Internal calculi for substructural logics via syntactic proofs of conservativity

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BiFL_e algebras—for the 'big' logic

A structure $\mathcal{A} = (A, \vee, \wedge, \otimes, \rightarrow, 1, \oplus, \prec, 0)$ is a BiFL_e algebra (short for *commutative Bi-Lambek algebra*) if:

1. $(A, \vee, \wedge)$ is a lattice ($\vee, \wedge$ are commutative, assoc., mutually absorptive)
2. (a) $(A, \otimes, 1)$ is a commutative monoid (i.e. $\otimes$ is associative with identity 1)
   (b) $(A, \oplus, 0)$ is a commutative monoid (i.e. $\oplus$ is associative with identity 0)
3. (a) $x \otimes y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightarrow z$, for every $x, y, z \in A$ $(\otimes, \rightarrow)$
   (b) $z \leq x \oplus y$ iff $z \prec x \leq y$ iff $z \prec y \leq x$, for every $x, y, z \in A$ $(\prec, \oplus)$

$\mathrm{BiFL}_e \subset \mathrm{form}(\mathrm{BiFL})$ denotes logic (set of formulae) and also its Hilbert calculus

Theorem (soundness, completeness wrt algebraic semantics)

Let $\alpha \in \mathrm{form}(\mathrm{BiFL})$. Then $\alpha \in \mathrm{BiFL}_e + Ax$ iff α is valid on BiFL_e algebras satisfying $\overline{Ax}$.

These semantics induce a display calculus (Goré, 1998) hence a proof-theory!

Display Calculus

1. (Belnap, 1982) introduced the display calculus as a proof-theoretic framework for presenting many different logics
2. Display calculi have been presented for substructural logics, modal and poly-modal logics, tense logic, bunched logics, bi-intuitionistic logic...
3. Key result. Belnap's general cut-elimination theorem applies when the rules of the calculus satisfy C1–C8 (*display conditions*).
   $\rightsquigarrow$ subformula property $\Rightarrow$ proofs with 'nice' structure (key to applications)
4. (more expressive that hypersequent calculus) Every hypersequent calculus induces a display calculus (RR, 2014), and yet there are display calculi for logics for which hypersequent calculi are unknown.

Theorem (Ciabattoni and RR, submitted 2014)

Let $\mathcal{C}$ be an amenable well-behaved display calculus for the logic L, and let L′ be an axiomatic extension of L. Then there is an analytic rule extension of $\mathcal{C}$ for L′ iff L′ is an extension of L by acyclic axioms.

By above theorem: we can compute display calculi for all acyclic axiomatic extensions of BiFL_e.

Constructing the display calculus $\delta\mathrm{BiFL}_e$ for BiFL_e

A display sequent $X \vdash Y$ is built from structures X and Y which are built from logical formulae using structural connectives.

- Use the residuation properties to define structural connectives:

$$\overbrace{x \otimes y}^{,} \leq z \quad\Leftrightarrow\quad x \leq \overbrace{y \to z}^{>} \quad\Leftrightarrow\quad y \leq \overbrace{x \to z}^{>}$$

$$z \leq \underbrace{x \oplus y}_{;} \quad\Leftrightarrow\quad \underbrace{z \prec x}_{<} \leq y \quad\Leftrightarrow\quad \underbrace{z \prec y}_{<} \leq x$$

- Introduce the (display rules) which capture the residuation properties

$$\frac{\dfrac{X, Y \vdash Z}{X \vdash Y > Z}}{Y \vdash X > Z} \qquad \frac{\dfrac{Z \vdash X ; Y}{Z < X \vdash Y}}{Z < Y \vdash X}$$

(notation: double lines mean the rule holds in both directions)

The display property for δBiFL$_e$

Definition (display property)

The calculus has the display property if for any sequent $X \vdash Y$ containing a substructure U, there is a sequent $U \vdash W$ or $W \vdash U$ for some W such that

$$\frac{X \vdash Y}{U \vdash W} \quad \text{or} \quad \frac{X \vdash Y}{W \vdash U}$$

We say that U is *displayed* in the lower sequent.

