Pretransitive modal logics of finite depth

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PLAN

1. The Problem
2. Pretransitive logics
3. Adding symmetry (depth=1)
4. Finite depth
5. Main theorem

THE PROBLEM

"Perhaps one of the most intriguing open problems in Modal Logic is the following:

PROBLEM. Do the logics of the form $\mathsf{K} + \Box^n p \to \Box^m p$ have the finite model property?"

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Known cases

$$\mathsf{K}_{\mathsf{n}}^{\mathsf{m}} = \mathsf{K} + \Box^m p \to \Box^n p$$

In some cases the FMP is known:

- $m = n$ is trivial
- $\mathsf{K4} = \mathsf{K}_2^1$
- K_1^2 — the density logic.
- $m \leq 1$ or $n \leq 1$ FMP via filtration [Gabbay, 1972]

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Language, logics and semantic

$$\phi ::= p \mid \bot \mid \phi \to \phi \mid \Box\phi.$$

$$\Box^0 A = A;\ \Box^1 A = \Box A;\ \Box^{n+1} A = \Box\Box^n A;$$

K denotes the minimal normal modal logic.

$$\mathsf{K}_\mathsf{n}^\mathsf{m} = \mathsf{K} + \Box^m p \to \Box^n p$$

For a Kripke frame $F = (W, R)$;

$$F \models \Box^m p \to \Box^n p \Leftrightarrow R^n \subseteq R^m.$$

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The pretransitive case: $m < n$

Definition

A logic L is called pretransitive (or conically expressive), if there exists a formula $\chi(p)$ with a single variable p such that for any Kripke model M with $M \models$ L and for any w in M we have:

$$M, w \models \chi(p) \Leftrightarrow \forall u(wR^*u \Rightarrow M, u \models p),$$

where R^* is the transitive reflexive closure of the acceptability relation on M.

By $\Box^*\varphi$ we mean $\chi(\varphi)$, $\Diamond^*\varphi = \neg\Box^*\neg\varphi$.

For $n > m \geq 1$, all logics K_n^m are pretransitive:

if L $\vdash \Box p \to \Box\Box p$, $\Box^*\varphi = \varphi \wedge \Box\varphi$;

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if L $\vdash \Box^m p \to \Box^n p$, $\Box^*\varphi = \Box^{\leq n-1}\varphi = \varphi \wedge \Box\varphi \wedge \cdots \wedge \Box^{n-1}\varphi$

Lemma (Shehtman, 2009)

L *is pretransitive iff for some* $m \geq 0$ *it contains the formula of* *m-transitivity* $\Box^{\leq m} p \to \Box^{m+1} p$, *where*

$$\Box^{\leq m} \varphi = \bigwedge_{i=0}^{m} \Box^{i} \varphi.$$

$$\mathsf{K}_{\leq m} = \mathsf{K} + \Box^{\leq m} p \to \Box^{m+1} p$$

$F \models \mathsf{K}_{\leq m} \Leftrightarrow R^0 \cup R^1 \cup \cdots \cup R^m$ is transitive.

Problem

For logics $\mathsf{K}_{\leq m} = \mathsf{K} + \Box^{\leq m} p \to \Box^{m+1} p$, the FMP is also unknown for all $m > 1$ (the weakly transitive logic $\mathsf{K}_{\leq 1} = \mathsf{wK4} = \mathsf{K} + \Box p \wedge p \to \Box\Box p$ is known to have the FMP).

Plan of the attack on the problem

To get the FMP we need to conquer 3 infinities:

1. infinite clusters [K & Sh, 2011]
2. infinite branching [K & Sh, 2015] (given that the depth is finite)
3. infinite depth — future work

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Known results with symmetry

If we add the symmetry axiom for $\Box^*$ then the FMP is known:

- $\mathsf{K}_{\leq \mathsf{m}} + p \to \Box^*\Diamond^* p$ FMP via minimal filtration

  [Jansana, 1994]

- $\mathsf{K}_{\mathsf{n}}^{\mathsf{m}} + p \to \Box^*\Diamond^* p$ FMP via filtration through special equivalence

  [K&Sh, TACL 2011]

We define the following formulas:

$$B_1 = p_1 \to \Box^* \Diamond^* p_1,$$
$$B_{n+1} = p_{n+1} \to \Box^* (\Diamond^* p_{n+1} \vee B_n).$$

Proposition

If L is a pretransitive logic and $F \models L$, then

$$F \models B_n \iff h(F) \le n.$$

where $h(F)$ is the depth of frame F.

Theorem

If L is a pretransitive logic then

1. $L + B_1 \supseteq L + B_2 \supseteq L + B_3 \supseteq \ldots \supseteq L.$
2. *If L is consistent then $L + B_1$ (and all $L + B_n$) are consistent.*
3. *If L is canonical then all $L + B_n$ are canonical*

Finite depth

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Corollary

Logics $\mathsf{K}_n^m + B_k$ and $\mathsf{K}_{\leq m} + B_k$ for $n > m \geq 1$ are Kripke complete.

NOTE B_1 is the symmetry axiom for $\Box^*$. So case $k = 1$ is already known.

