WITNESSED MODELS AND SKOLEMIZATION IN SUBSTRUCTURAL LOGICS

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PARALLEL SKOLEMIZATION

SKOLEMIZATION PROBLEM

In classical logic, Skolemization gives us

$$(\forall \bar{x})(\exists y)\varphi(\bar{x}, y) \text{ satisfiable} \iff (\forall \bar{x})\varphi(\bar{x}, f(\bar{x})) \text{ satisfiable}$$

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What is the situation in substructural logics?

Some problems:

- Formulas are not always equivalent to prenex formulas.
- Semantic consequence may not reduce to satisfiability.

An $\mathrm{FL_e}$-algebra is a structure $\boldsymbol{A} = \langle A, \&, \rightarrow, \wedge, \vee, \overline{0}, \overline{1} \rangle$ such that:

- $\langle A, \wedge, \vee \rangle$ is a lattice
- $\langle A, \&, \overline{1} \rangle$ is a commutative monoid
- $\rightarrow$ is the residuum of $\&$

Example

- Boolean algebras
- Heyting algebras
- Bounded residuated lattices
- MTL-algebras
- Gödel algebras
- MV-algebras

A predicate language $\mathcal{P} = \langle \mathsf{P}, \mathsf{F}, \mathsf{ar} \rangle$

A $\mathcal{P}$-structure $\mathfrak{M} = \langle \boldsymbol{A}, \mathsf{M} \rangle$

- $\boldsymbol{A}$ is a complete $\mathrm{FL_e}$-algebra
- $\mathsf{M} = \langle M, \langle P^{\mathsf{M}} \rangle_{P \in \mathsf{P}}, \langle f^{\mathsf{M}} \rangle_{f \in \mathsf{F}} \rangle$, where $P^{\mathsf{M}} : M^n \to A$ and $f^{\mathsf{M}} : M^n \to M$

Given an $\mathfrak{M}$-evaluation v mapping object variables to M,

$$
\begin{array}{rcll}
\|x\|^{\mathfrak{M}}_{\mathrm{v}} & = & \mathrm{v}(x) & \\
\|f(t_1, \dots, t_n)\|^{\mathfrak{M}}_{\mathrm{v}} & = & f^{\mathsf{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathfrak{M}}_{\mathrm{v}}) & \text{for } f \in \mathsf{F} \\
\|P(t_1, \dots, t_n)\|^{\mathfrak{M}}_{\mathrm{v}} & = & P^{\mathsf{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathfrak{M}}_{\mathrm{v}}) & \text{for } P \in \mathsf{P} \\
\|\varphi \circ \psi\|^{\mathfrak{M}}_{\mathrm{v}} & = & \|\varphi\|^{\mathfrak{M}}_{\mathrm{v}} \circ^{\boldsymbol{A}} \|\psi\|^{\mathfrak{M}}_{\mathrm{v}} & \text{for } \circ \in \{\&, \to, \wedge, \vee\} \\
\|\star\|^{\mathfrak{M}}_{\mathrm{v}} & = & \star^{\boldsymbol{A}} & \text{for } \star \in \{\overline{0}, \overline{1}\} \\
\|(\forall x)\varphi\|^{\mathfrak{M}}_{\mathrm{v}} & = & \bigwedge^{\boldsymbol{A}}_{a \in M} \|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x \to a]} & \\
\|(\exists x)\varphi\|^{\mathfrak{M}}_{\mathrm{v}} & = & \bigvee^{\boldsymbol{A}}_{a \in M} \{\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x \to a]} &
\end{array}
$$

For a fixed arbitrary class $\mathbb{K}$ of complete $\mathbf{FL_e}$-algebras,
a formula φ is a semantic consequence of a theory T in $\mathbb{K}$,

$$T \models_{\mathbb{K}} \varphi$$

if for each $A \in \mathbb{K}$ and each model $\mathfrak{M} = \langle A, M \rangle$ of T, $\mathfrak{M} \models \varphi$.

”NORMAL” SKOLEMIZATION FAILS IN SUBSTRUCTURAL LOGICS

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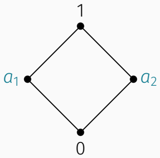
- Consider just a unary predicate symbol P.
- Consider the $\mathrm{FL_e}$-algebra $\boldsymbol{A}$ with
  - $1 \,\&\, x = x \,\&\, 1 = x$ and
  - $x \,\&\, y = 0$ for $x, y \in \{0, a_1, a_2\}$.



