Ideals and involutive filters in residuated lattices

Jiří Rachůnek   Dana Šalounová

Palacký University in Olomouc
VŠB–Technical University of Ostrava

Czech Republic

TACL 2015
Ischia, June 2015

The first author was supported by the IGA PřF 2015010.
The second author was supported by ESF Project CZ.1.07/2.3.00/20.0296.

Residuated lattices.

Definition

A bounded integral residuated lattice is an algebra $M = (M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0 \rangle$ satisfying:

- $(M;\ \odot, 1)$ is a monoid;
- $(M;\ \vee, \wedge, 0, 1)$ is a bounded lattice;
- $x \odot y \leq z$ iff $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$.

In what follows, a residuated lattice is a bounded integral residuated lattice.

Additional unary operations:

$x^- := x \to 0, \quad x^\sim := x \rightsquigarrow 0$

A residuated lattice M is called

- *good* if it satisfies $x^{-\sim} = x^{\sim-}$.
- *normal* if it satisfies

$$(x \odot y)^{-\sim} = x^{-\sim} \odot y^{-\sim}, \quad (x \odot y)^{\sim-} = x^{\sim-} \odot y^{\sim-}.$$

Residuated lattices.

Definition

A bounded integral residuated lattice is an algebra $M = (M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0 \rangle$ satisfying:

- $(M; \odot, 1)$ is a monoid;
- $(M; \vee, \wedge, 0, 1)$ is a bounded lattice;
- $x \odot y \leq z$ iff $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$.

In what follows, a residuated lattice is a bounded integral residuated lattice.

Additional unary operations:

$x^- := x \to 0, \ x^\sim := x \rightsquigarrow 0$

A residuated lattice M is called

- *good* if it satisfies $x^{-\sim} = x^{\sim -}$.
- *normal* if it satisfies

$$(x \odot y)^{-\sim} = x^{-\sim} \odot y^{-\sim}, \ (x \odot y)^{\sim -} = x^{\sim -} \odot y^{\sim -}.$$

Residuated lattices.

Definition

A bounded integral residuated lattice is an algebra $M = (M; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ of type $\langle 2,2,2,2,2,0,0 \rangle$ satisfying:

- $(M;\ \odot, 1)$ is a monoid;
- $(M;\ \vee, \wedge, 0, 1)$ is a bounded lattice;
- $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$.

In what follows, a residuated lattice is a bounded integral residuated lattice.

Additional unary operations:

$x^- := x \rightarrow 0, \ x^\sim := x \rightsquigarrow 0$

A residuated lattice M is called

- *good* if it satisfies $x^{-\sim} = x^{\sim -}$.
- *normal* if it satisfies

$$(x \odot y)^{-\sim} = x^{-\sim} \odot y^{-\sim}, \ (x \odot y)^{\sim -} = x^{\sim -} \odot y^{\sim -}.$$

Special cases of residuated lattices.

Terms of properties of residuated structures

(1)	$(x \to y) \vee (y \to x) \;=\; 1 \;=\; (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$	pre-linearity
(2)	$(x \to y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$	divisibility
(3)	$x^{-\sim} = x = x^{\sim -}$	involution
(4)	$x \odot x = x$	idempotency

A residuated lattice M is

- a pseudo MTL-algebra if M satisfies (1);
- an $R\ell$-monoid if M satisfies (2);
- a pseudo BL-algebra if M satisfies (1) and (2);
- a Heyting algebra if M satisfies (4);
- a GMV-algebra (pseudo MV-algebra equivalently) if M satisfies (2) and (3).

Special cases of residuated lattices.

