Bisimulation and path logic for sheaves

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Outline

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In general path categories are just assumed to have an initial object I, while presheaf models F are only assumed to be rooted, i.e $F(I) = \{*\}$.

From presheaves to labelled transition systems.

It was observed in a follow-up paper [3] that presheaves $F : \mathbf{P}^{op} \rightarrow \mathbf{Set}$ can in turn be made into relational structures:

- $W = \{(P, x) | P \in \mathbf{P}_0, x \in F(P)\}$;
- for every morphism m in $\mathbf{P}_1$ define $(P, x)R_m(P', x')$ iff $m : P \rightarrow P'$ in $\mathbf{P}$ and $F(m)(x') = x$.

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We can devise a notion of bisimulation on these relational structures that matches the bisimulation in terms of open maps. Such bisimulation is called **path bisimulation**.

Path logic $\mathbb{PL}_{\mathbf{P}}$

On presheaf models over $\mathbf{P}$, the logic that is characteristic for path-bisimulation is called **path logic** ([2]):

$$\varphi ::= \quad \bot \mid \neg\varphi \mid \bigwedge_{j \in J} \varphi_j \mid \langle m \rangle \varphi \mid \overline{\langle m \rangle} \varphi$$

where $m \in \mathbf{P}_1$ and J has cardinality $max\{|\mathbf{P}(X, Y)| | X, Y \in \mathbf{P}_0\}$.

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This logic is interpreted on the relational counterparts of presheaf models. Given an object P of **P**, a presheaf model F and $p \in F(P)$:

$$(P,p) \vDash \langle m\rangle\varphi \quad \text{iff} \quad \text{there exist } P', p', \ (P,x)R_m(P',x') \text{ and} (P',p') \vDash \varphi.$$

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The condition for $\overline{\langle m\rangle}\varphi$ is that of a backward-looking modality.

Outline

Definition

Call $\mathbf{T}_L$ the category of pointed transition systems with labels L, where morphisms preserve transitions and initial states. Call $\mathbf{Tree}_L$ the subcategory of $\mathbf{T}_L$ consisting of trees. Call $\mathbf{Bran}_L$ (a skeleton of) the subcategory consisting of only branches, i.e. linear paths.

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We can encode pointed transition systems with labels L into the presheaf category $\mathbf{Set}^{\mathbf{Bran}_L^{op}}$ ([2], [3]):

$$\begin{array}{ccc} \mathbf{T}_L & \xrightarrow{Pre} & rooted(\mathbf{Set}^{\mathbf{Bran}_L^{op}}) \\ & {\scriptstyle u}\searrow & \downarrow \simeq \\ & & \mathbf{Tree}_L \end{array}$$

Theorem ([2])

For transition systems we have that:

- *Open maps are p-morphisms*
- *given two transition systems M_1 and M_2, TFAE:*
  1. *M_1 and M_2 are bisimilar*
  2. *there is a span of open maps between (the presheaf models of) M_1 and M_2*
  3. *there is a path-bisimulation between (the relational counterpart of the presheaf models of) M_1 and M_2*

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can we **express interesting properties of sheaves** over topological spaces **with path logic**?

Given a topological space $\mathbb{X}$, we consider the sheaves from the poset category $\mathit{Open}(\mathbb{X})$, hence the modalities of path logic are labelled by the inclusions between opens.

A case study: Contextuality

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and on arrows as function restrictions.
Elements of O^U are called **sections**; they are a joint assignment of outcomes to the measurements in U. Elements of O^X are called **global sections**.

A case study: Contextuality

The authors single out two key properties of empirical systems:

1. An empirical system S is called **strongly contextual** if there are no global sections: $S(X) = \emptyset$.
2. An empirical system S is called **weakly contextual** if there is a section s over a subset U that cannot be extended to a global section.

