On One Embedding of Heyting Algebras

Alexei Muravitsky

Northwestern State University
alexeim@nsula.edu

TACL 2015 Ischia Island

Outline

Kuznetsov's Remark on an Embedding

Докл. Акад. Наук СССР
Том 283 (1985), № 1

Soviet Math. Dokl.
Vol. 32 (1985), No. 1

ON THE PROPOSITIONAL CALCULUS OF INTUITIONISTIC PROVABILITY

UDC 517.11:(51.01+519.48)

A. V. KUZNETSOV

A. Yu. Muravitskiĭ drew my attention to:

COROLLARY 2. *Every pseudo-Boolean algebra* $\mathfrak{A}$ *is a subalgebra of some* Δ*-enrichable pseudo-Boolean algebra in the variety generated by* $\mathfrak{A}$.

Enrichable Elment, Enrichable Heyting Algebra

Definition: enrichable element, enrichable Heyting algebra

An element a of Heyting algebra $\mathfrak{A}$ is said to be **enriched** by an element a^* (or a^* **enriches** a) if the following conditions are fulfilled:

$$\begin{array}{ll} (1) & a \leq a^*, \\ (2) & a^* \to a \leq a, \\ (3) & a^* \leq x \vee (x \to a), \text{ for any } x \in |\mathfrak{A}|. \end{array}$$

An element $a \in |\mathfrak{A}|$ is **enrichable** if there exists an element $b \in |\mathfrak{A}|$ that enriches a. A Heyting algebra is **enrichable** if each element of it is enrichable.

Note

Given a Heyting algebra $\mathfrak{A}$ and $a \in |\mathfrak{A}|$, it has been observed that if a is enriched by b and by c, then $b = c$

Kuznetsov's Theorem

Two propositional languages: $\mathcal{L}_a$ and $\mathcal{L}_m$

- The propositional variables — $Var = \{p_0, p_1, \ldots\}$
- $\mathcal{L}_a$-formulas $(A, B, \ldots) := A \in Var | (A \wedge B) | (A \vee B) | (A \to B) | \neg A$
- $\mathcal{L}_m$-formulas $(\alpha, \beta, \ldots) := \alpha \in Var | (\alpha \wedge \beta) | (\alpha \vee \beta) | (\alpha \to \beta) | \neg\alpha | \Box\alpha$
- **Int** denotes intuitionistic propositional logic in $\mathcal{L}_a$
- **KM** denotes proof-intuitionistic logic in $\mathcal{L}_m$

Kuznetsov's Theorem

$$\mathbf{KM} + A \vdash B \Longleftrightarrow \mathbf{Int} + A \vdash B.$$

Kuznetsov's Corollary 1

Every variety of Heyting algebras is generated by its enrichable members.

The Existence of embedding derived from the Kuznetsov Theorem

Keznetsov's Corollary 2

Any Heyting algebra is embedded into an enrichable one so that these algebras separately generate one and the same variety.

Proof [Muravitsky 1988]

- Step 1: Let us take a Heyting algebra $\mathfrak{A}$ and let K be the enrichable Heyting algebras of $\mathrm{HSP}(\mathfrak{A})$.
- Step 2: $\mathfrak{A} \in \mathrm{HSP}(K)$ [Kuznetsov's Corollary 1]
- Step 3: $\mathfrak{A} \in \mathrm{SHP}(K)$ [since any Heyting algebra has CEP]
- Step 4: "algebra … is enrichable" is a property that can be expressed by a conjunction of Horn positive formulas. Hence each of these formulas is stable under formation of direct products and homomorphic images. Therefore, $\mathrm{HP}(K) \subseteq K$.
- Step 5: Conclusion: $\mathfrak{A}$ is embedded into an enrichable Heyting algebra $\mathfrak{B}$ such that $\mathrm{HSP}(\mathfrak{A}) = \mathrm{HSP}(\mathfrak{B})$.

Question: How intimately are $\mathfrak{A}$ and $\mathfrak{B}$ related to one another? Which properties of $\mathfrak{A}$ are preserved in $\mathfrak{B}$?