Terms of properties of residuated structures

(1)	$(x \to y) \vee (y \to x) \;=\; 1 \;=\; (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$	pre-linearity
(2)	$(x \to y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$	divisibility
(3)	$x^{-\sim} = x = x^{\sim -}$	involution
(4)	$x \odot x = x$	idempotency

A residuated lattice M is

- a pseudo MTL-algebra if M satisfies (1);
- an $R\ell$-monoid if M satisfies (2);
- a pseudo BL-algebra if M satisfies (1) and (2);
- a Heyting algebra if M satisfies (4);
- a GMV-algebra (pseudo MV-algebra equivalently) if M satisfies (2) and (3).

Special cases of residuated lattices.

Terms of properties of residuated structures

(1)	$(x \to y) \vee (y \to x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$	pre-linearity
(2)	$(x \to y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$	divisibility
(3)	$x^{-\sim} = x = x^{\sim -}$	involution
(4)	$x \odot x = x$	idempotency

A residuated lattice M is

- a pseudo MTL-algebra if M satisfies (1);
- an $R\ell$-monoid if M satisfies (2);
- a pseudo BL-algebra if M satisfies (1) and (2);
- a Heyting algebra if M satisfies (4);
- a GMV-algebra (pseudo MV-algebra equivalently) if M satisfies (2) and (3).

Special cases of residuated lattices.

Terms of properties of residuated structures

(1)	$(x \to y) \vee (y \to x) \;=\; 1 \;=\; (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$	pre-linearity
(2)	$(x \to y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$	divisibility
(3)	$x^{-\sim} = x = x^{\sim -}$	involution
(4)	$x \odot x = x$	idempotency

A residuated lattice M is

- a pseudo MTL-algebra if M satisfies (1);
- an $R\ell$-monoid if M satisfies (2);
- a pseudo BL-algebra if M satisfies (1) and (2);
- a Heyting algebra if M satisfies (4);
- a GMV-algebra (pseudo MV-algebra equivalently) if M satisfies (2) and (3).

Special cases of residuated lattices.

Terms of properties of residuated structures

(1) $(x \to y) \vee (y \to x) \ = \ 1 \ = \ (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$ pre-linearity

(2) $(x \to y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$ divisibility

(3) $x^{-\sim} = x = x^{\sim -}$ involution

(4) $x \odot x = x$ idempotency

A residuated lattice M is

- a pseudo MTL-algebra if M satisfies (1);
- an $R\ell$-monoid if M satisfies (2);
- a pseudo BL-algebra if M satisfies (1) and (2);
- a Heyting algebra if M satisfies (4);
- a GMV-algebra (pseudo MV-algebra equivalently) if M satisfies (2) and (3).

Special cases of residuated lattices.

Terms of properties of residuated structures

(1)	$(x \to y) \vee (y \to x) \; = \; 1 \; = \; (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$	pre-linearity
(2)	$(x \to y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$	divisibility
(3)	$x^{-\sim} = x = x^{\sim -}$	involution
(4)	$x \odot x = x$	idempotency

A residuated lattice M is

- a pseudo MTL-algebra if M satisfies (1);
- an $R\ell$-monoid if M satisfies (2);
- a pseudo BL-algebra if M satisfies (1) and (2);
- a Heyting algebra if M satisfies (4);
- a GMV-algebra (pseudo MV-algebra equivalently) if M satisfies (2) and (3).

Special cases of residuated lattices.

Terms of properties of residuated structures

(1)	$(x \to y) \vee (y \to x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$	pre-linearity
(2)	$(x \to y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$	divisibility
(3)	$x^{-\sim} = x = x^{\sim -}$	involution
(4)	$x \odot x = x$	idempotency

A residuated lattice M is

- a pseudo MTL-algebra if M satisfies (1);
- an $R\ell$-monoid if M satisfies (2);
- a pseudo BL-algebra if M satisfies (1) and (2);
- a Heyting algebra if M satisfies (4);
- a GMV-algebra (pseudo MV-algebra equivalently) if M satisfies (2) and (3).

Filters and congruences.

A non-empty subset F of a residuated lattice M is called a *filter* of M if

(a) $x, y \in F$ imply $x \odot y \in F$;

(b) $x \in F,\ y \in M,\ x \leq y$ imply $y \in F$.