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It turns out that both properties can be captured in the path logic over the poset $\wp(X)$ (a discrete topology):

1. $\neg\langle\emptyset, X\rangle\top$
2. $\bigvee_{U\subseteq X}\langle\emptyset, U\rangle\neg\langle U, X\rangle\top$

Beyond Contextuality

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- the sheaf of sections of a covering map $\pi : \mathbb{X} \to \mathbb{X}'$: the satisfaction of the contextuality formulas tells us to what extent the covering space $\mathbb{X}$ 'looks like' $\mathbb{X}'$ globally.
- flabby sheaves over $Open(\mathbb{X})$: shaves whose restriction maps are surjective, can be defined in path logic by $\bigwedge_{U \in Open(\mathbb{X})} [\emptyset, U]\langle U, X\rangle \top$

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Path bisimulation on sheaves

We saw that path logic precisely captures path bisimulations and that given a pair of path bisimilar sheaves, we can construct a span of open maps connecting them. However, the presheaf at the "vertex" may not be a sheaf.

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Definition (Locality axiom for path bisimulations)

Suppose given a covering $(U_i)_{i \in I}$ of an open U, sheaves $Q_1, Q_2 : Open(\mathbb{X})^{op} \to \mathbf{Set}$ and a path bisimulation Z. We say Z satisfies **Locality** if for all $p \in Q_1(U)$ and $q \in Q_2(U)$ such that $(p|_{U_i}^{Q_1}, q|_{U_i}^{Q_2}) \in Z_{U_i}$ for all $i \in I$, we have $(p, q) \in Z_U$.

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Proposition

If two sheaves are related by a path bisimulation satisfying the Locality axiom then they are also related by a span of open maps.

Path bisimulation on sheaves

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Definition (Gluing axiom for path bisimulations)

Suppose given $(U_i)_{i\in I}$ covering of U, sheaves $Q_1, Q_2 : Open(\mathbb{X})^{op} \to \mathbf{Set}$ and a path bisimulation Z. We say Z satisfies **Glueing** if there are two families $(p_i)_{i\in I}$ and $(q_i)_{i\in I}$ with $p_i \in Q_1(U_i)$ and $q_i \in Q_2(U_i)$ for all i and moreover for all i, j $(p_i|^{Q_1}_{U_i\cap U_j}, q_j|^{Q_2}_{U_i\cap U_j}) \in Z$ then there exist two elements $p \in Q_1(U)$ and $q \in Q_2(U)$ such that $(p, q) \in Z$ and, for all i, $(p|^{Q_1}_{U_i}, q_i) \in Z$ and $(q|^{Q_2}_{U_i}, p_i) \in Z$.

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Definition

A path bisimulation Z is said to be **di-functional** if $(p,q) \in Z$, $(p',q) \in Z$ and $(p',q') \in Z$ entail $(p,q') \in Z$.

Path bisimulation on sheaves

Theorem

Two sheaves Q_1 and Q_2 are related by a co-span of open maps

$$Q_1 \to P \leftarrow Q_2$$

where P is a sheaf, if and only if they are related by a di-functional path bisimulation that satisfies the Gluing and Locality axioms.

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Theorem

Two sheaves Q_1 and Q_2 are related by a co-span of open maps

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where P is a sheaf, if and only if they are related by a di-functional path bisimulation that satisfies the Gluing and Locality axioms.

Proposition

Spans and co-spans of open maps are equivalent in the category of sheaves over a topological space if and only if the unit $\eta_P : P \to L(P)$ is open, where L is the sheafification functor.

Capturing sheaves in hybrid path logic

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Proposition

A rooted presheaf P is a sheaf iff P satisfies $Loc \wedge Glu$

Conclusions and further work

We have:

1. shown that interesting properties of sheaves can be expressed in path logic, taking as initial case study the Contextuality approach of [1]
2. studied how path bisimulation behaves on sheaves
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There are quite some open problems, besides those already mentioned:

- completeness, decidability, correspondence for path logic (some work on this already done)?
- how does path logic over topological spaces relate to the topological interpretation of modal logic?
- in light of the analysis of contextuality: is path logic a good logic for context change?

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