Ideals and involutive filters in residuated lattices

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Residuated lattices.

Definition

A bounded integral residuated lattice is an algebra $M = (M; \odot, \lor, \land, \to, \multimap, 0, 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0 \rangle$ satisfying:

- $(M; \odot, 1)$ is a monoid;
- $(M; \lor, \land, 0, 1)$ is a bounded lattice;
- $x \odot y \leq z$ iff $x \leq y \to z$ iff $y \leq x \multimap z$.

In what follows, a residuated lattice is a bounded integral residuated lattice.

Additional unary operations:

- $x^- := x \to 0$,
- $x^\sim := x \multimap 0$

A residuated lattice $M$ is called

- **good** if it satisfies $x^{\sim \sim} = x^{\sim \sim}$,
- **normal** if it satisfies
  
  $$
  (x \odot y)^{\sim \sim} = x^{\sim \sim} \odot y^{\sim \sim},
  (x \odot y)^{\sim \sim} = x^{\sim \sim} \odot y^{\sim \sim}.
  $$

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- $x \odot y \leq z$ iff $x \leq y \to z$ iff $y \leq x \rhd z$.

In what follows, a residuated lattice is a bounded integral residuated lattice.

Additional unary operations:

- $x^- := x \to 0$, $x^- := x \rhd 0$

A residuated lattice $M$ is called

- **good** if it satisfies $x^-^- = x^-^-$.  
- **normal** if it satisfies  
  $$(x \odot y)^-- = x^-^- \odot y^-^-, \quad (x \odot y)^-^- = x^-^- \odot y^-^-.$$
A bounded integral residuated lattice is an algebra $M = (M; \odot, \lor, \land, \rightarrow, \leftarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0 \rangle$ satisfying:

- $(M; \odot, 1)$ is a monoid;
- $(M; \lor, \land, 0, 1)$ is a bounded lattice;
- $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \leftarrow z$.

In what follows, a residuated lattice is a bounded integral residuated lattice.

Additional unary operations:

- $x^- := x \rightarrow 0$, $x^\sim := x \leftarrow 0$

A residuated lattice $M$ is called

- **good** if it satisfies $x^-\sim = x^\sim\sim$.
- **normal** if it satisfies

$$((x \odot y)^-\sim = x^-\sim \odot y^-\sim, (x \odot y)^\sim\sim = x^\sim\sim \odot y^\sim\sim).$$
Special cases of residuated lattices.

Terms of properties of residuated structures

(1) \((x \to y) \lor (y \to x) = 1 = (x \rightarrows y) \lor (y \rightarrows x)\) \hspace{1cm} \text{pre-linearity}
(2) \((x \to y) \circ x = x \land y = y \circ (y \rightarrows x)\) \hspace{1cm} \text{divisibility}
(3) \(x^{\sim \sim} = x = x^{\sim \sim}\) \hspace{1cm} \text{involution}
(4) \(x \circ x = x\) \hspace{1cm} \text{idempotency}

A residuated lattice \(M\) is

- a pseudo MTL-algebra if \(M\) satisfies (1);
- an \(R\ell\)-monoid if \(M\) satisfies (2);
- a pseudo BL-algebra if \(M\) satisfies (1) and (2);
- a Heyting algebra if \(M\) satisfies (4);
- a GMV-algebra (pseudo MV-algebra equivalently) if \(M\) satisfies (2) and (3).
Special cases of residuated lattices.

### Terms of properties of residuated structures

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A residuated lattice \(M\) is

- a **pseudo MTL-algebra** if \(M\) satisfies (1);
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- a **pseudo BL-algebra** if \(M\) satisfies (1) and (2);
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- a **GMV-algebra** (pseudo \(MV\)-algebra equivalently) if \(M\) satisfies (2) and (3).
Special cases of residuated lattices.