Filters and congruences.

A non-empty subset F of a residuated lattice M is called a *filter* of M if

(a) $x, y \in F$ imply $x \odot y \in F$;

(b) $x \in F,\ y \in M,\ x \leq y$ imply $y \in F$.

A filter F is called *normal* if for each $x, y \in M$

(c) $x \to y \in F \iff x \rightsquigarrow y \in F$.

normal filters of M $\longleftrightarrow$ kernels (i.e. 1-classes) of congruences on M

$$\begin{aligned} \langle x, y\rangle \in \theta_F &\iff (x \to y) \odot (y \to x) \in F \\ &\iff (x \rightsquigarrow y) \odot (y \rightsquigarrow x) \in F. \end{aligned}$$

M/F . . . a quotient residuated lattice;

$x/F = x/\theta_F$ . . . the class of M/F containing x.

Filters and congruences.

A non-empty subset F of a residuated lattice M is called a *filter* of M if

(a) $x, y \in F$ imply $x \odot y \in F$;

(b) $x \in F,\ y \in M,\ x \leq y$ imply $y \in F$.

A filter F is called *normal* if for each $x, y \in M$

(c) $x \to y \in F \iff x \rightsquigarrow y \in F$.

normal filters of M $\longleftrightarrow$ kernels (i.e. 1-classes) of congruences on M

$$\langle x, y\rangle \in \theta_F \iff (x \to y) \odot (y \to x) \in F$$
$$\iff (x \rightsquigarrow y) \odot (y \rightsquigarrow x) \in F.$$

M/F ... a quotient residuated lattice;

$x/F = x/\theta_F$ ... the class of M/F containing x.

Filters and congruences.

A non-empty subset F of a residuated lattice M is called a *filter* of M if

(a) $x, y \in F$ imply $x \odot y \in F$;

(b) $x \in F,\ y \in M,\ x \leq y$ imply $y \in F$.

A filter F is called *normal* if for each $x, y \in M$

(c) $x \to y \in F \iff x \rightsquigarrow y \in F$.

normal filters of M $\longleftrightarrow$ kernels (i.e. 1-classes) of congruences on M

$$\begin{aligned} \langle x, y\rangle \in \theta_F &\iff (x \to y) \odot (y \to x) \in F \\ &\iff (x \rightsquigarrow y) \odot (y \rightsquigarrow x) \in F. \end{aligned}$$

M/F . . . a quotient residuated lattice;

$x/F = x/\theta_F$ . . . the class of M/F containing x.

GMV-algebras (pseudo *MV*-algebras).

Definition

A *GMV-algebra* is an algebra $M = (M; \oplus, ^-, ^\sim, 0, 1)$ of type $\langle 2, 1, 1, 0, 0 \rangle$, where $x \odot y := (x^- \oplus y^-)^\sim$ for any $x, y \in M$, satisfying:

- $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- $x \oplus 0 = x = 0 \oplus x$;
- $x \oplus 1 = 1 = 1 \oplus x$;
- $1^- = 0 = 1^\sim$;
- $(x^\sim \oplus y^\sim)^- = (x^- \oplus y^-)^\sim$;
- $x \oplus (y \odot x^\sim) = y \oplus (x \odot y^\sim) = (y^- \odot x) \oplus y = (x^- \odot y) \oplus x$;
- $(x^- \oplus y) \odot x = y \odot (x \oplus y^\sim)$;
- $x^{-\sim} = x$.

$$x \leq y \ \text{ iff } \ x^- \oplus y = 1,$$

GMV-algebras (pseudo MV-algebras).

