# Towards a Riesz Representation Theorem for Finite Heyting Algebras

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# A classical conceptual framework

Fundamental concepts involving classical logic and classical probability.

Consider classical logic through its algebraic semantics, *i.e.* Boolean algebras $\mathcal{B}$.
Consider classical probability theory: Boolean $\sigma$-algebras.

1. Stone duality: yields
   $[\varphi] \in \mathcal{B}$ iff $\chi^{-1}_{[\varphi]}(1) \subseteq \operatorname{MaxSpec} \mathcal{B}$.

2. de Finetti's coherence criterion: yields
   $P \colon \mathcal{B} \to [0,1]$ finitely additive probability measure iff bettors have no winning strategy against (reversible) bookmaker selling bets at $P$.

3. Riesz representation theorem: yields
   $P([\varphi]) = \int_{\operatorname{MaxSpec} \mathcal{B}} \chi_{[\varphi]} d\mu$.

# Generalising this conceptual framework

To develop a theory of probability of non-classical events.

Events are identified with (logically equivalent classes of) formulas in a logic $L$:

1. A Stone-like duality theorem between the category of $\mathbb{L}$-algebras and their homomorphisms and a suitable category of spaces.

2. A de Finetti-like coherence criterion defining probability of events subject to the logical constraints imposed by $L$.

3. A Riesz-like representation theorem to express probabilities as integral measures over the dual spaces of $\mathbb{L}$-algebras.

# A non-classical case

What about propositional intuitionistic logic, and Heyting algebras?

1. Stone-like duality: Esakia spaces.
2. de Finetti's-like coherence criterion: ?
3. Riesz-like integral representation: ?

In this work in progress, we are proposing a way to define probabilities over finite Heyting algebras as integrals of certain functions.

# Another non-classical case

Propositional Gödel-Dummett logic is intuitionism plus prelinearity.
Algebraic semantics: Gödel algebras $\mathcal{G}$ (Heyting algebras plus prelinearity).

Consider finite Gödel algebras.

1. Stone-like duality: finite forests with open, order-preserving maps.
2. de-Finetti's like coherence criterion: a betting scheme over events evolving in time.
3. Riesz-like representation theorem: Free $n$-generated Gödel algebras are represented as algebras of functions $f\colon [0,1]^n \to [0,1]$.
   (Finitely additive) Probabilities are defined as integrals of them.

# The Gödel-Dummett case Algebraic semantics

Gödel-Dummett propositional logic $G$:
Intuitionistic propositional logic + $(\varphi \to \psi) \vee (\psi \to \varphi)$.

The equivalent algebraic semantics of $G$ is given by the variety $\mathbb{G}$ of Gödel algebras $(A, \wedge, \to, \bot)$, that is prelinear Heyting algebras.

$\mathbb{G}$ is generated by the standard Gödel algebra:

$([0,1], \min, \to, 0)$, where $x \to y = 1$ iff $x \leq y$; $x \to y = y$ iff $x > y$.

The free $n$-generated Gödel algebra $F_n(\mathbb{G})$ is the subalgebra of $[0,1]^{[0,1]^n}$ generated by the projections $\pi_i \colon (t_1, \ldots, t_n) \mapsto t_i$.

$\mathbb{G}$ is locally finite, whence the classes of finitely generated, finitely presented and finite Gödel algebras coincide.

# The Gödel-Dummett case $F_n(\mathbb{G})$

$\Omega = X_1 < X_2 < \cdots < X_h$: ordered partition of $\{0, 1, \ldots, n, n+1\}$ such that $0 \in X_1$ and $n+1 \in X_h$.

$(X_1 < \cdots < X_h) \preceq (Y_1 < \cdots < Y_k)$ iff $X_i = Y_i$ for all $i < h$ and $X_h = \bigcup_{i>=h} Y_i$.

