On decidabilty of some classes of Stone alebras

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Boolean and Stone algebras

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They form an equational class [Grätzer 1967].

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A is completely determined by $(\operatorname{Sk} A, \operatorname{Dn} A, h_A)$.

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$$B \rtimes_h D = \{(b, d) \in B \times D \mid h(b) \geq d\}$$

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and structural map $\widetilde{h} : \mathrm{Sk}(B \rtimes_h D) \to \mathrm{Dn}(B \rtimes_h D)$

$$\widetilde{h}\big(b, h(b)\big) = \big(b, h(b)\big) \vee (1, 0) = \big(1, h(b)\big)$$

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$\mathrm{Th}\,\mathbf{SA}_{n+1} = \mathrm{Th}\,\mathbf{SA} \cup (\mathrm{Th}\,\mathbf{SA}_n)^{\mathrm{Dn}} \quad$ for $n \geq 1$

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The class $\mathbf{PA}_n$ of all Post algebras of degree n is definitionally equivalent to $\mathbf{SA}_n^{\mathrm{i}} \cap \mathbf{SA}_n^{\mathrm{s}}$.

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- an L'-formula $f^I(\mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^n)$ for each n-ary functional symbol f in L (constants are nullary operation symbols)

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$\underline{\mathbf{a}}$ denotes the equivalence class of $\mathbf{a} \in U_{\mathfrak{M}}$ w.r.t. $E_{\mathfrak{M}}$

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A class $\mathbf{K}$ of L-structures **definable** in a class $\mathbf{K}'$ of L'-structures if there is an interpretation $I : L \to L'$ such that every structure $\mathfrak{A} \in \mathbf{K}$ is isomorphic to the structure $\mathfrak{M}^I$ for some $\mathfrak{M} \in \mathbf{K}'$ which admits I.

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An L-theory T is **interpretable** in an L'-theory T' if the class $\operatorname{Mod} T$ is definable in the class $\operatorname{Mod} T'$.

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Proving (un)decidability 2

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Let $n \geq 2$. Then

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Undecidability of $\mathbf{SA}^{\mathrm{i}}_{n}$

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Every Boolean pair (B, S) can be regarded as a triple $(S, B, \mathrm{id} : S \to B)$ and that way it gives rise to a Stone algebra $S \rtimes_{\mathrm{id}} B \in \mathbf{SA}^{\mathrm{i}}_2$.

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Conversely, every $A \in \mathbf{SA}_2^{\mathrm{i}}$ can be obtained in this way from the Boolean pair (B, S), with

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This enables to define a semantic embedding of the hereditarily undecidable class **BP** into $\mathbf{SA}^{\mathrm{i}}_2$.

Decidability of $\mathbf{SA}^{\mathrm{s}}_{n}$

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with canonic projections $p_1 : B \to B/J_1,\ p_k : B/J_{k-1} \to B/J_k$.

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Conversely, every $A \in \mathbf{SA}_2^{\mathrm{s}}$ can be obtained in this way from its P-product representation

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This enables to define a semantic embedding of $\mathbf{SA}_n^{\mathrm{s}}$ into the finitely axiomatizable decidable class class $\mathbf{BI}_{n-1}$.

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The last two items follow from the observation that all the algebras in $\mathbf{SA}_2$ (hence in every $\mathbf{SA}_n$) are relatively pseudocomplemented and satisfy $(x \to y) \vee (y \to x) = 1$ [Katriňák, Mitschke 1972], [Balbes, Dwinger 1974].

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