Definition

A GMV-algebra is an algebra $M = (M; \oplus, ^-, ^\sim, 0, 1)$ of type $\langle 2, 1, 1, 0, 0\rangle$, where $x \odot y := (x^- \oplus y^-)^\sim$ for any $x, y \in M$, satisfying:

- $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- $x \oplus 0 = x = 0 \oplus x$;
- $x \oplus 1 = 1 = 1 \oplus x$;
- $1^- = 0 = 1^\sim$;
- $(x^\sim \oplus y^\sim)^- = (x^- \oplus y^-)^\sim$;
- $x \oplus (y \odot x^\sim) = y \oplus (x \odot y^\sim) = (y^- \odot x) \oplus y = (x^- \odot y) \oplus x$;
- $(x^- \oplus y) \odot x = y \odot (x \oplus y^\sim)$;
- $x^{-\sim} = x$.

$x \leq y$ iff $x^- \oplus y = 1$, $(M, \leq)$ is a bounded distributive lattice,
$x \vee y = x \oplus (y \odot x^\sim)$, $x \wedge y = x \odot (y \oplus x^\sim)$

GMV-algebras (pseudo MV-algebras).

Definition

A GMV-algebra is an algebra $M = (M; \oplus, ^-, ^\sim, 0, 1)$ of type $\langle 2, 1, 1, 0, 0\rangle$, where $x \odot y := (x^- \oplus y^-)^\sim$ for any $x, y \in M$, satisfying:

- $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- $x \oplus 0 = x = 0 \oplus x$;
- $x \oplus 1 = 1 = 1 \oplus x$;
- $1^- = 0 = 1^\sim$;
- $(x^\sim \oplus y^\sim)^- = (x^- \oplus y^-)^\sim$;
- $x \oplus (y \odot x^\sim) = y \oplus (x \odot y^\sim) = (y^- \odot x) \oplus y = (x^- \odot y) \oplus x$;
- $(x^- \oplus y) \odot x = y \odot (x \oplus y^\sim)$;
- $x^{-\sim} = x$.

$$x \leq y \ \text{ iff } \ x^- \oplus y = 1, \quad (M, \leq) \text{ is a bounded distributive lattice,}$$
$$x \vee y = x \oplus (y \odot x^\sim), \ \ x \wedge y = x \odot (y \oplus x^\sim)$$

Ideals of GMV-algebras.

A non-empty subset I of a GMV-algebra M is called an *ideal* of M if

(a) $x, y \in I$ imply $x \oplus y \in I$;

(b) $x \in I,\ y \in M,\ y \leq x$ imply $y \in I$.

Ideals of *GMV*-algebras.

A non-empty subset I of a GMV-algebra M is called an *ideal* of M if

(a) $x, y \in I$ imply $x \oplus y \in I$;

(b) $x \in I,\ y \in M,\ y \leq x$ imply $y \in I$.

An ideal I is called *normal* if for each $x, y \in M$

(c) $x^- \odot y \in I \iff y \odot x^\sim \in I$.

normal ideals of M $\longleftrightarrow$ kernels of congruences on M

$$\langle x, y\rangle \in \theta_I \iff (x^- \odot y) \oplus (y^- \odot x) \in I$$
$$\iff (x \odot y^\sim) \oplus (y \odot x^\sim) \in I.$$

Ideals of GMV-algebras.

A non-empty subset I of a GMV-algebra M is called an *ideal* of M if

(a) $x, y \in I$ imply $x \oplus y \in I$;

(b) $x \in I,\ y \in M,\ y \leq x$ imply $y \in I$.

An ideal I is called *normal* if for each $x, y \in M$

(c) $x^- \odot y \in I \iff y \odot x^\sim \in I$.

normal ideals of M $\longleftrightarrow$ kernels of congruences on M

$$\begin{aligned} \langle x, y \rangle \in \theta_I &\iff (x^- \odot y) \oplus (y^- \odot x) \in I \\ &\iff (x \odot y^\sim) \oplus (y \odot x^\sim) \in I. \end{aligned}$$

Motivation.

Filter and ideal theories of GMV-algebras are mutually dual.

Motivation.