The Embedding

Tools for the embedding: Given a Heyting algebra $\mathfrak{A}$, let

- $\mu_{\mathfrak{A}}$ be the partially ordered set of the prime filters of the algebra $\mathfrak{A}$, arranged by set inclusion;
- $\mathcal{H}(\mathfrak{A})$ be the Heyting algebra of all upward sets of $\mu_{\mathfrak{A}}$;
- $h : \mathfrak{A} \to \mathcal{H}(\mathfrak{A})$ be a Stone embedding.
- Next we define for any $X \in \mathcal{H}(\mathfrak{A})$,

  $$\triangle X = \{F \mid F \in \mu_{\mathfrak{A}}, (\forall F' \in \mu_{\mathfrak{A}})(F \subset F' \Rightarrow F' \in X)\};$$

  in particular,

  $$\triangle h(x) = \{F \mid F \in \mu_{\mathfrak{A}}, (\forall F')(F \subset F' \Rightarrow x \in F')\}.$$

- Let $\mathcal{B}^{\triangle}(\mathfrak{A})$ be the subalgebra of $\mathcal{H}(\mathfrak{A})$, generated by the set $\{h(x) \mid x \in \mathfrak{A}\} \cup \{\triangle h(x) \mid x \in \mathfrak{A}\}$.

Definition of the embedding

Given a Heyting algebra $\mathfrak{A}$, we define the denumerable sequence of algebras as follows:

$$\begin{aligned} &\mathfrak{A}_0 = \mathfrak{A}, \\ &\mathfrak{A}_{i+1} = \mathcal{B}^{\triangle}(\mathfrak{A}_i) \quad (i < \omega). \end{aligned}$$

Along with the sequence $\{\mathfrak{A}_i\}_{i<\omega}$, we also have the embeddings:

$$\begin{aligned} &\varphi_{ii} : \mathfrak{A}_i \to \mathfrak{A}_i, i < \omega, \quad \text{(the identity embedding of } \mathfrak{A}_i\text{)} \\ &\varphi_{i(i+1)} : \mathfrak{A}_i \to \mathfrak{A}_{i+1}, i < \omega, \quad \text{(Stone embedding } h : \mathfrak{A}_i \to \mathcal{B}^{\triangle}(\mathfrak{A}_i)\text{)} \\ &\varphi_{ij} = \varphi_{i(i+1)} \circ \varphi_{(i+1)(i+2)} \circ \ldots \circ \varphi_{(j-1)j}, \text{ where } i < j. \end{aligned}$$

Thus the sequence $\{\mathfrak{A}_i\}_{i<\omega}$ along with the embeddings φ_{ij}, $i \leq j$, form a direct family. Let $\vec{\mathfrak{A}}$ be the direct limit of this family.

Properties of the embedding

- $\vec{\mathfrak{A}}$ is an enrichable Heyting algebra.
- Each $\mathfrak{A}_i$ is embedded into $\vec{\mathfrak{A}}$.
- If $\mathfrak{A}$ is finite, then $\mathfrak{A}$ and $\vec{\mathfrak{A}}$ are isomorphic.
- If $\mathfrak{A}$ is countable, then $\vec{\mathfrak{A}}$ is also countable.
- If $\mathfrak{A}$ is subdirectly irreducible, so is $\vec{\mathfrak{A}}$.

The main question: Is it true that $\mathrm{HSP}(\mathfrak{A}) = \mathrm{HSP}(\vec{\mathfrak{A}})$?

Remarks:

1. It must be clear that the answer to this question is affirmative if $\mathrm{HSP}(\mathfrak{A}) = \mathrm{HSP}(\mathfrak{A}_a^{\Delta})$, where $\mathfrak{A}_a^{\Delta}$ is a subalgebra of $\mathcal{H}(\mathfrak{A})$ generated by $\{h(x) \mid x \in |\mathfrak{A}|\} \cup \{\triangle h(a)\}$, where a is any fixed element of $\mathfrak{A}$.
2. Since $\triangle h(a)$ enriches $h(a)$ in $\mathfrak{A}_a^{\Delta}$, we have to focus on enrichment of one element of a given Heyting algebra in general setting.

