X \\
 & & & & \\
\zeta_{\mathcal{F}} FH' & \overset{Fr_1}{\underset{Fr_2}{\rightrightarrows}} & \zeta_{\mathcal{F}} FH & \xrightarrow{q_{F^*Y}} & F^*Y
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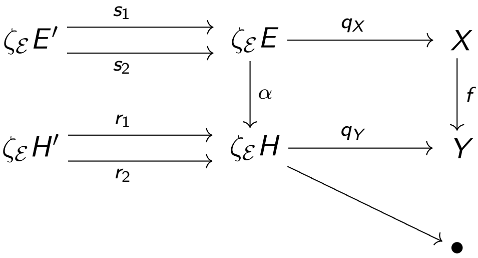

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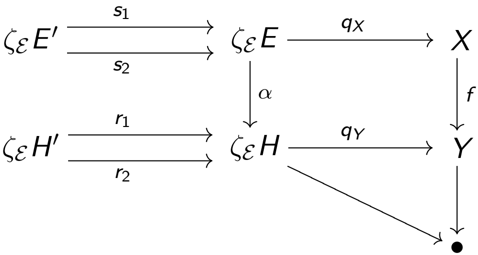

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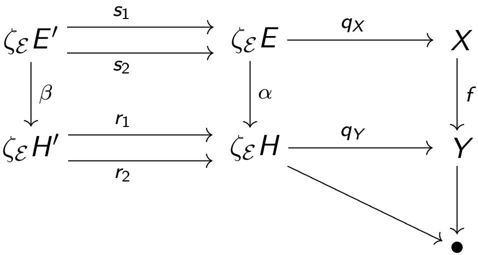

Consider f_1, f_2 with $F_{ex/reg} f_1 = F_{ex/reg} f_2$, f_i induced by α_i ,β_i

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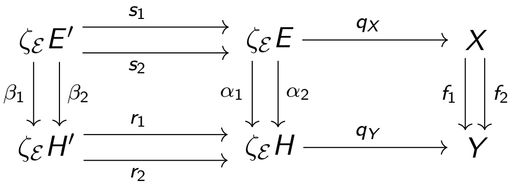

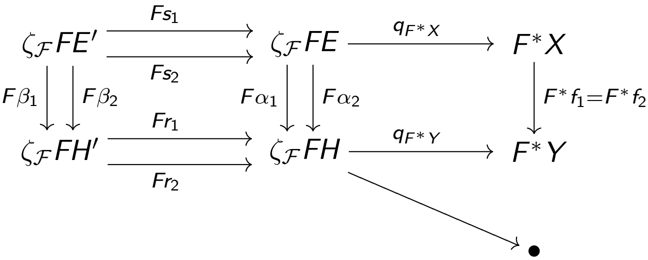

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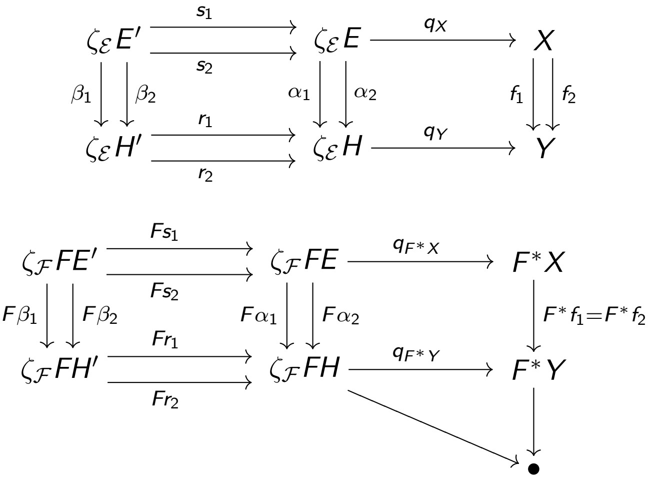

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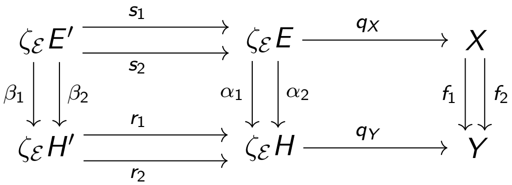

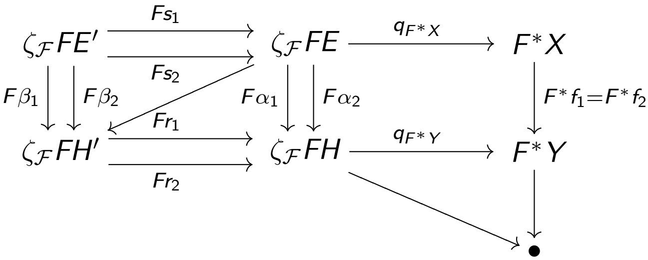

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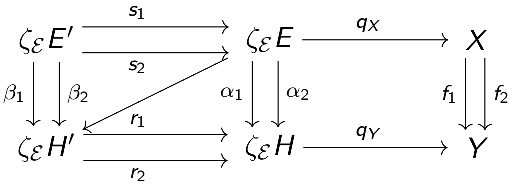
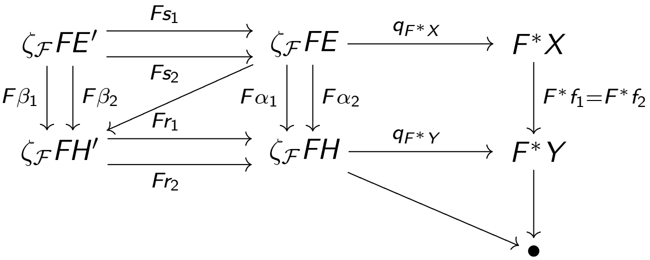
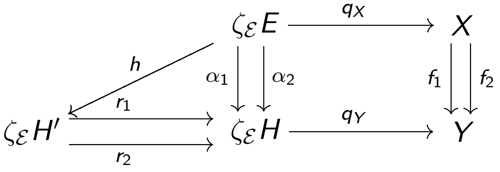