1. In GMV-algebra M, $x \odot y := (x^- \oplus y^-)^\sim$, put $x \to y := x^- \oplus y$, $x \rightsquigarrow y := y \oplus x^\sim$, then $(M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ is a residuated lattice with (2), (3).
2. Conversely, let $(M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ be a residuated lattice with (2), (3). Put $x^- := x \to 0$, $x^\sim := x \rightsquigarrow 0$, $x \oplus y := (x^- \odot y^-)^\sim = (x^\sim \odot y^\sim)^-$. Then $M = (M; \oplus, ^-, ^\sim, 0, 1)$ is a GMV-algebra.

Filter and ideal theories of GMV-algebras are mutually dual.

Motivation.

1. In GMV-algebra M, $x \odot y := (x^- \oplus y^-)^\sim$, put $x \to y := x^- \oplus y$, $x \rightsquigarrow y := y \oplus x^\sim$, then $(M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ is a residuated lattice with (2), (3).
2. Conversely, let $(M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ be a residuated lattice with (2), (3). Put $x^- := x \to 0$, $x^\sim := x \rightsquigarrow 0$, $x \oplus y := (x^- \odot y^-)^\sim = (x^\sim \odot y^\sim)^-$. Then $M = (M; \oplus, ^-, ^\sim, 0, 1)$ is a GMV-algebra.

Filter and ideal theories of GMV-algebras are mutually dual.

Motivation.

1. In GMV-algebra M, $x \odot y := (x^- \oplus y^-)^\sim$, put $x \to y := x^- \oplus y$, $x \rightsquigarrow y := y \oplus x^\sim$, then $(M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ is a residuated lattice with (2), (3).
2. Conversely, let $(M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ be a residuated lattice with (2), (3). Put $x^- := x \to 0$, $x^\sim := x \rightsquigarrow 0$, $x \oplus y := (x^- \odot y^-)^\sim = (x^\sim \odot y^\sim)^-$. Then $M = (M; \oplus, ^-, ^\sim, 0, 1)$ is a GMV-algebra.

   Filter and ideal theories of GMV-algebras are mutually dual.

A dual binary operation to multiplication in residuated lattices does not exist.

Motivation.

1. In GMV-algebra M, $x \odot y := (x^- \oplus y^-)^\sim$, put $x \to y := x^- \oplus y$, $x \rightsquigarrow y := y \oplus x^\sim$, then $(M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ is a residuated lattice with (2), (3).
2. Conversely, let $(M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ be a residuated lattice with (2), (3). Put $x^- := x \to 0$, $x^\sim := x \rightsquigarrow 0$, $x \oplus y := (x^- \odot y^-)^\sim = (x^\sim \odot y^\sim)^-$. Then $M = (M; \oplus, ^-, ^\sim, 0, 1)$ is a GMV-algebra.

   Filter and ideal theories of GMV-algebras are mutually dual.

A dual binary operation to multiplication in residuated lattices does not exist.

The aim.

- To fill the gap by introducing the notion of ideal in residuated lattices.
- To establish congruences in residuated lattices using ideals.

The aim.

- To fill the gap by introducing the notion of ideal in residuated lattices.
- To establish congruences in residuated lattices using ideals.

The aim.

- To fill the gap by introducing the notion of ideal in residuated lattices.
- To establish congruences in residuated lattices using ideals.

Ideals of residuated lattices.

Let M be a residuated lattice, we put

- $x \oslash y := y^- \rightsquigarrow x$ ... *left addition*,
- $x \obslash y := x^\sim \to y$ ... *right addition* on M.

Definition

A non-empty subset I of a residuated lattice M is called a *left ideal* of M if

(a) $x, y \in I \implies x \oslash y \in I$;

(b) $x \in I,\ z \in M,\ z \leq x \implies z \in I$.

Theorem 1

Let I be a subset of a residuated lattice M containing 0. Then I is a left ideal of M if and only if

$$\forall x, y \in M;\ x^- \odot y \in I, x \in I \implies y \in I.$$

Ideals of residuated lattices.