Localization of Enrichment

Defition: $\mathcal{E}$-pair and relation $\mathcal{E}$

Given an algebra $\mathfrak{A}$ and $a, a^* \in \mathfrak{A}$, (a, a^*) is an $\mathcal{E}$-**pair** (in/of $\mathfrak{A}$) if a is enriched by a^* in $\mathfrak{A}$. Then, we define:

$$\mathcal{E}_{\mathfrak{A}} = \{(a, a^*) \mid (a, a^*) \text{ is an } \mathcal{E}\text{-pair in } \mathfrak{A}\}.$$

Definition: $\sim$-negation, $\sim$-expansion

A unary operation $\sim x$ in a Heyting algebra is called **tilde-negation** (or $\sim$-**negation** for short) if the following identities hold:

$$\begin{array}{llll} (a) & x \to y \leq \sim y \to \sim x; & (b) & x \wedge \sim x \leq \sim \mathbf{1}; \\ (c) & \sim \mathbf{0} \leq x \vee \sim x; & (d) & \sim \mathbf{0} \to \sim \mathbf{1} \leq \sim \mathbf{1}. \end{array}$$

A Heyting $\mathfrak{A}$ with a $\sim$-negation is called a $\sim$-**expansion** of $\mathfrak{A}$ and denoted by $(\mathfrak{A}, \sim)$. The abstract class of all $\sim$-expansions is denoted by K.

Consequences from the two last definitions

Proposition 1

Class K is a variety.

Proposition 2

Given a $\sim$-negation, $(\sim\mathbf{1}, \sim\mathbf{0})$ is an $\mathcal{E}$-pair.

Proposition 3

Given a Heyting algebra $\mathfrak{A}$, if (a, a^*) is an $\mathcal{E}$-pair, then the operation

$$\sim x = (x \to a) \wedge a^*$$

is a $\sim$-negation in $\mathfrak{A}$ so that $a = \sim\mathbf{1}$ and $a^* = \sim\mathbf{0}$.

Definition: τ-expansion, $\tau\sim$-expansion, τ-reduct

Let $\mathfrak{A}$ be a Heyting algebra. We enrich the signature of $\mathfrak{A}$ with a nullary operation τ and call $\mathfrak{A}_\tau = (\mathfrak{A}, \tau)$ a τ-**expansion** of $\mathfrak{A}$. The $\sim$-expansion of $\mathfrak{A}_\tau$ satisfying in additional to the $\sim$-identities the identity $\sim\mathbf{1} = \tau$ is called a $\tau\sim$-**expansion** (of $\mathfrak{A}$). Accordingly, the equational class of all $\tau\sim$-expansions is denoted by K_τ. If $(\mathfrak{A}_\tau, \sim)$ is a $\tau\sim$-expansion, we call $\mathfrak{A}_\tau$ a τ-**reduct** of the former.

Definition: packing, $\tau\sim$-embrace, relation $\lhd$

Suppose $\mathfrak{A}_\tau$ is a subalgebra of $\mathfrak{B}_\tau$ and $(\mathfrak{B}_\tau, \sim)$ is a $\tau\sim$-expansion generated by $|\mathfrak{A}|$. Then we say that $\mathfrak{A}_\tau$ **is packed in** $\mathfrak{B}_\tau$, or $\mathfrak{B}_\tau$ is a $\tau\sim$-**embrace** of $\mathfrak{A}_\tau$; symbolically $\mathfrak{A}_\tau \lhd \mathfrak{B}_\tau$. If $\mathfrak{A}_\tau$ is packed in $\mathfrak{B}_\tau$ and $(\mathfrak{A}_\tau, \sim)$, regarded as a partial algebra w.r.t. $\sim$, is a relative subalgebra of a (full) algebra $(\mathfrak{B}_\tau, \sim)$ (in the sense of G. Grätzer, Universal Algebra, § 13), we also say that $(\mathfrak{A}_\tau, \sim)$ **is packed in** $(\mathfrak{B}_\tau, \sim)$, denoting this by $(\mathfrak{A}_\tau, \sim) \lhd (\mathfrak{B}_\tau, \sim)$.

Proposition 4

If $\mathfrak{A}_\tau \lhd \mathfrak{B}_\tau$, then $\mathfrak{B}_\tau$ is generated by Heyting operations from $|\mathfrak{A}_\tau| \cup \{\sim\tau\}$.

