On Completeness of Logics Enriched with Transitive Closure Modality

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Transitive closure modality

The transitive closure of a binary relation R is denoted by R^+. Given a frame $F = (W, R)$, we write $F^{(+)} = (W, R, R^+)$. For a class of frames $\mathcal{F}$, denote $\mathcal{F}^{(+)} = \{F^{(+)} \mid F \in \mathcal{F}\}$.

The extension of a normal unimodal logic $\mathbf{L}$ with the *transitive closure modality* $\boxplus$ is the minimal normal bimodal logic $\mathbf{L}^{\boxplus}$ that contains $\mathbf{L}$ and the axioms:

$$\text{(A1)} \quad \boxplus p \to \Box p, \qquad \text{(A2)} \quad \boxplus p \to \Box \boxplus p,$$
$$\text{(A3)} \quad \Box p \wedge \boxplus(p \to \Box p) \to \boxplus p.$$

Fact

$(W, R, S) \models \text{(A1)} \wedge \text{(A2)} \wedge \text{(A3)}$ *iff* $S = R^+$.

Fact

$$\mathrm{Frames}(\mathbf{L}^{\boxplus}) = \mathrm{Frames}(\mathbf{L})^{(+)}.$$

Question

Which properties transfer from $\mathbf{L}$ to $\mathbf{L}^{\boxplus}$?

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Our result:

Kripke completeness, and moreover — the finite model property (FMP), are preserved in the case when

- $\mathbf{L}$ is canonical and
- $\mathbf{L}$-frames *admit* what we call *definable filtrations*.

For a model M and a set of formulas Γ,
$x \sim_\Gamma y \leftrightharpoons \forall A \in \Gamma\ (M, x \models A \Leftrightarrow M, y \models A)$.

Definition (Filtration)

Let Γ be a finite *sub-closed* set of formulas. A *Γ-filtration of* $M = (W, R, V)$ is a finite model $\widehat{M} = (\widehat{W}, \widehat{R}, \widehat{V})$ s.t.

1. $\widehat{W} = W/\!\sim$ for some equivalence relation $\sim$ such that $\sim\ \subseteq\ \sim_\Gamma$, i.e., $\sim$ respects all formulas from Γ
2. $\widehat{x} \models p \Leftrightarrow x \models p$ for all $p \in \Gamma$
3. $R^{\min} \subseteq \widehat{R} \subseteq R_A^{\max}$, where

$$\widehat{x}\, R^{\min}\, \widehat{y} \leftrightharpoons \exists x' \sim x\ \exists y' \sim y: \quad x'\, R\, y'$$
$$\widehat{x}\, R_A^{\max}\, \widehat{y} \leftrightharpoons \forall \text{ formula } \Box B \in \mathrm{Sub}(A)\ \big(x \models \Box B \implies y \models B\big)$$

If $\sim\ =\ \sim_\Delta$ for some finite sub-closed set of formulas $\Delta \supseteq \Gamma$, then $\widehat{M}$ is called a *definable Γ-filtration* of the model M.

Filtration lemma

$$\forall B \in \Gamma \quad \big(M, x \models B \Leftrightarrow \widehat{M}, \widehat{x} \models B\big)$$

Definition

A class of frames $\mathcal{F}$ *admits (definable) filtration* if, for every $\mathcal{F}$-model M and every finite sub-closed set of formulas Γ, there exists an $\mathcal{F}$-model $\widehat{M}$ that is a (definable) Γ-filtration of M.

Observation

If a logic **L** *is complete w.r.t. some class of frames that admits filtration, then* **L** *has the FMP.*

Example (Lemmon, Scott, Segerberg, Gabbay)

Frames(**L**) admits definable filtration for the following logics **L**:

- **K**, **T**, **K4**, **S4**, **B**, **S5**, **S4.1**;
- $\mathsf{K} + \Box p \to \Box^m p$, for $m \geq 0$;
- $\mathsf{K} + \Box^m p \to \Box p$, for $m \geq 0$.

Theorem (Kikot + Sh + Zolin, 2014)

If a class of frames $\mathcal{F}$ admits (definable) filtration, then the class $\mathcal{F}^{(+)}$ admits (definable) filtration too.

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Corollary

Suppose

- $\mathrm{Frames}(\mathbf{L})$ *admits filtration*
- $\mathbf{L}^{\boxplus}$ *is Kripke complete.*

Then $\mathbf{L}^{\boxplus}$ *has the FMP.*

Proof.

Indeed, let $\mathcal{F} = \mathrm{Frames}(\mathbf{L})$. Then $\mathcal{F}^{(+)} = \mathrm{Frames}(\mathbf{L}^{\boxplus})$.
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Prove the Kripke completeness, get the FMP for free!

Theorem (Main result)

Suppose

- $\mathbf{L}$ *is canonical*
- $\mathrm{Frames}(\mathbf{L})$ *admits definable filtration*

Then

- $\mathbf{L}^{\boxplus}$ *is Kripke complete*
- $\mathrm{Frames}(\mathbf{L}^{\boxplus})$ *admits definable filtration*

(and hence $\mathbf{L}^{\boxplus}$ *has the FMP).*

Example

Let $\mathbf{L}$ be the density logic $\mathbf{K} + \Box\Box p \to \Box p$
(or even $\mathbf{K} + \Box^m p \to \Box p$ for some $m \geq 0$).
Then $\mathbf{L}^{\boxplus}$ has the FMP.

- $\mathbf{L}$ is canonical
- $\mathrm{Frames}(\mathbf{L})$ admits definable filtration

$\Longrightarrow$

- $\mathbf{L}^{\boxplus}$ is Kripke complete
- $\mathrm{Frames}(\mathbf{L}^{\boxplus})$ admits definable filtration

This result also holds for the polymodal case, where the new modality $\boxplus$ corresponds to the transitive closure of some relation R_i^+ or of the union of a finite set of relations $(R_i \cup \ldots \cup R_j)^+$.
Can we iterate the construction?

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By the way, it seems that

$\mathbf{L}^{\boxplus}$ is canonical iff $\boxplus$ is expressible in $\mathbf{L}$, i.e., $\mathbf{L}$ is *pretransitive* (e.g., $\mathbf{L}$ is $\mathbf{K4}$, or $\mathbf{K} + \Box p \to \Box\Box\Box p$, or $\mathbf{K} + \Box\Box p \to \Box\Box\Box p$).

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- $\mathrm{Frames}(\mathbf{L})$ admits definable filtration

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We cannot use our result to iterate this operation: the premise of the theorem for $\mathbf{L}$ is stronger than its conclusion for the resulting logic. Nevertheless...

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Theorem (Main result-2)

If the class of all $\mathbf{L}$*-models admits definable filtration, then the class of all* $\mathbf{L}^{\boxplus}$*-models admits definable filtration (so* $\mathbf{L}^{\boxplus}$ *has the FMP).*

Now we can use the new theorem to iterate the operation in the polymodal case.

Upcoming result

Suppose that Φ is a finite set of n-modal formulas such that

$$\mathrm{Models}(\mathsf{K}_n + \Phi) \text{ admits definable filtration.}$$

Then the extension of the *PDL* with axioms Φ axiomatizing atomic modalities has the finite model property.

Thank you!