Conuclear images of substructural logics

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From an algebraic point of view, Heyting algebras can be represented as Boolean algebras endowed with an interior operator.

In particular, given a Boolean algebra **B** and an interior operator σ on **B**, $\sigma(\mathbf{B})$ is a Heyting algebra.

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Interior operator	$\leftrightsquigarrow$	Conucleus

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Conuclear images of substructural logics

A **residuated lattice** is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ such that

- $\langle A, \cdot, 1\rangle$ is a monoid;
- $\langle A, \wedge, \vee\rangle$ is a lattice;
- the *residuation laws* hold: for all $x, y, z \in A$

$$x \cdot y \leq z \text{ iff } x \leq z/y \text{ iff } y \leq x\backslash z.$$

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If $\cdot$ is commutative, then $\backslash = / = \rightarrow$ and $\neg x = x \rightarrow 0$.

A **conucleus** σ on a residuated lattice $\mathbf{A}$ is an interior operator, that is for all $x, y \in A$

- $\sigma(x) \leq x$;
- $\sigma(x) = \sigma(\sigma(x))$;
- if $x \leq y$, then $\sigma(x) \leq \sigma(y)$;

and, furthermore, it satisfies the following properties:

- $\sigma(1) = 1$;
- $\sigma(x) \cdot \sigma(y) \leq \sigma(x \cdot y)$.

Let $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ be a residuated lattice and σ a conucleus on it. Then the conuclear image $\sigma(\mathbf{A})$ of $\mathbf{A}$ is a residuated lattice:

$$\sigma(\mathbf{A}) = \langle \sigma(A), \wedge_\sigma, \vee, \cdot, \backslash_\sigma, /_\sigma, 1\rangle$$

where $\vee, \cdot, 1$ are the same as in $\mathbf{A}$, while, for all $x, y \in \sigma(A)$,

$$x \wedge_\sigma y = \sigma(x \wedge y),$$

$$x\backslash_\sigma y = \sigma(x\backslash y), \quad x/_\sigma y = \sigma(x/y)$$

Conuclear image

Let L be a substructural logic. We denote by L_σ the logic L with an additional unary operator σ which satisfies the following axioms:

1. $\sigma(F) \to F$
2. $\sigma(F) \to \sigma(\sigma(F))$
3. $(\sigma(F) \cdot \sigma(G)) \to \sigma(F \cdot G)$

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L_σ: conuclear extension of L.

We define the following interpretation ${}^{\sigma}$: $\mathsf{L} \to \mathsf{L}_{\sigma}$:

- $p^{\sigma} = \sigma(p)$ where p is a propositional variable,
- $0^{\sigma} = \sigma(0)$,
- $1^{\sigma} = 1$,
- $(F \circ G)^{\sigma} = F^{\sigma} \circ G^{\sigma}$, for $\circ \in \{\vee, \cdot\}$,
- $(F \circ G)^{\sigma} = \sigma(F^{\sigma} \circ G^{\sigma})$, for $\circ \in \{\backslash, /, \wedge\}$.

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$\sigma(\mathsf{L})$: *conuclear image of* L: logic whose theorems are those formulas F such that $\mathsf{L}_{\sigma} \vdash F^{\sigma}$.

Conuclear image

Let $\mathcal{V}$ be a variety of FL-algebras. We denote by $\mathcal{V}_\sigma$ the variety consisting of all the algebras $(\mathbf{A}, \sigma)$, where $\mathbf{A} \in \mathcal{V}$ and σ is a conucleus on $\mathbf{A}$.

$\sigma(\mathcal{V})$ (the conuclear image of $\mathcal{V}$) is the variety generated by all the algebras $\sigma(\mathbf{A})$, where $(\mathbf{A}, \sigma) \in \mathcal{V}_\sigma$.

Examples

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Task

Investigating the relationship between L and $\sigma(\mathrm{L})$, whatever the substructural logic L is.

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2. Which properties are excluded to hold in $\sigma(\mathsf{L})$, whatever L is?
3. Which theorems are not necessarily preserved as in (1), nor excluded to hold as in (2)?

Disjunction property (DP)

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The conuclear image of **every** consistent substructural logic has the DP.

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The conuclear image of **every** consistent substructural logic has the DP.

σ(L) is always a **constructive** logic.

Complexity

R.Horčík, K.Terui, *Disjunction property and complexity of substructural logics*, TCS, 2011.

Complexity

Theorem

If L is a consistent substructural logic, then $\sigma(\mathsf{L})$ and L_σ are **PSPACE-hard**.

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Properties which never hold

- **excluded middle**: $x \vee \neg x \geq 1$;
- **prelinearity**: $(x \backslash y) \vee (y \backslash x) \geq 1$;
- **weak excluded middle**: $\neg x \vee \neg\neg x \geq 1$.

Properties which never hold

FL + $(\neg\neg x = x)$ has DP.

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Theorem

For any consistent substructural logic L, its conuclear image does not satisfy the double negation principle.

Proof

Let $\mathcal{V}$ be a nontrivial variety of FL-algebras and $\mathbf{A}$ and $\mathbf{C}$ two algebras in $\mathcal{V}$, where $\mathbf{C}$ is nontrivial. We define a particular conucleus σ on $\mathbf{A} \times \mathbf{C}$ and we prove that the double negation law fails in $\sigma(\mathbf{A} \times \mathbf{C}) \in \sigma(\mathcal{V})$.