Let M be a residuated lattice, we put

- $x \oslash y := y^{-} \rightsquigarrow x$ ... *left addition*,
- $x \odot y := x^{\sim} \to y$ ... *right addition* on M.

Definition

A non-empty subset I of a residuated lattice M is called a *left ideal* of M if

(a) $x, y \in I \implies x \oslash y \in I$;

(b) $x \in I,\ z \in M,\ z \leq x \implies z \in I$.

Theorem 1

Let I be a subset of a residuated lattice M containing 0. Then I is a left ideal of M if and only if

$$\forall x, y \in M;\ x^{-} \odot y \in I, x \in I \implies y \in I.$$

Ideals of residuated lattices.

Let M be a residuated lattice, we put

- $x \oslash y := y^- \rightsquigarrow x$ ... *left addition*,
- $x \obslash y := x^\sim \to y$ ... *right addition* on M.

Definition

A non-empty subset I of a residuated lattice M is called a *left ideal* of M if

(a) $x, y \in I \implies x \oslash y \in I$;

(b) $x \in I,\ z \in M,\ z \leq x \implies z \in I$.

Theorem 1

Let I be a subset of a residuated lattice M containing 0. Then I is a left ideal of M if and only if

$$\forall x, y \in M;\ x^- \odot y \in I, x \in I \implies y \in I.$$

Ideals of residuated lattices.

Definition

A non-empty subset I of a residuated lattice M is called a *right ideal* of M if

(a') $x, y \in I \implies x \oslash y \in I$;

(b) $x \in I,\ z \in M,\ z \leq x \implies z \in I$.

Theorem 1'

Let I be a subset of a residuated lattice M containing 0. Then I is a right ideal of M if and only if

$$\forall x, y \in M;\ y \odot x^{\sim} \in I, x \in I \implies y \in I.$$

Every left ideal as well as every right ideal of a residuated lattice M is a lattice ideal.

Ideals of residuated lattices.

Definition

A non-empty subset I of a residuated lattice M is called a *right ideal* of M if

(a') $\quad x, y \in I \implies x \oslash y \in I$;

(b) $\quad x \in I,\ z \in M,\ z \leq x \implies z \in I$.

Theorem 1'

Let I be a subset of a residuated lattice M containing 0. Then I is a right ideal of M if and only if

$$\forall x, y \in M;\ y \odot x^{\sim} \in I, x \in I \implies y \in I.$$

Every left ideal as well as every right ideal of a residuated lattice M is a lattice ideal.

Ideals of residuated lattices.

Definition

A non-empty subset I of a residuated lattice M is called a *ideal* of M if it is both left and right ideal of M, i.e.

(a) $x, y \in I \implies x \oslash y \in I$;

(a') $x, y \in I \implies x \obslash y \in I$;

(b) $x \in I,\ z \in M,\ z \leq x \implies z \in I$.

For an ideal I of a residuated lattice M,

$$\langle x, y \rangle \in \theta_I \ :\Longleftrightarrow\ x^- \odot y \in I,\ y^- \odot x \in I,\ x \odot y^\sim \in I,\ y \odot x^\sim \in I.$$

θ_I ... an equivalence on M.

Theorem 2

a) Let M be a residuated lattice and I an ideal of M. Then the equivalence θ_I is a congruence on the reduct $(M; \odot, \vee, \to, \rightsquigarrow, 0, 1)$ of the residuated lattice M.

b) If M is a pseudo BL-algebra then θ_I is a congruence on M.

Ideals of residuated lattices.

Definition

A non-empty subset I of a residuated lattice M is called a *ideal* of M if it is both left and right ideal of M, i.e.

(a) $x, y \in I \implies x \oslash y \in I$;

(a') $x, y \in I \implies x \varobslash y \in I$;

(b) $x \in I,\ z \in M,\ z \leq x \implies z \in I$.

