# Undecidability in abstract algebraic logic

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Leibniz hierarchy $\longmapsto$ definability of equivalence and of truth predicates

Frege hierarchy $\longmapsto$ replacement properties

- Can we classify mechanically logics of Hilbert-style calculi in these hierarchies?
- We begin by the Leibniz hierarchy.

# Definability of equivalence

- Given an algebra $\boldsymbol{A}$, the Leibniz congruence of $F \subseteq A$ is

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- A logic $\mathcal{L}$ is protoalgebraic if equivalence is definable, i.e., if there is a set of formulas $\Delta(x, y, \overline{z})$ such that for every model $\langle \boldsymbol{A}, F \rangle$ of $\mathcal{L}$:

$$\langle a, b \rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F \iff \Delta(a, b, \overline{c}) \subseteq F \text{ for every } \overline{c} \in A.$$

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- A logic $\mathcal{L}$ is equivalential if it is protoalgebraic and $\Delta(x, y)$ has only variables $x, y$.

## Definability of truth predicates

- The reduced models of a logic $\mathcal{L}$ are

$$\mathrm{Mod}^{*}\mathcal{L} = \{\langle \boldsymbol{A}, F \rangle : F \text{ is a filter of } \mathcal{L} \text{ and } \boldsymbol{\Omega}^{\boldsymbol{A}} F = \mathrm{Id}_{\boldsymbol{A}}\}.$$

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- A logic $\mathcal{L}$ is truth-equational if truth predicates in $\text{Mod}^*\mathcal{L}$ are definable, i.e., if there is a set of equations $\boldsymbol{\tau}(x)$ such that for every $\langle \boldsymbol{A}, F\rangle \in \text{Mod}^*\mathcal{L}$:

$$F = \{a \in A : \boldsymbol{A} \vDash \boldsymbol{\tau}(a)\}.$$

# The Leibniz hierarchy

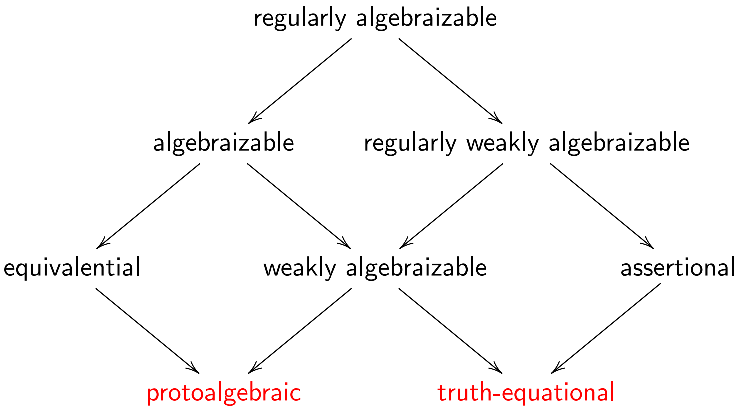


Figure: Some classes of the Leibniz hierarchy.

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# Basic logic of a variety

## Definition

Let V be a non-trivial variety. The basic logic $\mathcal{L}_{\mathsf{V}}$ of $V$ is determined by the following class of matrices:

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Let V be a non-trivial variety and $\Gamma \cup \{\varphi\} \subseteq Fm$.

1. $\mathrm{Alg}\,\mathcal{L}_{\mathsf{V}} = \mathsf{V}$.
2. $\Gamma \vdash_{\mathsf{V}} \varphi$ if and only if there is $\gamma \in \Gamma$ such that $\mathsf{V} \models \gamma \approx \varphi$.

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- An explicit and finite axiomatization of $\mathcal{L}_{\mathsf{CR}}$.

Unfortunately, in general:

- No clever way to axiomatize $\mathcal{L}_{\mathsf{V}}$ out of a base for V.
- Even if V is finitely based, $\mathcal{L}_{\mathsf{V}}$ need not to be finitely axiomatizable.