$B_\Omega$: set of all $(t_1, \ldots, t_n) \in [0,1]^n$ such that

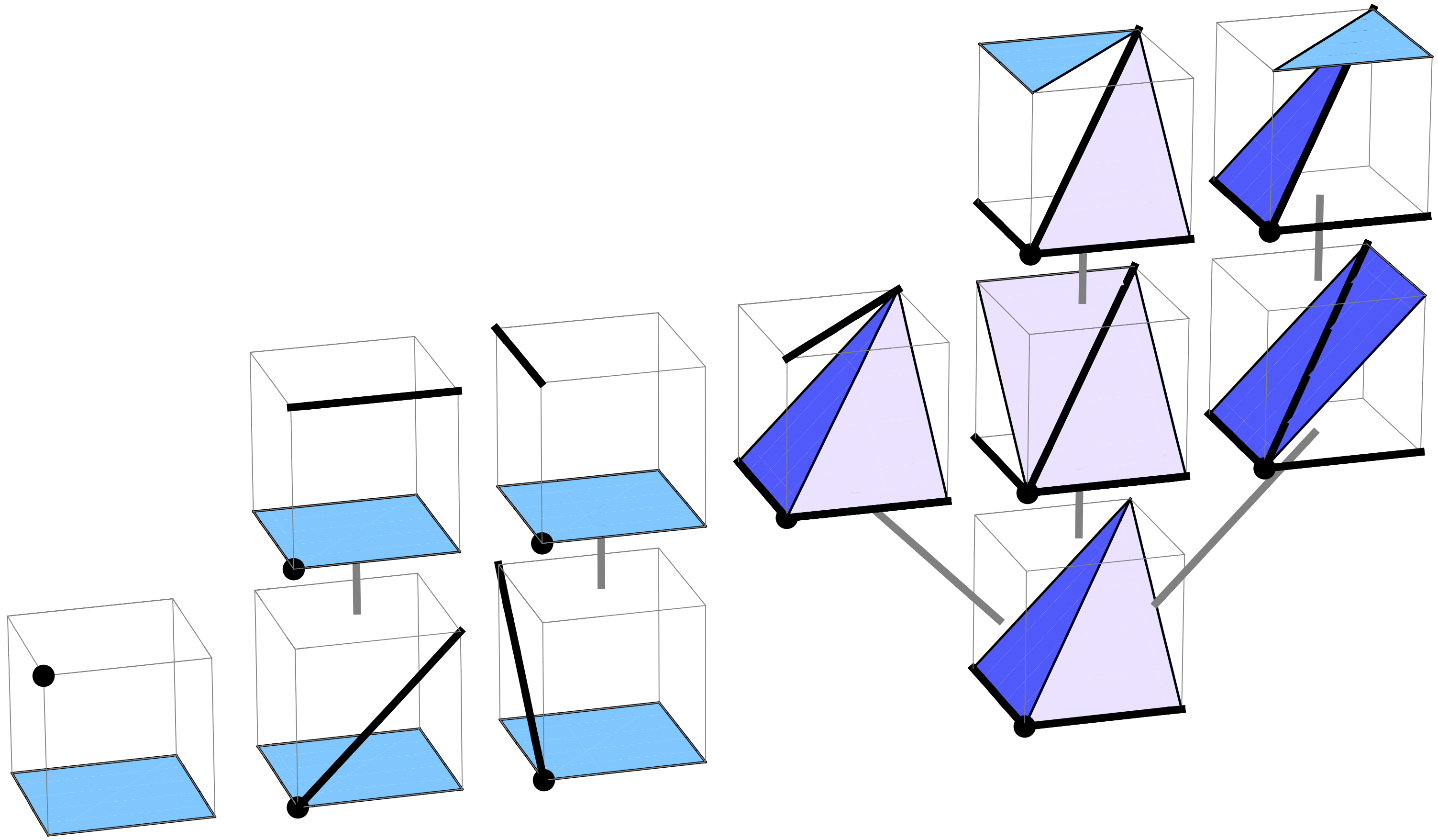
- $t_i = 0$ iff $i \in X_1$, $\qquad t_i = 1$ iff $i \in X_h$,
- $t_i = t_j$ iff there is $l$ such that $i, j \in X_l$,
- $t_i < t_j$ iff there are $l < m$ such that $i \in X_l$ and $j \in X_m$.