- Example of display property (*display* occurrence of r):

$$\dfrac{\dfrac{\dfrac{\dfrac{(p < (q; r)), s \vdash z}{p < (q; r) \vdash s > z}\ \text{push } s \text{ to rhs}}{p \vdash q; r; (s > z)}\ \text{push } q; r \text{ to rhs}}{p < q \vdash r; (s > z)}\ \text{push } q \text{ to lhs}}{(p < q) < (s > z) \vdash r}\ \text{push } s > z \text{ to lhs}$$

Add structural connectives for logical connectives

- Add structural connectives to interpret logical connectives (rewrite rules):

$$0 \vdash \qquad\qquad \vdash 1$$

$$\frac{A, B \vdash Y}{A \otimes B \vdash Y}\otimes l \qquad \frac{X \vdash A ; B}{X \vdash A \oplus B}\oplus r \qquad \cfrac{\cfrac{L, \Phi \vdash M}{L \vdash M}}{\Phi, L \vdash M}$$

$$\frac{A < B \vdash Y}{A \prec B \vdash Y}\prec l \qquad \frac{X \vdash A > B}{X \vdash A \to B}\to r \qquad \cfrac{\cfrac{L \vdash M ; \Phi}{L \vdash M}}{L \vdash \Phi ; M}$$

- Obtain missing decoding rules:

$$\frac{X \vdash A \qquad Y \vdash B}{X, Y \vdash A \otimes B}\otimes r \qquad \frac{A \vdash X \qquad B \vdash Y}{A \oplus B \vdash X ; Y}\oplus l$$

$$\frac{X \vdash A \qquad B \vdash Y}{X < Y \vdash A \prec B}\prec r \qquad \frac{X \vdash A \qquad B \vdash Y}{A \to B \vdash X > Y}\to l$$

Rules for lattice connectives, structural rules, cut-rule

- Rules for lattice connectives

$$\frac{A \vdash X \qquad B \vdash X}{A \vee B \vdash X}\ \vee l \qquad \frac{X \vdash A}{X \vdash A \vee B}\ \vee r$$

$$\frac{A \vdash Y}{A \wedge B \vdash Y}\ \wedge l \qquad \frac{X \vdash A \qquad X \vdash B}{X \vdash A \wedge B}\ \wedge r$$

- Structural rules (i.e. only structural variables and connectives)

$$\frac{X, Y \vdash Z}{Y, X \vdash Z}\ \text{le} \qquad \frac{X \vdash Y; Z}{X \vdash Z; Y}\ \text{re}$$

$$\frac{X, (Y, Z) \vdash U}{(X, Y), Z \vdash U}\ \text{la} \qquad \frac{X \vdash (U; V); W}{X \vdash U; (V; W)}\ \text{ra}$$

- Initial sequent and cut-rule (notice the lack of *context*):

$$p \vdash p \qquad \frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\ \text{cut}$$

$\mathrm{FL}_e^{\oplus}$ algebras—for the ‘small’ logic

A structure $\mathcal{A} = (A, \vee, \wedge, \otimes, \rightarrow, 1, \oplus,$ ~~$-\!<$~~$, 0)$ is an $\mathrm{FL}_e^{\oplus}$ algebra (short for *commutative $FL^{\oplus}$ algebra*) if:

1. $(A, \vee, \wedge)$ is a lattice ($\vee, \wedge$ are commutative, assoc., mutually absorptive)
2. (a) $(A, \otimes, 1)$ is a commutative monoid (i.e. $\otimes$ is associative with identity 1)
   (b) $(A, \oplus, 0)$ is a commutative monoid (i.e. $\oplus$ is associative with identity 0)
3. (a) $x \otimes y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightarrow z$, for every $x, y, z \in A$

   ~~(b) $z \leq x \oplus y$ iff $z -\!< x \leq y$ iff $z -\!< y \leq x$, for every $x, y, z \in A$.~~

   (b) $x \oplus (u \wedge v) = (x \oplus u) \wedge (x \oplus v)$

Conservativity of BiFL_e over $\mathrm{FL}_e^{\oplus}$ is the statement:

For $A \in \mathrm{form}(\mathrm{FL}^{\oplus})$: $A \in \mathrm{BiFL}_e$ implies $A \in \mathrm{FL}_e^{\oplus}$?