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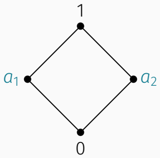

$$\{(\exists x)P(x)\} \models_{\boldsymbol{A}} (\exists x)(P(x) \& P(x)) \quad \overset{?}{\Longleftrightarrow} \quad \{P(c)\} \models_{\boldsymbol{A}} (\exists x)(P(x) \& P(x))$$

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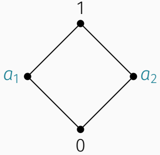

$$\{P(c)\} \models_A (\exists x)(P(x) \& P(x))$$

Clearly since

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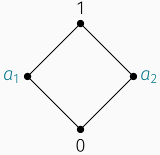

$$\{(\exists x)P(x)\} \not\models_{\boldsymbol{A}} (\exists x)(P(x)\&P(x))$$

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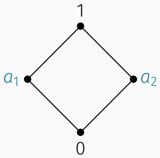

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Consider the structure $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M} \rangle$

- $M = \{d_1, d_2\}$
- $P^{\mathbf{M}}(d_i) = a_i$ for $i = 1, 2$

$\mathfrak{M} \models_{\boldsymbol{A}} (\exists x)P(x)$
$\mathfrak{M} \not\models_{\boldsymbol{A}} (\exists x)(P(x)\&P(x))$

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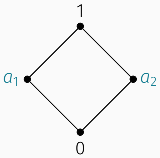

$$\{(\exists x)P(x)\} \not\models_{\boldsymbol{A}} (\exists x)(P(x)\&P(x)) \quad \not\Leftrightarrow \quad \{P(c)\} \models_{\boldsymbol{A}} (\exists x)(P(x)\&P(x))$$

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ALTERNATIVE WAY TO SAVE SKOLEMIZATION

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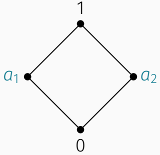

$$\{(\exists x)P(x)\} \not\models_{\boldsymbol{A}} (\exists x)(P(x)\&P(x)) \quad \Longleftrightarrow \quad \{P(c_1) \vee P(c_2)\} \not\models_{\boldsymbol{A}} (\exists x)(P(x)\&P(x))$$

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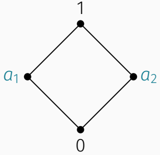

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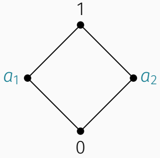

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$$T \cup \{(\forall \bar{y})(\exists x)\varphi(x, \bar{y})\} \models_{\boldsymbol{A}} \psi \iff T \cup \{(\forall \bar{y})(\varphi(f_1(\bar{y}), \bar{y}) \vee \varphi(f_2(\bar{y}), \bar{y}))\} \models_{\boldsymbol{A}} \psi$$

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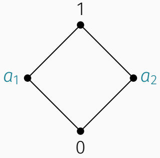

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Parallel Skolemization was introduced for intermediate logics in:

M. Baaz, R. Iemhoff
Skolemization in intermediate logics with the finite model property, submitted.

STRONG AND WEAK OCCURRENCES OF QUANTIFIERS

An occurrence of ψ in φ is positive (negative) if:

- φ is ψ;
- φ is $\varphi_1 \wedge \varphi_2$, $\varphi_2 \wedge \varphi_1$, $\varphi_1 \vee \varphi_2$, $\varphi_2 \vee \varphi_1$, $\varphi_1 \mathbin{\&} \varphi_2$, $\varphi_2 \mathbin{\&} \varphi_1$, $(\forall x)\varphi_1$, $(\exists x)\varphi_1$, or $\varphi_2 \to \varphi_1$, and ψ is positive (negative) in $\varphi_1[\psi]$;
- φ is $\varphi_1 \to \varphi_2$ and ψ is negative (positive) in $\varphi_1[\psi]$.

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- φ is $\varphi_1 \to \varphi_2$ and ψ is negative (positive) in $\varphi_1[\psi]$.

An occurrence of $(Qx)\psi$ in φ is strong if

it is positive and $Q = \forall$ or it is negative and $Q = \exists$.

It is called weak otherwise.

Procedure P : Replace a subformula $(Qx)\psi(x,\bar{y})$ in a $\mathcal{P}$-sentence φ by

$$\bigvee_{i=1}^{n} \psi(f_i(\bar{y}),\bar{y}) \text{ if } Q = \exists \qquad \text{and} \qquad \bigwedge_{i=1}^{n} \psi(f_i(\bar{y}),\bar{y}) \text{ if } Q = \forall,$$

where $f_1,\ldots,f_n \notin \mathcal{P}$ are function symbols of arity $|\bar{y}|$.