Terms of properties of residuated structures

\[(1) \ (x \rightarrow y) \vee (y \rightarrow x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) \quad \text{pre-linearity}\]
\[(2) \ (x \rightarrow y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x) \quad \text{divisibility}\]
\[(3) \ x \rightsquigarrow \neg = x = x \neg \rightsquigarrow \quad \text{involution}\]
\[(4) \ x \odot x = x \quad \text{idempotency}\]

A residuated lattice \(M\) is

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Special cases of residuated lattices.

Terms of properties of residuated structures

1. \((x \rightarrow y) \lor (y \rightarrow x) = 1 = (x \leadsto y) \lor (y \leadsto x)\) pre-linearity
2. \((x \rightarrow y) \odot x = x \land y = y \odot (y \leadsto x)\) divisibility
3. \(x \dashv \vdash = x = x \dashv \vdash\) involution
4. \(x \odot x = x\) idempotency

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Terms of properties of residuated structures

(1) \((x \rightarrow y) \lor (y \rightarrow x) = 1 = (x \sim y) \lor (y \sim x)\) \hspace{1cm} \text{pre-linearity}

(2) \((x \rightarrow y) \odot x = x \land y = y \odot (y \sim x)\) \hspace{1cm} \text{divisibility}

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- a \(GMV\)-algebra (pseudo \(MV\)-algebra equivalently) if \(M\) satisfies (2) and (3).
A non-empty subset $F$ of a residuated lattice $M$ is called a filter of $M$ if

(a) $x, y \in F$ imply $x \circ y \in F$;

(b) $x \in F$, $y \in M$, $x \leq y$ imply $y \in F$. 

A filter $F$ is called normal if for each $x, y \in M$

(c) $x \rightarrow y \in F \iff x \leadsto y \in F$. 

Normal filters of $M$ $\leftrightarrow$ kernels (i.e. 1-classes) of congruences on $M$

$\langle x, y \rangle \in \theta F \iff (x \rightarrow y) \circ (y \rightarrow x) \in F \iff (x \leadsto y) \circ (y \leadsto x) \in F$. 

$M/F$ ... a quotient residuated lattice; $x/F = x/\theta F$ ... the class of $M/F$ containing $x$. 

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Filters and congruences.

A non-empty subset $F$ of a residuated lattice $M$ is called a \textit{filter} of $M$ if

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$\langle x, y \rangle \in \theta_F \iff (x \rightarrow y) \circ (y \rightarrow x) \in F$

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normal filters of $M \longleftrightarrow$ kernels (i.e. 1-classes) of congruences on $M$

$$\langle x, y \rangle \in \theta_F \iff (x \rightarrow y) \circ (y \rightarrow x) \in F$$
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Filters and congruences.

A non-empty subset \( F \) of a residuated lattice \( M \) is called a \textit{filter} of \( M \) if

(a) \( x, y \in F \) imply \( x \circ y \in F \);
(b) \( x \in F, \ y \in M, \ x \preceq y \) imply \( y \in F \).

A filter \( F \) is called \textit{normal} if for each \( x, y \in M \)

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normal filters of \( M \) \( \iff \) kernels (i.e. 1-classes) of congruences on \( M \)

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\langle x, y \rangle \in \theta_F \iff (x \rightarrow y) \circ (y \rightarrow x) \in F
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\]

\( M/F \) \( \ldots \) a quotient residuated lattice;

\( x/F = x/\theta_F \) \( \ldots \) the class of \( M/F \) containing \( x \).
**GMV-algebras (pseudo MV-algebras).**

**Definition**

A *GMV-algebra* is an algebra $M = (M; \oplus, -, \sim, 0, 1)$ of type $\langle 2, 1, 1, 0, 0 \rangle$, where $x \circ y := (x^- \oplus y^-)^\sim$ for any $x, y \in M$, satisfying:

- $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- $x \oplus 0 = x = 0 \oplus x$;
- $x \oplus 1 = 1 = 1 \oplus x$;
- $1^- = 0 = 1^\sim$;
- $(x^- \oplus y^-)^\sim = (x^- \oplus y^-)^\sim$;
- $x \oplus (y \circ x^-) = y \oplus (x \circ y^-) = (y^- \circ x) \oplus y = (x^- \circ y) \oplus x$;
- $(x^- \oplus y) \circ x = y \circ (x \oplus y^-)$;
- $x^- = x$.