Two (important) examples

1. Given a τ-expansion $\mathfrak{A}_\tau$, $\mathfrak{A}^{\Delta}_\tau$ is a $\tau\sim$-embrace of the former.
2. Let Z be a 2-element Heyting algebra. Then $\mathcal{E}_Z = \{(\mathbf{0},\mathbf{1}),((\mathbf{1},\mathbf{1})\}$. Accordingly, we have two choices to define $\sim$:

x	$\sim x$
$\mathbf{0}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$

x	$\sim x$
$\mathbf{0}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{1}$

Proposition 5

Any nontrivial subvariety of K_τ contains Z_τ.

Proposition 6

Any finite $\mathfrak{A}_\tau$ is packed in itself. Hence, Z_τ is packed in itself.

Expansions of Int

Extending $\mathcal{L}_a$

- The atomic formulas — $Atom = Var \cup \{\tau\}$
- $\mathcal{L}_\tau$-formulas $(A, B, \ldots) := A \in Atom|(A \wedge B)|(A \vee B)|(A \to B)|\neg A$
- $\mathcal{L}_{\tau\sim}$-formulas $(\alpha, \beta, \ldots) :=$
  $\alpha \in Atom|(\alpha \wedge \beta)|(\alpha \vee \beta)|(\alpha \to \beta)|\neg\alpha|{\sim}\alpha$

Expanding **Int**

- $\mathbf{Int}_\tau$ denotes intuitionistic propositional logic in $\mathcal{L}_\tau$
- $\mathbf{Int}_{\tau\sim}$ denotes intuitionistic logic in $\mathcal{L}_{\tau\sim}$ plus the following axioms:
  1. ${\sim}p_0 \leftrightarrow (p_0 \to \tau) \wedge {\sim}\tau$,
  2. $({\sim}\tau \to \tau) \to \tau$,
  3. ${\sim}\tau \to (p_0 \vee (p_0 \to \tau))$,
  4. $\tau \to {\sim}\tau$.

Proposition 7

$\mathbf{Int}_{\tau\sim} \vdash \alpha \iff \alpha$ is valid in any $\tau\sim$-expansion.

Proposition 8

For any $\mathcal{L}_\tau$-formulas A and B,

$$\mathbf{Int}_{\tau\sim} + A \vdash B \Longleftrightarrow \mathbf{Int}_\tau + A \vdash B.$$

Corollary 8.1

Any variety of τ-expansions is generated by those algebras of the variety which are τ-reducts of a $\tau\sim$-embraces.

Remark

In the proof of Corollary 8.1, we use that any variety of τ-expansions contains τ-reducts of some $\tau\sim$-embraces, for example, Z_τ.

Corollary 8.2

Any τ-expansion $\mathfrak{A}_\tau$ is embedded into the τ-reduct $\mathfrak{B}_\tau$ of $\tau\sim$-expansion $(\mathfrak{B}_\tau, \sim)$ so that $\mathsf{HSP}(\mathfrak{A}_\tau) = \mathsf{HSP}(\mathfrak{B}_\tau)$. Hence, for any $\mathfrak{A}_\tau$, there is a $\mathfrak{B}_\tau$ such that $\mathfrak{A}_\tau \lhd \mathfrak{B}_\tau$ and $\mathsf{HSP}(\mathfrak{A}_\tau) = \mathsf{HSP}(\mathfrak{B}_\tau)$.

Conclusive Step

Propositin 9

Let a Heyting algebra $\mathfrak{A}$ be a subalgebra of $\mathfrak{B}$ and let $a \in \mathfrak{A}$. Assume that a is enriched in $\mathfrak{B}$ by and an element b. If $\mathfrak{B}$ is generated by $|\mathfrak{A}| \cup \{b\}$, then $\mathfrak{B}$ is isomorphic to $\mathfrak{A}_a^{\Delta}$.

Using Corollary 8.2 and Proposition 9, we obtain the following.

Proposition 10

Let $\mathfrak{A}$ be a Heyting algebra and $a \in \mathfrak{A}$. Then $\mathrm{HSP}(\mathfrak{A}) = \mathrm{HSP}(\mathfrak{A}_a^{\Delta})$.

Theorem

For any Heyting algebra $\mathfrak{A}$, $\mathrm{HSP}(\mathfrak{A}) = \mathrm{HSP}(\vec{\mathfrak{A}})$.

Thank You