The last induced arrow comes from some $E \to H'$ since F (and $\zeta_{\mathcal{E}}$) is full, and since F is faithful it fits in a commutative diagram



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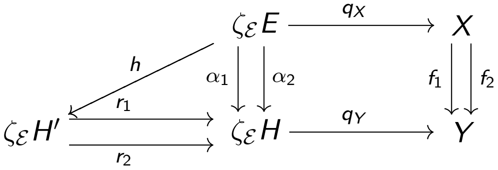

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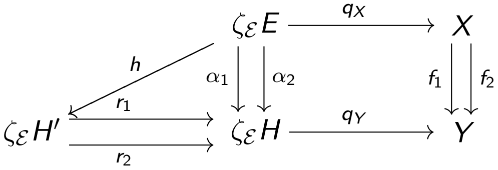
$$f_1 \cdot q_X = q_Y \cdot \alpha_1 = q_Y \cdot r_1 \cdot h = q_Y \cdot r_2 \cdot h = q_Y \cdot \alpha_2 = f_2 \cdot q_X$$

The last induced arrow comes from some $E \to H'$ since F (and $\zeta_{\mathcal{E}}$) is full, and since F is faithful it fits in a commutative diagram



$r_i \cdot h = \alpha_i$, $i = 1, 2$, hence

$$f_1 \cdot q_X = q_Y \cdot \alpha_1 = q_Y \cdot r_1 \cdot h = q_Y \cdot r_2 \cdot h = q_Y \cdot \alpha_2 = f_2 \cdot q_X$$

q_X is an epi so $f_1 = f_2$

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The only written account of the details (Makkai, Ultraproducts in Categorical Logic) seems unnecessarily complicated: Perform the factorization in the category of small categories with finite limits, obtain a coherent category, then complete to a pretopos.

Instead we can simplify using Benabou's notion of regular congruence for studying quotients in the category of small regular categories (forming **categories of fractions** such that the passage to them are regular functors) combined with the following result of general interest

Proposition: If $\mathcal{C}$ is a category with coproducts, Σ is a class of morphisms of $\mathcal{C}$ admitting a calculus of right fractions and such that, for all $s\colon A \to B$ and $t\colon C \to D$ in Σ, $s \sqcup t\colon A \sqcup C \to B \sqcup D$ is in Σ, then the category of fractions $\mathcal{C}[\Sigma^{-1}]$ has coproducts and the quotient functor $P_\Sigma\colon \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ preserves them.

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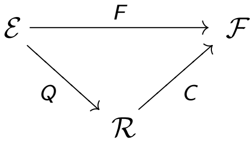
Proof: Given objects A, B in $\mathcal{C}[\Sigma^{-1}]$, their coproduct is given by $A \sqcup B$, because if $A \xleftarrow{s} I \xrightarrow{f} C$ and $B \xleftarrow{t} J \xrightarrow{g} C$ are two arrows from A to C and from B to C, respectively, in $\mathcal{C}[\Sigma^{-1}]$, then $A \sqcup B \xleftarrow{s \sqcup t} I \sqcup J \xrightarrow{[f,g]} C$ is the required unique factorization through $A \sqcup B$ making the two triangles commutative. ∎

We can now perform the factorization first in REG

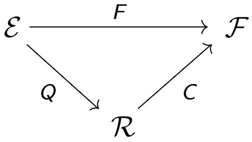


We can now perform the factorization first in REG

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad F \quad} & \mathcal{F} \\ & {\scriptstyle Q}\searrow \quad \nearrow{\scriptstyle C} & \\ & \mathcal{R} & \end{array}$$

obtaining a regular extensive category,

We can now perform the factorization first in REG



obtaining a regular extensive category, then completing it to $\mathcal{R}_{ex/reg}$ to get the factorization in pretoposes

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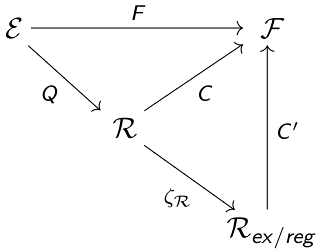

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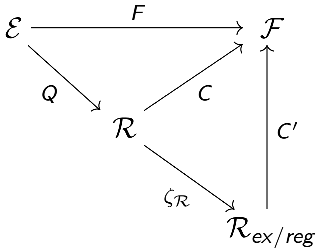

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Note: The passage from an exact category to a regular category of fractions does not preserve exactness: The coreflection to the inclusion of Stone spaces into the dual category of presheaves on finite Boolean algebras provides a counterexample.