$x \leq y$ iff $x^- \oplus y = 1$, 

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A **GMV-algebra** is an algebra $M = (M; \oplus, -, \sim, 0, 1)$ of type $\langle 2, 1, 1, 0, 0 \rangle$, where $x \odot y := (x^- \oplus y^-)^\sim$ for any $x, y \in M$, satisfying:

- $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- $x \oplus 0 = x = 0 \oplus x$;
- $x \oplus 1 = 1 = 1 \oplus x$;
- $1^- = 0 = 1^\sim$;
- $(x^\sim \oplus y^\sim)^- = (x^- \oplus y^-)^\sim$;
- $x \oplus (y \odot x^\sim) = y \oplus (x \odot y^\sim) = (y^- \odot x) \oplus y = (x^- \odot y) \oplus x$;
- $(x^- \oplus y) \odot x = y \odot (x \oplus y^\sim)$;
- $x^{\sim \sim} = x$.

$x \leq y$ iff $x^- \oplus y = 1$, $(M, \leq)$ is a bounded distributive lattice,

$$x \lor y = x \oplus (y \odot x^\sim), \quad x \land y = x \odot (y \oplus x^\sim)$$
**GMV-algebras (pseudo MV-algebras).**

**Definition**

A **GMV-algebra** is an algebra $M = (M; \oplus, -, \sim, 0, 1)$ of type $\langle 2, 1, 1, 0, 0 \rangle$, where $x \odot y := (x^- \oplus y^-)\sim$ for any $x, y \in M$, satisfying:

- $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- $x \oplus 0 = x = 0 \oplus x$;
- $x \oplus 1 = 1 = 1 \oplus x$;
- $1^- = 0 = 1\sim$;
- $(x^- \oplus y^-)^- = (x^- \oplus y^-)\sim$;
- $x \oplus (y \odot x^-) = y \oplus (x \odot y^-) = (y^- \odot x) \oplus y = (x^- \odot y) \oplus x$;
- $(x^- \oplus y) \odot x = y \odot (x \oplus y^-)$;
- $x^{\sim \sim} = x$.

\[ x \leq y \text{ iff } x^- \oplus y = 1, \quad (M, \leq) \text{ is a bounded distributive lattice,} \]
\[ x \lor y = x \oplus (y \odot x^-), \quad x \land y = x \odot (y \oplus x^-) \]
Ideals of $\text{GMV}$-algebras.

A non-empty subset $I$ of a $\text{GMV}$-algebra $M$ is called an ideal of $M$ if

(a) $x, y \in I$ imply $x \oplus y \in I$;
(b) $x \in I$, $y \in M$, $y \leq x$ imply $y \in I$.
Ideals of $GMV$-algebras.

A non-empty subset $I$ of a $GMV$-algebra $M$ is called an ideal of $M$ if

(a) $x, y \in I$ imply $x \oplus y \in I$;
(b) $x \in I$, $y \in M$, $y \leq x$ imply $y \in I$.

An ideal $I$ is called normal if for each $x, y \in M$

(c) $x^- \circ y \in I \iff y \circ x^- \in I$.

normal ideals of $M \iff$ kernels of congruences on $M$

$\langle x, y \rangle \in \theta_1 \iff (x^- \circ y) \oplus (y^- \circ x) \in I$

$\iff (x \circ y^-) \oplus (y \circ x^-) \in I.$
Ideals of $GMV$-algebras.

A non-empty subset $I$ of a $GMV$-algebra $M$ is called an ideal of $M$ if

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(b) $x \in I$, $y \in M$, $y \leq x$ imply $y \in I$.