## Some examples

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$$x \dashv\vdash x \wedge x \qquad x \wedge y \dashv\vdash y \wedge x \qquad x \wedge (y \wedge z) \dashv\vdash (x \wedge y) \wedge z$$

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is a reduced model of $\mathcal{R}$. A complete axiomatization of $\mathcal{L}_{\mathsf{SL}}$ is obtained by adding:

$$u \wedge x \dashv\vdash u \wedge (x \wedge x) \quad u \wedge (x \wedge y) \dashv\vdash u \wedge (y \wedge x)$$

$$u \wedge (x \wedge (y \wedge z)) \dashv\vdash u \wedge ((x \wedge y) \wedge z)$$

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$\mathcal{L}_{\mathrm{CM}}$ is not finitely axiomatizable:

- Let $\Sigma$ be a finite set of deductions holding in $\mathcal{L}_{\mathrm{CM}}$.
- There is a natural $n \geq 2$ that bounds the number of occurrences of (possibly equal) variables in terms appearing in the rules of $\Sigma$.

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Then consider the algebra $\boldsymbol{A} = \langle \{0, 1, 2, \ldots, n\}, \cdot \rangle$ with a binary operation such that $1 \cdot 2 := 2$ and $2 \cdot 1 := 1$ and

$$a \cdot b = b \cdot a := \begin{cases} a & \text{if } a \neq n \text{ and } b = 0 \\ 0 & \text{if } a = n \text{ and } b = 0 \\ a & \text{if } b = a - 1 \text{ and } a \geq 3 \\ a - 1 & \text{if } b = a - 2 \text{ and } a \geq 3 \\ 1 & \text{otherwise} \end{cases}$$

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- Why? It is reduced: if $a, b \in A \smallsetminus \{0\}$ and $a < b$, we consider the polynomial

  $$p(x) := (\dots((\dots(\dots((1 \cdot 2) \cdot 3) \cdot \dots a) \cdot \dots \cdot b - 1) \cdot x) \cdot \dots \cdot n) \cdot 0.$$

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  Then

  $$p(b) = 0 \text{ and } p(a) \neq 0.$$

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# A logic for commutative rings

## Definition

Let $\mathcal{CR}$ be the logic axiomatized by the rules:

$$
\begin{aligned}
w + (u \cdot ((x \cdot y) \cdot z)) &\dashv\vdash w + (u \cdot (x \cdot (y \cdot z)) && \text{(A)}\\
w + (u \cdot (x \cdot y)) &\dashv\vdash w + (u \cdot (y \cdot x)) && \text{(B)}\\
w + (u \cdot (x \cdot 1)) &\dashv\vdash w + (u \cdot x) && \text{(C)}\\
w + (u \cdot ((x + y) + z)) &\dashv\vdash w + (u \cdot (x + (y + z))) && \text{(D)}\\
w + (u \cdot (x + y)) &\dashv\vdash w + (u \cdot (y + x)) && \text{(E)}\\
w + (u \cdot (x + 0)) &\dashv\vdash w + (u \cdot x) && \text{(F)}\\
w + (u \cdot (x + -x)) &\dashv\vdash w + (u \cdot 0) && \text{(G)}\\
w + (u \cdot (x \cdot (y + z))) &\dashv\vdash w + (u \cdot ((x \cdot y) + (x \cdot z))) && \text{(H)}\\
w + (u \cdot -(x + y)) &\dashv\vdash w + (u \cdot (-x + -y)) && \text{(I)}\\
w + (u \cdot -(x \cdot y)) &\dashv\vdash w + (u \cdot (-x \cdot y)) && \text{(L)}\\
w + (u \cdot -(x \cdot y)) &\dashv\vdash w + (u \cdot (x \cdot -y)) && \text{(M)}\\
0 + x &\dashv\vdash x && \text{(N)}\\
x + (1 \cdot y) &\dashv\vdash x + y && \text{(O)}
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# Completeness

**Theorem**

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## Proof.

- The relation $\dashv\vdash_{\mathcal{CR}}$ is a congruence. Then:

$$\begin{aligned} \alpha \approx \beta \text{ is in the base of } CR &\implies \alpha \dashv\vdash_{\mathcal{CR}} \beta \\ &\implies \mathsf{Alg}\mathcal{CR} \vDash \alpha \approx \beta \\ &\implies \mathsf{Alg}\mathcal{CR} \subseteq CR. \end{aligned}$$

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- Since $\langle \boldsymbol{A}, F \rangle$ is a model of $\mathcal{L}_{CR}$ for every $\boldsymbol{A} \in CR$, we conclude that $\mathcal{L}_{\mathsf{CR}} \leq \mathcal{CR}$.