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Properties compatible with conuclear images but not necessarily preserved

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- **Distributivity**: $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$;
- **Divisibility**: if $x \leq y$, then there are u and z such that $z \cdot y = y \cdot u = x$;

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P.Bahls, J.Cole, N.Galatos, P.Jipsen, C.Tsinakis, *Cancellative residuated lattices*, AU, 2003.

Preservation under conuclear images

An inequation $f \leq g$ is **preserved under conuclear images** when, given an FL-algebra $\mathbf{A}$ and a conucleus σ on $\mathbf{A}$, if $f \leq g$ holds in $\mathbf{A}$, then $f \leq g$ holds in $\sigma(\mathbf{A})$.

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Examples

Commutativity: $x \cdot y = y \cdot x$;

Integrality: $x \leq 1$;

Contraction: $x \leq x \cdot x$;

Idempotence: $x = x \cdot x$;

Cancellativity: $xy/y = x = y\backslash yx$;

Weak Contraction: $x \wedge \neg x \leq 0$.

Theorem

If $f \in P_2$ and $g \in N_2$, then $f \leq g$ is preserved under conuclear images.

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P_n, N_n give the **Substructural Hierarchy**

(0) $P_0 = N_0 =$set of variables.

(P1) 1 and all terms of N_n belong to P_{n+1}.

(P2) If $t, u \in P_{n+1}$, then $t \vee u, t \cdot u \in P_{n+1}$.

(N1) 0 and all terms of P_n belong to N_{n+1}.

(N2) If $t, u \in N_{n+1}$, then $t \wedge u \in N_{n+1}$.

(N3) If $t \in P_{n+1}$ and $u \in N_{n+1}$, then $t \backslash u$, $u/t \in N_{n+1}$.

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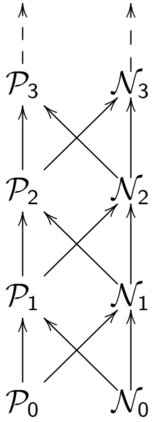

Figure: The Substructural Hierarchy.

Generalization of the theorem

$\mathbf{P_2^*}$ is the smallest class such that :

- $P_2 \subseteq P_2^*$;
- if $t, u \in P_2^*$, then $t \wedge u, t \vee u, t \cdot u \in P_2^*$;
- if $f \in P_2^*$ and $g \in P_1$, then $g \backslash f, f/g \in P_2^*$.

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$\mathbf{N_2^*}$ is obtained from N_2 replacing (N3) with the following axiom:

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Theorem

If $f \in P_2^*$ and $g \in N_2^*$, then $f \leq g$ is preserved under conuclear images.

An equation ε is a $N_n(P_n)$-*equation* if it is equivalent to an inequation $t \geq 1$ for some t in N_n (P_n).

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Each N_2^*-equation is preserved under conuclear images.

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Theorem

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Proof

As it is done for N_2-equations by Ciabattoni, Galatos and Terui, we prove that each N_2^*-equation is equivalent to a finite set of particular quasi-equations, which are proved to be preserved under conuclear images.

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  - $g = x \vee x \backslash 0 \notin \mathbf{N_2^*}$ but $\in \mathbf{P_2}$.
- Prelinearity: $(x \backslash y) \vee (y \backslash x) \geq 1$:
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Other counterexamples...

- Distributivity: $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$:
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  - $g = (x \wedge y) \vee (x \wedge z) \in \mathbf{P_2}$ but $\notin \mathbf{N_2^*}$.
- Double negation: $\neg\neg x \leq x$:
  - $f = (x \backslash 0) \backslash 0 \in \mathbf{N_2}$ but $\notin \mathbf{P_2^*}$
  - $g = x \in N_2^*$.

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  - $g = x \in N_2^*$.
- Divisibility: $x(x\backslash(x \wedge y)) = x \wedge y$:
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  - $g = x(x\backslash(x \wedge y)) \in \mathbf{P_2}$ but $\notin \mathbf{N_2^*}$.

Some properties

Equation	Name	Behaviour
$xy \leq yx$	Commutativity	Preserved
$x \leq 1$	Left weakening	Preserved
$0 \leq x$	Right weakening	Preserved
$x \leq xx$	Contraction	Preserved
$x = xx$	Idempotence	Preserved
$x^n \leq x^m$	Knotted $(n, m \geq 0)$	Preserved
$x \wedge \neg x \leq 0$	Weak contraction	Preserved
$xy/y = x = y\backslash yx$	Cancellativity	Preserved
$1 \leq x \vee \neg x$	Excluded middle	Never preserved
$1 \leq (x\backslash y) \vee (y\backslash x)$	Prelinearity	Never preserved
$1 \leq \neg x \vee \neg\neg x$	Weak excluded middle	Never preserved
$\neg\neg x \leq x$	Double negation	Never preserved
$x(x\backslash(x \wedge y)) = x \wedge y = ((x \wedge y)/x)x$	Divisibility	Not preserved but compat.
$x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$	Distributivity	Not preserved but compat.

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