An ideal $I$ is called normal if for each $x, y \in M$

(c) $x^- \odot y \in I \iff y \odot x^\sim \in I$.

normal ideals of $M \iff$ kernels of congruences on $M$

$\langle x, y \rangle \in \theta_I \iff (x^- \odot y) \oplus (y^- \odot x) \in I$

$\iff (x \odot y^\sim) \oplus (y \odot x^\sim) \in I$. 
Motivation.

In $\text{GMV}$-algebra $M$, $x \odot y := (x - \oplus y - \sim) \sim$, put $x \rightarrow y := x - \oplus y$, $x \Rightarrow y := y \oplus x \sim$, then $(M; \odot, \lor, \land, \rightarrow, \Rightarrow, 0, 1)$ is a residuated lattice with (2), (3).

Conversely, let $(M; \odot, \lor, \land, \rightarrow, \Rightarrow, 0, 1)$ be a residuated lattice with (2), (3). Put $x - := x \rightarrow 0$, $x \sim := x \Rightarrow 0$, $x \oplus y := (x - \odot y - \sim) \sim = (x \sim \odot y \sim) -$. Then $M = (M; \oplus, -, \sim, 0, 1)$ is a $\text{GMV}$-algebra.

Filter and ideal theories of $\text{GMV}$-algebras are mutually dual.
Motivation.

1. In GMV-algebra $M$, $x \odot y := (x^- \oplus y^-)^\sim$, put $x \to y := x^- \oplus y$, $x \leadsto y := y \oplus x^\sim$, then $(M; \odot, \vee, \wedge, \to, \leadsto, 0, 1)$ is a residuated lattice with (2), (3).

2. Conversely, let $(M; \odot, \vee, \wedge, \to, \leadsto, 0, 1)$ be a residuated lattice with (2), (3). Put $x^- := x \to 0$, $x^\sim := x \leadsto 0$, $x \oplus y := (x^- \odot y^-)^\sim = (x^\sim \odot y^\sim)^\sim$. Then $M = (M; \oplus, ^-, ^\sim, 0, 1)$ is a GMV-algebra.

Filter and ideal theories of GMV-algebras are mutually dual.
Motivation.

1. In $GMV$-algebra $M$, $x ∘ y := (x^- ⊕ y^-)\sim$, put $x → y := x^- ⊕ y$, $x ⇝ y := y ⊕ x\sim$, then $(M; ∘, ∨, ∧, →, ⇝, 0, 1)$ is a residuated lattice with (2), (3).

2. Conversely, let $(M; ∘, ∨, ∧, →, ⇝, 0, 1)$ be a residuated lattice with (2), (3). Put $x^- := x → 0$, $x^\sim := x ⇝ 0$, $x ⊕ y := (x^- ∘ y^-)\sim = (x^\sim ∘ y^\sim)^-$. Then $M = (M; ⊕, ^-, ^\sim, 0, 1)$ is a $GMV$-algebra.

Filter and ideal theories of $GMV$-algebras are mutually dual.
Motivation.

1. In GMV-algebra $M$, $x \odot y := (x^- \oplus y^-)^\sim$, put $x \rightarrow y := x^- \oplus y$, $x \leftrightharpoons y := y \oplus x^\sim$, then $(M; \odot, \lor, \land, \rightarrow, \leftrightharpoons, 0, 1)$ is a residuated lattice with (2), (3).

2. Conversely, let $(M; \odot, \lor, \land, \rightarrow, \leftrightharpoons, 0, 1)$ be a residuated lattice with (2), (3). Put $x^- := x \rightarrow 0$, $x^\sim := x \leftrightharpoons 0$, $x \oplus y := (x^- \odot y^-)^\sim = (x^\sim \odot y^\sim)^\sim$. Then $M = (M; \oplus, ^\sim, \sim, 0, 1)$ is a GMV-algebra.

Filter and ideal theories of GMV-algebras are mutually dual.

A dual binary operation to multiplication in residuated lattices does not exist.
In GMV-algebra $M$, $x \odot y := (x^- \oplus y^-)\sim$, put $x \rightarrow y := x^- \oplus y$, $x \leftrightsquigarrow y := y \oplus x^\sim$, then $(M; \odot, \vee, \wedge, \rightarrow, \leftrightsquigarrow, 0, 1)$ is a residuated lattice with (2), (3).