$C_\Omega = \bigcup_{\Omega' \preceq \Omega} B_{\Omega'}$.

$F_n(\mathbb{G})$ : all functions $f\colon [0,1]^n \to [0,1]$ such that for each $\preceq$-maximal $\Omega$ there is $g_\Omega \in \{0, \pi_1, \ldots, \pi_n, 1\}$ with $f \restriction C_\Omega = g \restriction C_\Omega$
(equipped with the pointwise operations of the standard algebra).

# Example: $F_2(\mathbb{G})$

All join-irreducible elements of $F_2(\mathbb{G})$ by pointwise order:



# The Gödel-Dummett case States (?)

$A$ is a finite Gödel algebra.

$A$ is a homomorphic image of some $F_n(\mathbb{G})$.

$A$ is isomorphic to the algebra of restrictions of functions of $F_n(\mathbb{G})$ to a suitable subset $D_A \subseteq [0,1]^n$.

A state of $A$ is a functional $s\colon A \to [0,1]$ such that

1. $s(\bot) = 0$, $s(\top) = 1$;
2. $s(a) + s(b) = s(a \vee b) + s(a \wedge b)$;
3. If $a \leq b$ then $s(a) \leq s(b)$;

States coincide with integrals of functions of $A$.
For each state $s$ there is a (regolar Borel probability) measure $\mu$ (and viceversa) such that

$$s(a) = \int_{[0,1]^n} a \, d\mu, \quad \text{for all } a \in A.$$

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States coincide with integrals of functions of $A$. $\Longleftarrow$ **Wrong!**

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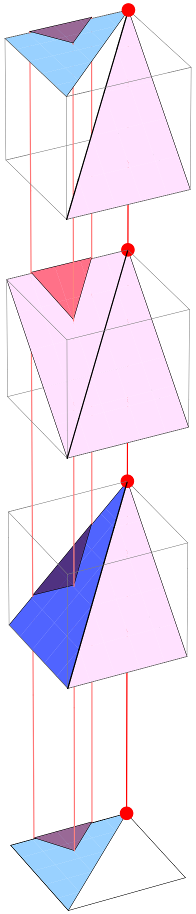
# The Gödel-Dummett case States



Let $f, g, h$ be three join irreducibles or $\bot$ with $f < g < h$, and consider $C_\Omega$ included in the *support* of $g$.

$$\text{If} \qquad \int_{[0,1]^n} f \, \mathrm{d}\mu = \int_{[0,1]^n} g \, \mathrm{d}\mu$$

then all mass of $\mu$ in $C_\Omega$ is concentrated in the 1-set $f^{-1}(1)$ of $f$.

That is: $\forall B \in \mathfrak{B}([0,1]^n)\colon B \subseteq C_\Omega,\ B \cap f^{-1}(1) = \emptyset \Rightarrow \mu(B) = 0\,.$

$$\text{Whence,} \qquad \int_{[0,1]^n} f \, \mathrm{d}\mu = \int_{[0,1]^n} g \, \mathrm{d}\mu = \int_{[0,1]^n} h \, \mathrm{d}\mu\,.$$

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4. If $a \leq b$ and $s(b) = s(a)$ then $s(b \to a) = 1$.