For an ideal I of a residuated lattice M,

$$\langle x, y\rangle \in \theta_I \;:\Longleftrightarrow\; x^- \odot y \in I,\ y^- \odot x \in I,\ x \odot y^\sim \in I,\ y \odot x^\sim \in I.$$

θ_I ... an equivalence on M.

Theorem 2

a) Let M be a residuated lattice and I an ideal of M. Then the equivalence θ_I is a congruence on the reduct $(M; \odot, \vee, \to, \rightsquigarrow, 0, 1)$ of the residuated lattice M.

b) If M is a pseudo BL-algebra then θ_I is a congruence on M.

Ideals of residuated lattices.

Definition

A non-empty subset I of a residuated lattice M is called a *ideal* of M if it is both left and right ideal of M, i.e.

(a) $x, y \in I \implies x \oslash y \in I$;

(a') $x, y \in I \implies x \obslash y \in I$;

(b) $x \in I,\ z \in M,\ z \leq x \implies z \in I$.

For an ideal I of a residuated lattice M,

$$\langle x, y\rangle \in \theta_I \ :\Longleftrightarrow\ x^- \odot y \in I,\ y^- \odot x \in I,\ x \odot y^\sim \in I,\ y \odot x^\sim \in I.$$

θ_I ... an equivalence on M.

Theorem 2

a) Let M be a residuated lattice and I an ideal of M. Then the equivalence θ_I is a congruence on the reduct $(M; \odot, \vee, \to, \rightsquigarrow, 0, 1)$ of the residuated lattice M.

b) If M is a pseudo BL-algebra then θ_I is a congruence on M.

Ideals of residuated lattices.

Theorem 3

a) If M is a pseudo BL-algebra and I is an ideal of M, then M/θ_I is a GMV-algebra.

b) If M is any residuated lattice then M/θ_I is an involutive residuated lattice.

Involutive filter, Glivenko property

If F is a normal filter of a residuated lattice M, then we say that F is an *involutive filter* if the quotient residuated lattice M/F is involutive.

Glivenko property

A residuated lattice M satisfies the *Glivenko property* if for any $x, y \in M$

$$(x \to y)^{-\sim} = x \to y^{-\sim}, \quad (x \rightsquigarrow y)^{\sim -} = x \rightsquigarrow y^{\sim -}.$$

Involutive filter, Glivenko property

If F is a normal filter of a residuated lattice M, then we say that F is an *involutive filter* if the quotient residuated lattice M/F is involutive.

Glivenko property

A residuated lattice M satisfies the *Glivenko property* if for any $x, y \in M$

$$(x \to y)^{-\sim} = x \to y^{-\sim}, \ \ (x \rightsquigarrow y)^{\sim -} = x \rightsquigarrow y^{\sim -}.$$

The Glivenko property was introduced (Cignoli, Torrens 2004) for commutative residuated lattices

$$(x \to y)^{--} = x \to y^{--}.$$

For a good residuated lattice M, the following conditions are equivalent:

(i) $(x^{-\sim} \to x)^{-\sim} = 1 = (x^{\sim -} \rightsquigarrow x)^{\sim -}$,

(ii) $(x \to y)^{-\sim} = x^{-\sim} \to y^{-\sim}$, $(x \rightsquigarrow y)^{\sim -} = x^{\sim -} \rightsquigarrow y^{\sim -}$,

(iii) (GP).

Involutive filter, Glivenko property

If F is a normal filter of a residuated lattice M, then we say that F is an *involutive filter* if the quotient residuated lattice M/F is involutive.