Conversely, let $(M; \odot, \vee, \wedge, \rightarrow, \leftrightsquigarrow, 0, 1)$ be a residuated lattice with (2), (3). Put $x^- := x \rightarrow 0$, $x^\sim := x \leftrightsquigarrow 0$, $x \oplus y := (x^- \odot y^-)\sim = (x^\sim \odot y^\sim)^-$.
Then $M = (M; \oplus, ^-, ^\sim, 0, 1)$ is a GMV-algebra.

Filter and ideal theories of GMV-algebras are mutually dual.

A dual binary operation to multiplication in residuated lattices does not exist.
The aim.

- To fill the gap by introducing the notion of ideal in residuated lattices.
- To establish congruences in residuated lattices using ideals.
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- To fill the gap by introducing the notion of ideal in residuated lattices.
- To establish congruences in residuated lattices using ideals.
Ideals of residuated lattices.

Let $M$ be a residuated lattice, we put

- $x \odot y := y \sim x \ldots$ \textit{left addition},
- $x \odot y := x \sim \rightarrow y \ldots$ \textit{right addition} on $M$.

\textbf{Definition}

A non-empty subset $I$ of a residuated lattice $M$ is called a \textit{left ideal} of $M$ if

(a) $x, y \in I \implies x \odot y \in I$;
(b) $x \in I, z \in M, z \leq x \implies z \in I$.

\textbf{Theorem 1}

Let $I$ be a subset of a residuated lattice $M$ containing 0. Then $I$ is a left ideal of $M$ if and only if

$\forall x, y \in M; x^{-} \odot y \in I, x \in I \implies y \in I$. 
Ideals of residuated lattices.

Let \( M \) be a residuated lattice, we put

- \( x \odot y := y^{-} \simrightarrow x \ldots \text{left addition} \),
- \( x \odot y := x^{-} \rightarrow y \ldots \text{right addition} \) on \( M \).

**Definition**

A non-empty subset \( I \) of a residuated lattice \( M \) is called a *left ideal* of \( M \) if

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**Theorem 1**

Let $I$ be a subset of a residuated lattice $M$ containing $0$. Then $I$ is a left ideal of $M$ if and only if
$$\forall x, y \in M; x^\sim \ominus y \in I, x \in I \implies y \in I.$$
Ideals of residuated lattices.

**Definition**
A non-empty subset \( I \) of a residuated lattice \( M \) is called a *right ideal* of \( M \) if

(a') \[ x, y \in I \implies x \square y \in I; \]
(b) \[ x \in I, \ z \in M, \ z \leq x \implies z \in I. \]

**Theorem 1’**

*Let \( I \) be a subset of a residuated lattice \( M \) containing 0. Then \( I \) is a right ideal of \( M \) if and only if*

\[ \forall x, y \in M; \ y \square x^\sim \in I, \ x \in I \implies y \in I. \]

Every left ideal as well as every right ideal of a residuated lattice \( M \) is a lattice ideal.

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Ideals of residuated lattices.

Definition
A non-empty subset $I$ of a residuated lattice $M$ is called a right ideal of $M$ if

(a') $x, y \in I \implies x \odot y \in I$;
(b) $x \in I, z \in M, z \leq x \implies z \in I$.

Theorem 1'
Let $I$ be a subset of a residuated lattice $M$ containing 0. Then $I$ is a right ideal of $M$ if and only if

$$\forall x, y \in M; y \odot x\sim \in I, x \in I \implies y \in I.$$ 

Every left ideal as well as every right ideal of a residuated lattice $M$ is a lattice ideal.
Ideals of residuated lattices.

Definition

A non-empty subset \( I \) of a residuated lattice \( M \) is called a ideal of \( M \) if it is both left and right ideal of \( M \), i.e.