States coincide with integrals of functions of $A$.
For each state $s$ there is a (regolar Borel probability) measure $\mu$ (and viceversa) such that

$$s(a) = \int_{[0,1]^n} a \, d\mu \,, \qquad \text{for all } a \in A \,.$$

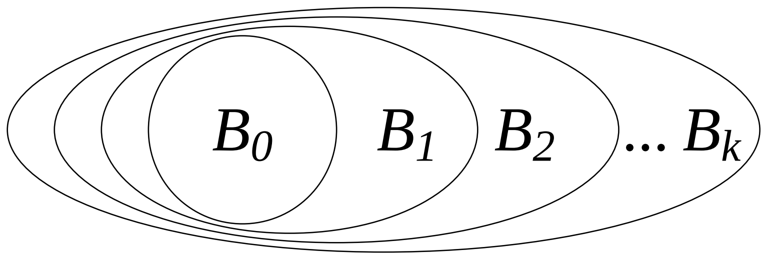
# States: Condition (4)

(4) If $a \leq b$ and $s(b) = s(a)$ then $s(b \to a) = 1$.

$$\frac{\vdash \varphi \to \psi \qquad P(\psi) \leq P(\varphi)}{P(\psi \to \varphi) = 1} \,. \qquad (\dagger)$$

Intuitive classical reading:
*If $A \subseteq B$ and $P(B) = P(A)$ then $P(B \setminus A) = 0$.*

($\dagger$) is automatically true in Boolean algebras (and in some non-classical case, *e.g.* MV-algebras), but it must be required explicitly in the case of Gödel algebras (and it reads: *for certain families $B_0, \ldots, B_k$ of nested sets, if $P(B_i) = P(B_{i+1})$ for some $i$ then $P(B_{j+1} \setminus B_j) = 0$ for all $j \geq i$*).



# The Gödel-Dummett case Duality

A finite forest is a finite poset such that the downset of each point is linearly ordered.

A map between forests is open (or a p-morphism) if it carries downsets to downsets.

$$\mathsf{G}_{fin} \equiv \mathsf{FF}^{\partial} \qquad \text{via} \qquad \mathrm{Sub}\colon \mathsf{FF} \to \mathsf{G}_{fin}, \qquad \mathrm{Spec}\colon \mathsf{G}_{fin} \to \mathsf{FF}\,,$$

Spec $A$ is the prime spectrum of $A$, ordered by reverse inclusion (equivalently, is the poset of the join irreducibles of $A$, ordered by restriction).

Sub $F = (\{X \subseteq F \mid X = \downarrow X\}, X \cap Y, F \setminus \uparrow (X \setminus Y), \emptyset)$ is the algebra of all downsets of $F$, equipped with suitably defined operations.

Spec $h\colon \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is given by $(\mathrm{Spec}\, h)(\mathfrak{p}) = h^{-1}(\mathfrak{p})$.

Sub $f\colon \mathrm{Sub}\, G \to \mathrm{Sub}\, F$ is given by $(\mathrm{Sub}\, f)(X) = f^{-1}(X)$.

# The Gödel-Dummett case — States and labelings

$A$ is a finite Gödel algebra.

$F = \mathrm{Spec}\, A$ is its dual finite forest.

States of $A$ are in bijective correspondence with labelings of $F$.

A labeling of a finite forest $F$ is a map $l\colon F \to [0,1]$ such that

1. $\sum_{a \in F} l(a) = 1$
2. $l^{-1}(0)$ is an upward closed subset of $F$.

(Condition (2) translates state:Condition (4)).

For each $a \in A$ , $s(a) = \sum\{l(b) \mid b \in \text{J.I.}(A),\ b \leq a\}$.

# On weakly Gödelian maps

D a (bounded) distibutive lattice.

C a linearly ordered distributive lattice.

$q\colon D \twoheadrightarrow C$ a quotient map in the category of bounded distributive lattices.

The regular epi $q$ is locally Gödelian iff for all factorisations $q = k \circ h$, $h\colon D \twoheadrightarrow E$ and $k\colon E \to C$ as bounded distributive lattices, with $E$ linearly ordered, it holds that $k$ is a morphism of Heyting algebras.

A morphism $g\colon D \to C$ of bounded distributive lattices, with $C$ lineary ordered, is locally Gödelian iff so is $q\colon D \twoheadrightarrow \operatorname{Im} g$ (where $g = e \circ q$ is the regular epi, mono factorisation of $g$).