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Example

$$\varphi = (\forall x)((\exists y)P(x,y) \to (\exists z)Q(x,z))$$

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$$\begin{aligned} \varphi &= (\forall x)((\exists y)P(x,y) \to (\exists z)Q(x,z)) \\ sk_1^l(\varphi) &= (\forall x)((\exists y)P(x,y) \to Q(x,g(x))) \end{aligned}$$

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$$sk^r_2(\varphi) = ((P(c_1, d^1_1) \vee P(c_1, d^1_2)) \to (\exists z)Q(c_1, z)) \wedge$$
$$((P(c_2, d^2_1) \vee P(c_2, d^2_2)) \to (\exists z)Q(c_2, z))$$

Let us fix an arbitrary class of complete $\mathrm{FL_e}$-algebras $\mathbb{K}$.

$\models_{\mathbb{K}}$ admits parallel Skolemization left of degree n for a sentence φ if for any theory $T \cup \{\psi\}$,

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Lemma

If $\models_{\mathbb{K}}$ admits parallel Skolemization left of degree n for all sentences, then $\models_{\mathbb{K}}$ admits parallel Skolemization right of degree n for all sentences.

WITNESSED MODELS

N-COMPACTENESS

Let $\boldsymbol{L}$ be a lattice and $\mathcal{X} \subseteq \mathfrak{P}(L)$.

$\mathcal{X}$ is *n*-compact for some $n \in \mathbb{N}^+$ if for each $A \in \mathcal{X}$,

$$\bigvee A = a_1 \vee \ldots \vee a_n \qquad \text{for some } a_1, \ldots, a_n \in A$$
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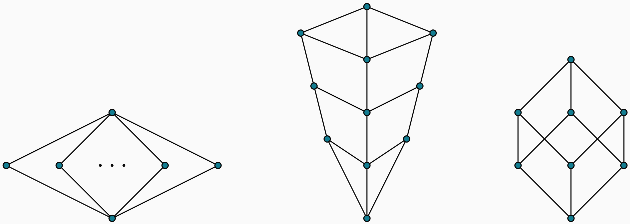

2-compact   2-compact   3-compact

A structure $\mathfrak{M} = \langle \mathbf{A}, \mathsf{M} \rangle$ is *n*-witnessed if the system

$$\{\{||\varphi(b, \bar{a})||^{\mathfrak{S}} \mid b \in M\} \mid \varphi(x, \bar{y}) \text{ a } \mathcal{P}\text{-formula and } \bar{a} \in M\}$$

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is *n*-compact.

$\models_{\mathbb{K}}$ has the *n*-witnessed model property if for any theory $T \cup \{\varphi\}$,

$$T \models_{\mathbb{K}} \varphi \iff \text{each } n\text{-witnessed model } \mathfrak{M} \text{ of } T \text{ is a model of } \varphi.$$

Example

Let $\mathbb{K}$ consists of the standard Łukasiewicz algebra on $[0, 1]$.

The powerset of $[0, 1]$ is clearly not n-compact for any $n \in \mathbb{N}^+$.

However, $\models_{Ł}$ has the 1-witnessed model property, as shown by Hájek.

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Let $\mathbb{K}$ be a class of $\mathrm{FL_e}$-algebras whose underlying lattices

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Example

Let $\mathbb{K}$ be a class of $\mathrm{FL_e}$-algebras whose underlying lattices

- *either* have height bounded by some fixed $n + 1$,
- *or* contain no infinite chain and have width bounded by some fixed n.

Then $\models_{\mathbb{K}}$ has the n-witnessed model property.

Theorem

If $\models_{\mathbb{K}}$ has the n-witnessed model property, then $\models_{\mathbb{K}}$ admits parallel Skolemization left and right of degree n for all sentences.

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Moreover, if $\models_{\mathbb{K}}$ is finitary, i.e. for any theory $T \cup \{\varphi\}$,

$$T \models_{\mathbb{K}} \varphi \quad \Leftrightarrow \quad T' \models_{\mathbb{K}} \varphi \text{ for some finite } T' \subseteq T.$$

and admits parallel Skolemization left and right of degree n for all sentences, then $\models_{\mathbb{K}}$ has the n-witnessed model property.

Thank you!

WITNESSED MODELS AND SKOLEMIZATION IN SUBSTRUCTURAL LOGICS

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