On everywhere strongly logifiable algebras

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Algebraizable logic

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- In this case K is unique. K is called the equivalent algebraic semantics of $\mathcal{L}$.

Equivalent algebraic semantics

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Examples:

- classical logic $\longleftrightarrow$ Boolean algebras
- intuitionistic logic $\longleftrightarrow$ Heyting algebras
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- Hájek's basic logic $\longleftrightarrow$ BL-algebras
- full Lambek calculus $\longleftrightarrow$ residuated lattices

Variety problem

Definition

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Find logically meaningful sufficient conditions under which an algebraizable logic is strongly algebraizable too.

Definition

1. $\mathcal{L}$ is selfextensional when the relation $\dashv\vdash_{\mathcal{L}}$ is a congruence.
2. $\mathcal{L}$ is Fregean when the relation $\{\langle\varphi,\psi\rangle : \varphi, \Gamma \dashv\vdash_{\mathcal{L}} \Gamma, \psi\}$ is a congruence for every Γ.

Variety problem

Theorem (Czelakowski and Pigozzi)

1. If $\mathcal{L}$ is a finitary Fregean logic with the uniterm DDT, then it is strongly algebraizable wrt a variety of Hilbert algebras expanded with compatible operators.

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Theorem (Font and Jansana)

Let $\mathcal{L}$ be a finitary selfextensional logic. If $\mathcal{L}$ has either a conjunction or the uniterm DDT, then $\mathrm{Alg}\,\mathcal{L}$ is a variety.

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Definition

A finite non-trivial algebra $\boldsymbol{A}$ is everywhere strongly logifiable if the matrix $\langle \boldsymbol{A}, F \rangle$ determines a strongly algebraizable logic with equivalent algebraic semantics $\mathbb{V}(\boldsymbol{A})$, for every $F \in \mathcal{P}(A) \smallsetminus \{\emptyset, A\}$.

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Let $\boldsymbol{A}$ be a non-trivial primal algebra.

- Pick $F \in \mathcal{P}(A) \smallsetminus \{\emptyset, A\}$.
- Choose any $1 \in F$ and $0 \notin F$.

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Let $\boldsymbol{A}$ be a non-trivial primal algebra.

- Pick $F \in \mathcal{P}(A) \smallsetminus \{\emptyset, A\}$.
- Choose any $1 \in F$ and $0 \notin F$.
- Consider the functions $\Box \colon A \to A$ and $\lhd\!\rhd \colon A^2 \to A$ s.t.

$$\Box(a) := \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{otherwise} \end{cases} \qquad a \lhd\!\rhd b := \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases}$$

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By primality $\Box(x)$ and $x \lhd\rhd y$ are term functions.

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  By primality $\Box(x)$ and $x \lhd\!\rhd y$ are term functions.
- The logic of $\langle \boldsymbol{A}, F \rangle$ is algebraizable with equivalent algebraic semantics $\mathbb{V}(\boldsymbol{A})$ via

$$\boldsymbol{\tau}(x) = \{\Box(x) \approx 1(x)\} \text{ and } \boldsymbol{\rho}(x, y) = \{x \lhd\!\rhd y\}.$$

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2. An algebra $\boldsymbol{A}$ is n-permutable for $n \geq 2$ if

$$\phi \vee \eta = \theta_1 \circ \cdots \circ \theta_n \text{ where } \theta_i = \begin{cases} \phi & \text{if } i \text{ is even} \\ \eta & \text{otherwise} \end{cases}$$

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   for every $\phi, \eta \in \mathrm{Con}\boldsymbol{A}$.
3. A variety V is n-permutable when so are its members.
4. A variety V is point regular if there is a constant 1 such that for every $\boldsymbol{A} \in V$ and $\theta, \phi \in \boldsymbol{A}$:

$$\text{if } 1/\theta = 1/\phi, \text{ then } \theta = \phi.$$

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Theorem

Let $\boldsymbol{A}$ be a non-trivial finite algebra. The following conditions are equivalent:

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(iii) $\boldsymbol{A}$ is simple, without proper subalgebras and $\mathbb{V}(\boldsymbol{A})$ is minimal and point regular.

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(iii) $\boldsymbol{A}$ is simple, without proper subalgebras and $\mathbb{V}(\boldsymbol{A})$ is minimal and point regular.

(iv) $\boldsymbol{A}$ is simple, constantive and $\mathbb{V}(\boldsymbol{A})$ is congruence distributive and n-permutable for some $n \geq 2$.

Sketch of the proof (ii)⇒(iii)

Remark

If

the logic of $\langle \boldsymbol{A}, \{a\}\rangle$ is strongly algebraizable with equivalent algebraic semantics $\mathbb{V}(\boldsymbol{A})$ for every $a \in A$,

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- $\boldsymbol{A}$ is constantive.
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- $\mathbb{V}(\boldsymbol{A}) = \mathbb{Q}(\boldsymbol{A})$. Hence $\mathbb{V}(\boldsymbol{A})_{\mathsf{si}} \subseteq \mathbb{ISP}_{\mathsf{u}}(\boldsymbol{A}) = \mathbb{I}\{\boldsymbol{A}\}$.