A morphism $g\colon D_1 \to D_2$ of bounded distributive lattices is weakly Gödelian iff for each locally Gödelian quotient $q\colon D_2 \to C_2$, the composition $q \circ g\colon D_1 \to C_2$ is locally Gödelian.

# On weakly Gödelian maps Gödelian hulls

$D$ is a distributive lattice.

$G$ is a Gödel algebra.

$\iota\colon D \to U(G)$ is weakly Gödelian.

$\iota$ is a Gödelian hull of $D$ iff for any weakly Gödelian morphism $f\colon D \to U(G')$, there is a unique $g\colon G \to G'$ of Gödel algebras such that $f = U(g) \circ \iota$.

Then:

- Any finite distributive lattice $D$ has a unique Gödelian hull $\iota\colon D \to U(G)$;
- $\iota$ is monic;
- $G$ is finite;
- If $D \cong U(G)$ for some Gödel algebra $G$ then $\mathrm{id}_D$ is its Gödelian hull.

# On weakly Gödelian maps the dual side

P is a finite poset.

F a forest.

Saturated path in $P$: $[a_1, a_2, \ldots, a_h]$, with $a_1 \in \min P$, and $a_{i+1}$ covers $a_i$ in $P$.

$[a_1, \ldots, a_h] \preceq [b_1, \ldots b_k]$ iff $h \leq k$ and $a_i = b_i$.

Path $P$ is the forest of saturated paths, ordered by $\preceq$.

e.o.p.: Path $P \to P$: $[a_1, \ldots, a_h] \mapsto a_h$ is the end-of-path map.

Then:

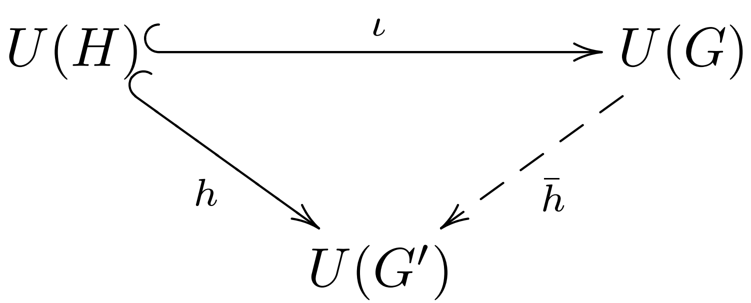
Sub e.o.p. : $\mathrm{Sub}\, P \to U(\mathrm{Sub}\,\mathrm{Path}\, P)$ is the Gödelian hull of Sub $P$.

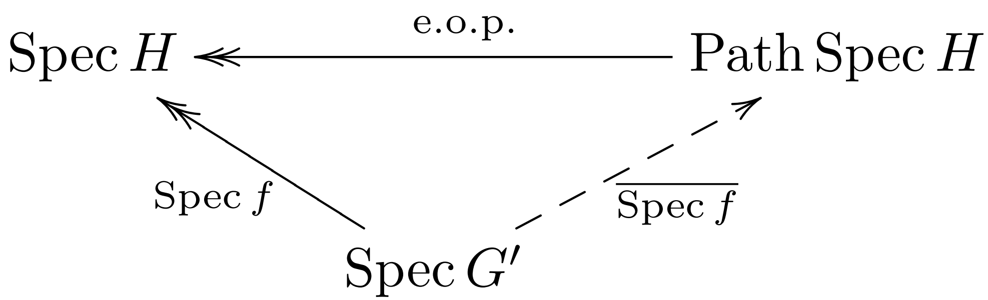
# Gödelian hull of a finite Heyting algebra

$H$: finite Heyting algebra.

$G'$: finite Gödel algebra.

$\iota\colon U(H) \to U(G)$: Gödelian hull of $U(H)$.