Informally: every formula of the small language that is a theorem of the big logic is also a theorem of the small logic

Aim: syntactic conservativity and internal calculi

1. We want to obtain syntactic proofs of conservativity of $\mathrm{BiFL}_e + \mathrm{Ax}$ over $\mathrm{FL}_e^{\oplus} + \mathrm{Ax}$ via the display calculus for axioms Ax

   For $A \in \mathrm{form}(\mathrm{FL}^{\oplus})$: $A \in \mathrm{BiFL}_e + \mathrm{Ax}$ implies $A \in \mathrm{FL}_e^{\oplus} + \mathrm{Ax}$?

   Conservativity can be obtained arguing via the algebraic semantics

2. However: syntactic conservativity will yield a proof calculus for the smaller logic $\mathrm{FL}_e^{\oplus} + \mathrm{Ax}$ (specifically, a proof calculus where every sequent is interpretable in the small logic). The semantic proof does not yield a proof calculus.

3. Ultimately, we want to use the internal calculi for $\mathrm{FL}_e^{\oplus} + \mathrm{Ax}$ to investigate properties such as decidability and complexity.

Syntactic proof that BiFL_e is conservative over $\mathrm{FL}_e^{\oplus}$

1. Let δ be a cutfree derivation in $\delta\mathrm{BiFL}_e$ of $\mathbf{I} \vdash A$ where $A \in \mathrm{form}(\mathrm{FL}^{\oplus})$.
2. Let us try to interpret every sequent in δ in the language of $\mathrm{FL}_e^{\oplus}$...
3. The sequent $(X < Y), (U < V) \vdash Z$ cannot be interpreted in $\mathrm{FL}_e^{\oplus}$ because we cannot remove *all* occurrences of $<$ using the display rules
4. Call a sequent good if it can be interpreted in $\mathrm{FL}_e^{\oplus}$ (using the display rules) and contains no occurrences of logical connective $\prec$
   Otherwise the sequent is called bad
5. Observe: A bad sequent occurring in the derivation δ cannot become good later on (by inspection, there is no ameliorating rule that makes a bad sequent into good)
6. Since $\mathbf{I} \vdash A$ is a good sequent (recall we chose $A \in \mathrm{form}(\mathrm{FL}^{\oplus})$) every sequent in δ must be good
7. It remains to check that every rule instance in $\delta\mathrm{BiFL}_e$ with good premises and conclusion is a sound inference in $\mathrm{FL}_e^{\oplus}$ when the principal formula is nested deeply. This can be done.

Obtain internal calculus by interpreting each sequent deeply

When bad sequents become good. . . trouble

1. The calculus $\delta\mathrm{BiFL}_e$ contains no ameliorating rules so the argument is straightforward. Now consider the Grishin rule below left (or equivalently using display rules, below right):

$$\frac{(X < Y), U \vdash Z}{(X, U) < Y \vdash Z}\ \mathrm{grn} \qquad \frac{X \vdash (U > Z), Y}{X \vdash U > (Z, Y)}\ \mathrm{grn\text{-}r}$$

2. Rule above right is equivalent to $(u \to z) \oplus y \leq u \to (z \oplus y)$.

3. Here is why the rule is ameliorating. . . bad sequents can be made good.

$$\frac{(A < B), (C < D) \vdash E}{(A, (C < D)) < B \vdash E}\ \mathrm{grn}$$

Since

$$\frac{(A, (C < D)) < B \vdash E}{C \vdash (A > (E, B)), D} \qquad \begin{array}{c}((A \otimes (C \prec D)) \prec B) \to E \\ C \to ((A \to (E \oplus B)) \oplus D)\end{array}$$

Conservativity of δBiFL$_e$ + *grn* over FL$_e^\oplus$ + *grn*

1. FILL ('full intuitionistic linear logic') is FL$_e^\oplus$ + *grn*
2. BiILL = BiFL$_e$ + *grn* is conservative over FILL (Clauston *et al.*, 2013)
3. When a bad sequent is obtained in the derivation of $\mathbf{I} \vdash A$, make it good by applying suitable grn rules (but not too many!).