Main theorem

Theorem

For all $n > m \geq 1,\ h \geq 1$, $\mathsf{K}_\mathsf{n}^\mathsf{m} + B_h$ *and* $\mathsf{K}_{\leq \mathsf{m}} + B_h$ *have the FMP.*

We prove it using filtration. But we need to filter the model several times using different equivalences.

Corollary

Let L *be one of the logics* $\mathsf{K}_\mathsf{n}^\mathsf{m}$, $\mathsf{K}_{\leq \mathsf{m}}$, $n > m \geq 1$.

$$\mathsf{L} \textit{ has the FMP iff } \mathsf{L} = \bigcap_{h \geq 1} (\mathsf{L} + B_h).$$

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Filtration with fine tuning

Let Ψ be a set of formulas, closed under subformulas. We define equivalence relation on model $M = (W, R, V)$

$$x \equiv_\Psi y \Leftrightarrow \forall A \in \Psi(M, x \models A \Leftrightarrow M, y \models A).$$

Consider $\sim \subseteq \equiv_\Psi$.
Model $M' = (W/\sim, R', V')$ is $(\sim, \Psi)$-filtration of $M = (W, R, V)$ if

$$[x]R'[y] \Leftrightarrow \exists x' \sim x \exists y' \sim y (x' R y')$$

$$V'(p) = \{[x] \mid x \in V(p)\}$$

Steps of the proof:

1. Filter to make all R^*-clusters finite.
2. Filter the upper level so that all cones of depth 2 are finite.

   $\vdots$

i. Filter the i-th level so that all cones of depth i are finite.

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Make all clusters finite

Consider an R^*-cluster V. For $r \in V$ we define

$$H(r) = \{\alpha \,|\, \alpha \text{ is a loop in } r\}.$$

$H(r)$ is a monoid with concatenation operation.
Function of taking the length of loop $l : H(r) \to \mathbb{Z}$ is a homomorphism of monoids.
Define $H^\sharp(r) = l(H(r))$.
Let $k = n - m$ and $\mathbb{Z}/k\mathbb{Z}$ be a cyclic group of order k.
$f : \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$ is the standard epimorphism.

Lemma

$f(H^\sharp(r))$ is a subgroup of $\mathbb{Z}/k\mathbb{Z}$.

Hence $f(H^\sharp(r)) = \{0, d, 2d, \ldots\}$ for d is the minimal positive element of $f(H^\sharp)$ and for any $\alpha \in H(r)$ $l(\alpha) \,\vdots\, d$.

Lemma

Group $f(H^\sharp(r))$ do not depend on r.

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Make all clusters finite-2

We define $\approx_d$ on V:

$$x \approx_d y \Leftrightarrow \text{there is a path of length } l \in d\mathbb{Z}.$$

$$\sim \; = \; \approx_d \cap \equiv_\Psi \; .$$

Lemma

Let $M' = (F', V')$ be the $(\sim, \Psi)$-filtration of M. Then F' is an (m, n)-frame. All R^*-clusters are finite of size no greater then $2^{d \cdot |\Psi|}$.

Steps of the proof:

1. Filter to make all R^*-clusters finite.
2. Filter the upper level so that all cones of depth 2 are finite.

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i. Filter the i-th level so that all cones of depth i are finite.

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h. Filter the bottom level.

Filtrating i-th level

Let M_1 is the model after 1st filtration with finite clusters. In M_2 all cones of depth 2 are finite. And so on.
Let us show how to get M_i.

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Filtrating level i

level i: $L(i) = \{x \mid h(F^x) = i\}$;

$X = \{x \in W \mid h(F^x) > i\}$, $Y = L_i(F)$, $Z = \{x \in W \mid h(F^x) < i\}$.

Suppose $x \in Y$, C is the cluster of x. Put for $z \in Z$, $\psi \in \Psi$:

$$P_z = R^{-1}(z) \cap C;\ P_\psi := \{y \in C \mid M, y \vDash \psi\}.$$

$$\mathbf{A}(x) := (C, R{\restriction}C, (P_z)_{z \in Z}, (P_\psi)_{\psi \in \Psi}), x).$$

$$x \sim y \quad \leftrightharpoons \quad \mathbf{A}(x) \cong \mathbf{A}(y)$$

If the size of any cluster is bounded by N, then $Y/{\sim}$ is finite:

$$|Y/{\sim}| \leq N \cdot 2^{N \times N} \cdot 2^{N \times |\Psi|} \cdot 2^{N \times |Z|} \cdot N$$

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Main lemma

Lemma

Let L *be one of the logics* $\mathsf{K}_\mathsf{n}^\mathsf{m}$, $\mathsf{K}_{\leq \mathsf{m}}$, $n > m \geq 1$. *If a formula is satisfiable in an* L*-frame of finite depth* h*, then it is satisfiable in a finite* L*-frame of depth* h*; the size of the finite frame is bounded by*

$$2^{2^{\cdot^{\cdot^{\cdot^{2^l}}}}} \Big\} h,$$

where l *is the length of the formula.*

Grazie!