Glivenko property

A residuated lattice M satisfies the *Glivenko property* if for any $x, y \in M$

$$(x \to y)^{-\sim} = x \to y^{-\sim}, \quad (x \rightsquigarrow y)^{\sim-} = x \rightsquigarrow y^{\sim-}.$$

The Glivenko property was introduced (Cignoli, Torrens 2004) for commutative residuated lattices

$$(x \to y)^{--} = x \to y^{--}.$$

For a good residuated lattice M, the following conditions are equivalent:

(i) $(x^{-\sim} \to x)^{-\sim} = 1 = (x^{\sim-} \rightsquigarrow x)^{\sim-}$,

(ii) $(x \to y)^{-\sim} = x^{-\sim} \to y^{-\sim}$, $(x \rightsquigarrow y)^{\sim-} = x^{\sim-} \rightsquigarrow y^{\sim-}$,

(iii) (GP).

Set of dense elements.

For a residuated lattice M:

$$D(M) := \{x \in M : x^{-\sim} = 1 = x^{\sim -}\}.$$

Theorem 4

a) If M is a good residuated lattice, then $D(M)$ is a filter of M.

b) If, moreover, M satisfies (GP), then $D(M)$ is a normal filter of M.

Theorem 5

Let M be a good residuated lattice satisfying (GP) and $x, y \in M$. Then $\langle x, y\rangle \in \theta_{D(M)}$ if and only if $x^{-\sim} = y^{-\sim}$. Moreover, $M/D(M)$ is an involutive residuated lattice, i.e. $D(M)$ is an involutive filter.

$D(M)$ and involutive normal filters.

Theorem 6

If a good residuated lattice M satisfies (GP) and F is an involutive normal filter of M, then $D(M) \subseteq F$.

Proposition 7

If F_1 and F_2 are normal filters of a residuated lattice M, $F_1 \subseteq F_2$ and F_1 is an involutive filter, then F_2 is also involutive.

Corollary 8

If M is a good residuated lattice satisfying (GP) then the involutive filters of M are exactly all normal filters of M containing $D(M)$.

$D(M)$ and involutive normal filters.

Theorem 6

If a good residuated lattice M satisfies (GP) and F is an involutive normal filter of M, then $D(M) \subseteq F$.

Proposition 7

If F_1 and F_2 are normal filters of a residuated lattice M, $F_1 \subseteq F_2$ and F_1 is an involutive filter, then F_2 is also involutive.

Corollary 8

If M is a good residuated lattice satisfying (GP) then the involutive filters of M are exactly all normal filters of M containing $D(M)$.

Connections among ideals, filters and congruences.

Proposition 9

If M is a residuated lattice and I is an ideal of M then I is the 0-class in M/θ_I.

Proposition 10

Let I be an ideal of a pseudo BL-algebra and $F = F_I = 1/\theta_I$. Then F is an involutive normal filter of M.

Proposition 11

If M is a residuated lattice and F is a normal filter of M, then the class $0/F$ is an ideal of M.

Connections among ideals, filters and congruences.

Proposition 9

If M is a residuated lattice and I is an ideal of M then I is the 0-class in M/θ_I.

Proposition 10

Let I be an ideal of a pseudo BL-algebra and $F = F_I = 1/\theta_I$. Then F is an involutive normal filter of M.

Proposition 11

If M is a residuated lattice and F is a normal filter of M, then the class $0/F$ is an ideal of M.

Connections among ideals, filters and congruences.

Proposition 9

If M is a residuated lattice and I is an ideal of M then I is the 0-class in M/θ_I.

Proposition 10

Let I be an ideal of a pseudo BL-algebra and $F = F_I = 1/\theta_I$. Then F is an involutive normal filter of M.

Proposition 11

If M is a residuated lattice and F is a normal filter of M, then the class $0/F$ is an ideal of M.

Connections among ideals, filters and congruences.

Theorem 12 (the main result)

If M is an arbitrary pseudo BL-algebra then there is a one-to-one correspondence between ideals and involutive normal filters of M.

Remark 13

Let M be a good pseudo BL-algebra. In the previous correspondence, the ideal $\{0\}$ corresponds to the filter $D(M)$. (In fact, this is also true for any good residuated lattice satisfying (GP).)