   Call this the merge operation (takes a bad sequent and returns the *set* of good sequents obtained by applying only those grn rules that make the sequent 'better') (interpret as transformation on grammar tree)
4. (Clauston *et al.*, 2013) obtain internal calculus with terminating proof-search for FILL and obtain NP-completeness

   (original proof differs from above)
5. We want to obtain similar results for other acyclic extensions of BiFL$_e$

Case studies: left contraction with and without grn rule

1. Another interesting ameliorating rule is the left contraction rule:

$$\frac{X, X \vdash Z}{X \vdash Z}\ \text{lc}$$

2. Here is the amelioration effect of the left contraction rule in action:

$$\dfrac{\dfrac{\overset{\text{good sequent}}{U < V \vdash A} \qquad \overset{\text{good sequent}}{U < V \vdash B}}{(U < V), (U < V) \vdash A \otimes B}\ \otimes\text{r}}{U < V \vdash A \otimes B}\ \text{lc}$$

3. Not clear how to obtain syntactic conservativity for δBiFL$_e$+left ctr (merge operation via grn is not sound here). Via semantics: conservativity holds.
4. NB. We cannot take left contraction as the new merge operation because not all bad sequents can be made good this way!
5. What about δBiFL$_e$+grn+left contraction? (still seems problematic)

$$\dfrac{\dfrac{\dfrac{(U < V), (U < V) \vdash A \otimes B}{(U, (U < V)) < V \vdash A \otimes B}\ \text{grn}}{((U, U) < V) < V \vdash A \otimes B}\ \text{grn}}{U, U \vdash A \otimes B, V, V}$$

δBiFL$_e$+grn+left and right contraction? (in progress)

1. Observation: A bad sequent s first made good via grn rule (merge operation) can simulate an immediate application of left contraction to s:

$$\dfrac{\dfrac{\dfrac{\dfrac{(U < V), (U < V) \vdash A \otimes B}{(U, (U < V)) < V \vdash A \otimes B}\text{grn}}{((U, U) < V) < V \vdash A \otimes B}\text{grn}}{\dfrac{U, U \vdash A \otimes B, V, V}{U \vdash A \otimes B, V}\text{left ctr, right ctr}}}{U < V \vdash A \otimes B}$$

2. Plan: given a derivation in δBiFL$_e$+grn+left and right contraction of $A \in \text{form}(\text{FL}^{\oplus})$, interpret each sequent as a formula in $\text{form}(\text{FL}^{\oplus})$.
3. Make bad sequent good via the merge operation. Use grn followed by left, right contraction to simulate an ameliorating contraction rule.
4. The internal calculus is obtained by using the merge operation in the conclusion of all rules that can introduce bad sequents eg:

$$\dfrac{X \vdash A \qquad Y \vdash B}{\text{merge}(X, Y) \vdash A \otimes B}\otimes\text{r}$$

5. Ultimately: obtain uniform proofs of syntactic conservativity for suitable substructural logics, and thus internal calculi for these logics.

- N.D. Belnap. Display Logic. *Journal of Philosophical Logic*, 11(4), 375–417, 1982.
- A. Ciabattoni, N. Galatos and K. Terui. From axioms to analytic rules in nonclassical logics. Proceedings of LICS 2008, pp. 229–240, 2008.
- A. Ciabattoni and R. Ramanayake. Structural rule extensions of display calculi: a general recipe. Proceedings of WOLLIC 2013.
- A. Ciabattoni and R. Ramanayake. Power and Limits of Structural Display Rules. Submitted.
- R. Clouston, R. Goré, and A. Tiu Annotation-Free Sequent Calculi for Full Intuitionistic Linear Logic. Proceedings of CSL 2013.
- G. Gentzen. The collected papers of Gerhard Gentzen. Edited by M. E. Szabo. Studies in Logic and the Foundations of Mathematics. Amsterdam, 1969.
- R. Goré. Substructural Logics On Display. Logic Journal of the IGPL, 6(3):451-504, 1998.
- R. Goré. Gaggles, Gentzen and Galois: how to display your favourite substructural logic. Logic Journal of the IGPL, 6(5):669-694, 1998.
- R. Ramanayake. Embedding the hypersequent calculus in the display calculus. Journal of Logic and Computation, 23(3):921-942, 2015.

The slides can be found at <www.logic.at/staff/revantha>