(a) \( x, y \in I \implies x \ominus y \in I; \)

(a') \( x, y \in I \implies x \ominus y \in I; \)

(b) \( x \in I, z \in M, z \leq x \implies z \in I. \)

For an ideal \( I \) of a residuated lattice \( M \),

\[
\langle x, y \rangle \in \theta_I :\iff x^- \ominus y \in I, y^- \ominus x \in I, x \ominus y^- \in I, y \ominus x^- \in I.
\]

\( \theta_I \) ... an equivalence on \( M. \)

Theorem 2

a) Let \( M \) be a residuated lattice and \( I \) an ideal of \( M \). Then the equivalence \( \theta_I \) is a congruence on the reduct \((M; \ominus, \lor, \rightarrow, \multimap, 0, 1)\) of the residuated lattice \( M \).

b) If \( M \) is a pseudo BL-algebra then \( \theta_I \) is a congruence on \( M \).
Ideals of residuated lattices.

Definition

A non-empty subset \( I \) of a residuated lattice \( M \) is called a *ideal* of \( M \) if it is both left and right ideal of \( M \), i.e.

\[(a) \quad x, y \in I \implies x \odot y \in I; \]
\[(a') \quad x, y \in I \implies x \odot y \in I; \]
\[(b) \quad x \in I, z \in M, z \leq x \implies z \in I. \]

For an ideal \( I \) of a residuated lattice \( M \),

\[\langle x, y \rangle \in \theta_I : \iff x^- \odot y \in I, y^- \odot x \in I, x \odot y^- \in I, y \odot x^- \in I.\]

\( \theta_I \) ... an equivalence on \( M \).

Theorem 2

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Ideals of residuated lattices.

Definition

A non-empty subset $I$ of a residuated lattice $M$ is called a *ideal* of $M$ if it is both left and right ideal of $M$, i.e.

(a) $x, y \in I \iff x \otimes y \in I$;

(a') $x, y \in I \iff x \ominus y \in I$;

(b) $x \in I, z \in M, z \leq x \Rightarrow z \in I$.

For an ideal $I$ of a residuated lattice $M$,

$$\langle x, y \rangle \in \theta_I :\iff x^{-} \circ y \in I, y^{-} \circ x \in I, x \circ y^\sim \in I, y \circ x^\sim \in I.$$ 

$\theta_I$ ... an equivalence on $M$.

Theorem 2

a) Let $M$ be a residuated lattice and $I$ an ideal of $M$. Then the equivalence $\theta_I$ is a congruence on the reduct $(M; \odot, \lor, \rightarrow, \multimap, 0, 1)$ of the residuated lattice $M$.

b) If $M$ is a pseudo BL-algebra then $\theta_I$ is a congruence on $M$. 
Theorem 3

a) If $M$ is a pseudo BL-algebra and $I$ is an ideal of $M$, then $M/\theta_I$ is a GMV-algebra.

b) If $M$ is any residuated lattice then $M/\theta_I$ is an involutive residuated lattice.
Involution filter, Glivenko property

If $F$ is a normal filter of a residuated lattice $M$, then we say that $F$ is an **involution filter** if the quotient residuated lattice $M/F$ is involutive.

Glivenko property

A residuated lattice $M$ satisfies the **Glivenko property** if for any $x, y \in M$

$$(x \rightarrow y)^\sim = x \rightarrow y^\sim,$$

$$(x \Join y)^\sim = x \Join y^\sim.$$
Involutive filter, Glivenko property

If $F$ is a normal filter of a residuated lattice $M$, then we say that $F$ is an \textit{involutive filter} if the quotient residuated lattice $M/F$ is involutive.