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Remark

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- There are $1 \in F$ and $0 \notin F$ such that $\boldsymbol{A}|_{\{0,1\}}$ is polynomially equivalent to the two-element Boolean algebra.

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- There are $1 \in F$ and $0 \notin F$ such that $\boldsymbol{A}|_{\{0,1\}}$ is polynomially equivalent to the two-element Boolean algebra.
- For every different $a, b \in A$ there is a polynomial p_{ab} such that

$$p_{ab}[A] = \{0, 1\} \text{ and } p_{ab}(a) \neq p_{ab}(b).$$

Sketch of the proof

- Let $F = \{a_1, \ldots, a_n\}$ and consider the terms

$$x \lhd\rhd y \coloneqq \bigwedge\{p_{ab}(x) \leftrightarrow p_{ab}(y) : a, b \in A \text{ and } a \neq b\}$$

$$\Box(x) \coloneqq \bigvee_{i \leq n} \Big(\bigwedge \Big\{ p_{a_i b}(x) \leftrightarrow p_{a_i b}(a_i(x)) : b \in A \smallsetminus \{a_i\} \Big\} \Big).$$

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- It is easy to prove that:

$$\mathit{\Gamma} \vdash_{\mathcal{L}} \varphi \iff \{\Box(\gamma) \approx 1(\gamma) : \gamma \in \mathit{\Gamma}\} \vDash_{\boldsymbol{A}} \Box(\varphi) \approx 1(\varphi)$$

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- $\mathcal{L}$ is algebraizable with equivalent algebraic semantics $\mathbb{Q}(\boldsymbol{A})$ through the structural transformers

$$\boldsymbol{\tau}(x) = \{\Box(x) \approx 1(x)\} \text{ and } \boldsymbol{\rho}(x, y) = \{x \lhd\rhd y\}.$$

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$$\boldsymbol{\tau}(x) = \{\Box(x) \approx 1(x)\} \text{ and } \boldsymbol{\rho}(x, y) = \{x \lhd\rhd y\}.$$

- By Jónsson's lemma $\mathbb{Q}(\boldsymbol{A}) = \mathbb{V}(\boldsymbol{A})$.

Contents

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Let $\boldsymbol{A}$ be finite and non-trivial. $\boldsymbol{A}$ is everywhere strongly logifiable if and only if $\boldsymbol{A}$ has no proper subalgebra and there is $a \in A$ such that the logic $\mathcal{L}$ of $\langle \boldsymbol{A}, \{a\} \rangle$ has theorems and $\mathrm{Alg}^*\mathcal{L} = \mathbb{V}(\boldsymbol{A})$.

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Corollary

Let $\boldsymbol{A}$ be finite and non-trivial. $\boldsymbol{A}$ is everywhere strongly logifiable if and only if $\boldsymbol{A}$ has no proper subalgebra and there is $a \in A$ such that the logic $\mathcal{L}$ of $\langle \boldsymbol{A}, \{a\} \rangle$ has theorems and $\mathrm{Alg}^{*}\mathcal{L} = \mathbb{V}(\boldsymbol{A})$.

Corollary

Let $\boldsymbol{A}$ be (finite and non-trivial) without proper subalgebras and a constant 1.

Variety problem

Corollary

Let $\boldsymbol{A}$ be finite and non-trivial. $\boldsymbol{A}$ is everywhere strongly logifiable if and only if $\boldsymbol{A}$ has no proper subalgebra and there is $a \in A$ such that the logic $\mathcal{L}$ of $\langle \boldsymbol{A}, \{a\} \rangle$ has theorems and $\mathrm{Alg}^*\mathcal{L} = \mathbb{V}(\boldsymbol{A})$.

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Let $\boldsymbol{A}$ be (finite and non-trivial) without proper subalgebras and a constant 1. For the logic $\mathcal{L}$ of $\langle \boldsymbol{A}, \{1\} \rangle$ the following are equivalent:

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(i) The class $\mathrm{Alg}^*\mathcal{L}$ is a variety.

(ii) $\boldsymbol{A}/\Omega\{1\}$ is everywhere strongly logifiable.

Some properties

- If $\boldsymbol{A}$ is everywhere strongly logifiable, then for every $F \in \mathcal{P}(A) \smallsetminus \{\emptyset, A\}$ the algebraizability of the logic of $\langle \boldsymbol{A}, F \rangle$ is witnessed by two single element structural transformers.

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- In congruence permutable varieties the notion of a everywhere strongly logifiable algebra coincides with the one of a primal algebra.
- For two-element algebras the notion of an everywhere strongly logifiable algebra and the one of a primal algebra coincide.

A counterexample

Example

- Pick a finite bounded poset $\langle A, \leq, 0, 1 \rangle$ such that $|A| \geq 3$.

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- Consider $\boldsymbol{A} = \langle A, \to, \Delta, \{a : a \in A\}\rangle$ where

$$x \to y := \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases} \qquad \Delta x := \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

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  for every $x, y \in A$.
- $\boldsymbol{A}$ is everywhere strongly logifiable.
- $\boldsymbol{A}$ is not primal:

$$\mathrm{Id}_A \cup (\{1\} \times A) \cup (A \times \{1\})$$

  is the universe of a subalgebra of $\boldsymbol{A} \times \boldsymbol{A}$.

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- More on the analogy with primal algebras...

Thank you!