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- Since $\langle \boldsymbol{A}, F \rangle$ is a model of $\mathcal{L}_{CR}$ for every $\boldsymbol{A} \in CR$, we conclude that $\mathcal{L}_{\mathsf{CR}} \leq \mathcal{CR}$.
- Easy to check that $\mathcal{CR} \leq \mathcal{L}_{\mathsf{CR}}$.

□

# Contents

# From equations to logics

## Definition

Given a Diophantine equation $p(z_1, \ldots, z_n) \approx 0$, we pick two new variables $x$ and $y$, a new binary symbol $\leftrightarrow$ and consider the logic $\mathcal{L}(p \approx 0)$ axiomatized by the rules:

$$\emptyset \vdash x \leftrightarrow x \tag{R}$$

$$x \leftrightarrow y \vdash y \leftrightarrow x \tag{S}$$

$$x \leftrightarrow y, y \leftrightarrow z \vdash x \leftrightarrow z \tag{T}$$

$$x \leftrightarrow y \vdash -x \leftrightarrow -y \tag{Re1}$$

$$x \leftrightarrow y, z \leftrightarrow u \vdash (x + z) \leftrightarrow (y + u) \tag{Re2}$$

$$x \leftrightarrow y, z \leftrightarrow u \vdash (x \cdot z) \leftrightarrow (y \cdot u) \tag{Re3}$$

$$x \leftrightarrow y, z \leftrightarrow u \vdash (x \leftrightarrow z) \leftrightarrow (y \leftrightarrow u) \tag{Re4}$$

$$p(z_1, \ldots, z_n) \leftrightarrow 0, x, x \leftrightarrow y \vdash y \tag{MP'}$$

$$p(z_1, \ldots, z_n) \leftrightarrow 0, x \dashv\vdash x \leftrightarrow (x \leftrightarrow x), p(z_1, \ldots, z_n) \leftrightarrow 0 \tag{A3'}$$

$$p(z_1, \ldots, z_n) \leftrightarrow 0, x, y \vdash x \leftrightarrow y \tag{G'}$$

plus the axioms of the form $\emptyset \vdash \alpha \leftrightarrow \beta$ for every $\alpha \dashv\vdash \beta \in \mathcal{CR}$.

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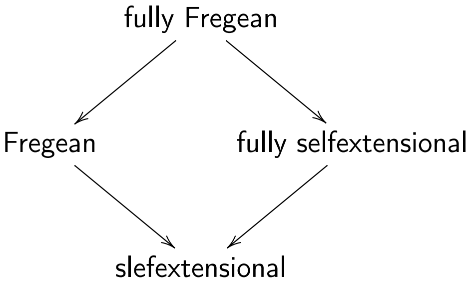
**Theorem**

Let $K$ a level of the Leibniz hierarchy. The problem of determining whether the logic of a finite Hilbert calculus in a finite language belongs to $K$ is undecidable.

# Frege hierarchy



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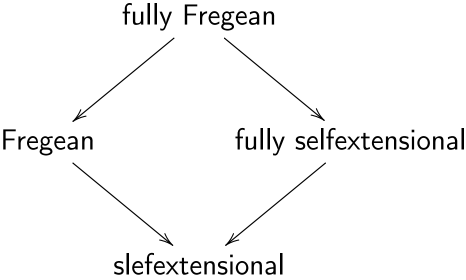


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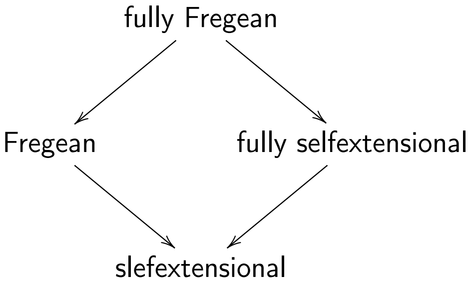


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Let $K$ a level of the Frege hierarchy. The problem of determining whether the logic of a finite Hilbert calculus in a finite language belongs to $K$ is undecidable.

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- Is it possible to do this for logics of a finite set of finite matrices?
- For the Leibniz hierarchy yes.
- The Frege hiearchy seems more complicated, since it involves semantic notions.
- We have a positive solution for selfextentionality and Fregeanity, but the problem for their fully-versions in open.

# Finally...

**Thank you!**