Glivenko property

A residuated lattice $M$ satisfies the \textit{Glivenko property} if for any $x, y \in M$

$$(x \rightarrow y)\sim = x \rightarrow y\sim, \quad (x \rightsquigarrow y)\sim\sim = x \rightsquigarrow y\sim\sim.$$ 

The Glivenko property was introduced (Cignoli, Torrens 2004) for commutative residuated lattices

$$(x \rightarrow y)\sim\sim = x \rightarrow y\sim\sim.$$ 

For a good residuated lattice $M$, the following conditions are equivalent:

(i) \quad $(x\sim \rightarrow x)\sim = 1 = (x\sim \rightsquigarrow x)\sim\sim$,

(ii) \quad $(x \rightarrow y)\sim = x\sim \rightarrow y\sim$, \quad $(x \rightsquigarrow y)\sim\sim = x\sim \rightsquigarrow y\sim\sim$,

(iii) \quad (GP).
Involution filter, Glivenko property

If $F$ is a normal filter of a residuated lattice $M$, then we say that $F$ is an **involution filter** if the quotient residuated lattice $M/F$ is involutive.

Glivenko property

A residuated lattice $M$ satisfies the **Glivenko property** if for any $x, y \in M$

$$ (x \to y)^\sim = x \to y^- \sim, \quad (x \To y)^\sim = x \To y^- \sim. $$

The Glivenko property was introduced (Cignoli, Torrens 2004) for commutative residuated lattices

$$ (x \to y)^\bowtie = x \to y^- \bowtie. $$

For a good residuated lattice $M$, the following conditions are equivalent:

(i) $$(x^- \to x)^\sim = 1 = (x^- \To x)^\sim,$$
(ii) $$(x \to y)^\sim = x^- \to y^- \sim, \quad (x \To y)^\sim = x^- \To y^- \sim,$$
(iii) (GP).
Set of dense elements.

For a residuated lattice $M$:

$$D(M) := \{ x \in M : x^{-\sim} = 1 = x^{\sim-} \}.$$ 

**Theorem 4**

a) If $M$ is a good residuated lattice, then $D(M)$ is a filter of $M$.

b) If, moreover, $M$ satisfies (GP), then $D(M)$ is a normal filter of $M$.

**Theorem 5**

Let $M$ be a good residuated lattice satisfying (GP) and $x, y \in M$. Then

$$\langle x, y \rangle \in \theta_{D(M)}$$

if and only if $x^{-\sim} = y^{-\sim}$. Moreover, $M/D(M)$ is an involutive residuated lattice, i.e. $D(M)$ is an involutive filter.
$D(M)$ and involutive normal filters.

**Theorem 6**

If a good residuated lattice $M$ satisfies (GP) and $F$ is an involutive normal filter of $M$, then $D(M) \subseteq F$.

**Proposition 7**

If $F_1$ and $F_2$ are normal filters of a residuated lattice $M$, $F_1 \subseteq F_2$ and $F_1$ is an involutive filter, then $F_2$ is also involutive.

**Corollary 8**

If $M$ is a good residuated lattice satisfying (GP) then the involutive filters of $M$ are exactly all normal filters of $M$ containing $D(M)$. 
$D(M)$ and involutive normal filters.

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**Corollary 8**

If $M$ is a good residuated lattice satisfying (GP) then the involutive filters of $M$ are exactly all normal filters of $M$ containing $D(M)$.
Connections among ideals, filters and congruences.

**Proposition 9**

*If $M$ is a residuated lattice and $I$ is an ideal of $M$ then $I$ is the 0-class in $M/\theta_I$.***

**Proposition 10**

*Let $I$ be an ideal of a pseudo BL-algebra and $F = F_I = 1/\theta_I$. Then $F$ is an involutive normal filter of $M$.***

**Proposition 11**

*If $M$ is a residuated lattice and $F$ is a normal filter of $M$, then the class $0/F$ is an ideal of $M$.***
Connections among ideals, filters and congruences.

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Connections among ideals, filters and congruences.

**Theorem 12 (the main result)**

*If M is an arbitrary pseudo BL-algebra then there is a one-to-one correspondence between ideals and involutive normal filters of M.*

**Remark 13**

Let $M$ be a good pseudo $BL$-algebra. In the previous correspondence, the ideal $\{0\}$ corresponds to the filter $D(M)$. (In fact, this is also true for any good residuated lattice satisfying (GP).)