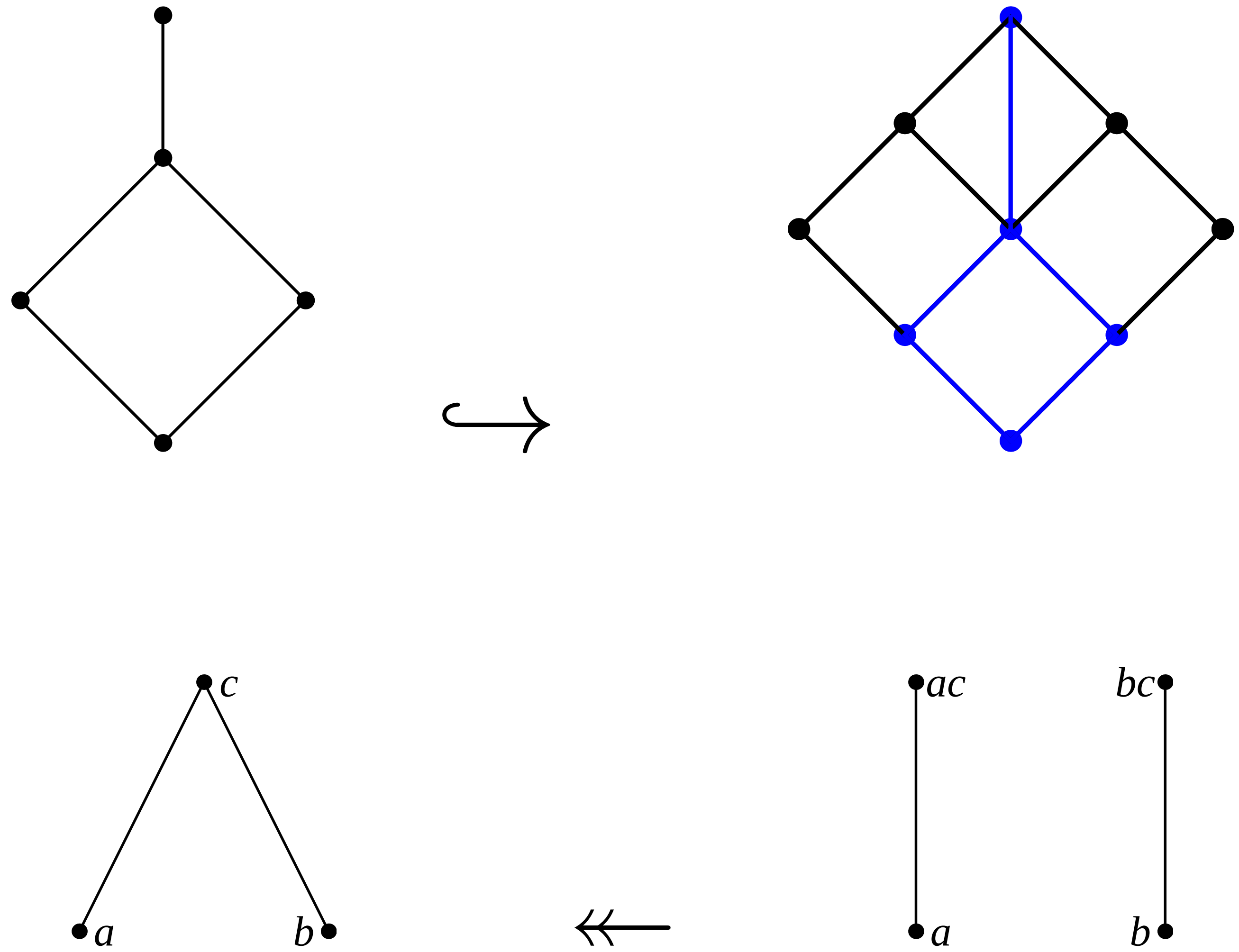



$G \cong \operatorname{Sub}\operatorname{Path}\operatorname{Spec} H$.

$\overline{\operatorname{Spec} f}\colon \operatorname{Spec} G' \to \operatorname{Path}\operatorname{Spec} H$ defined by
$(\overline{\operatorname{Spec} f})(x) = ((\operatorname{Spec} f)[\downarrow x], \leq_{\operatorname{Spec} H})$.

# The smallest example



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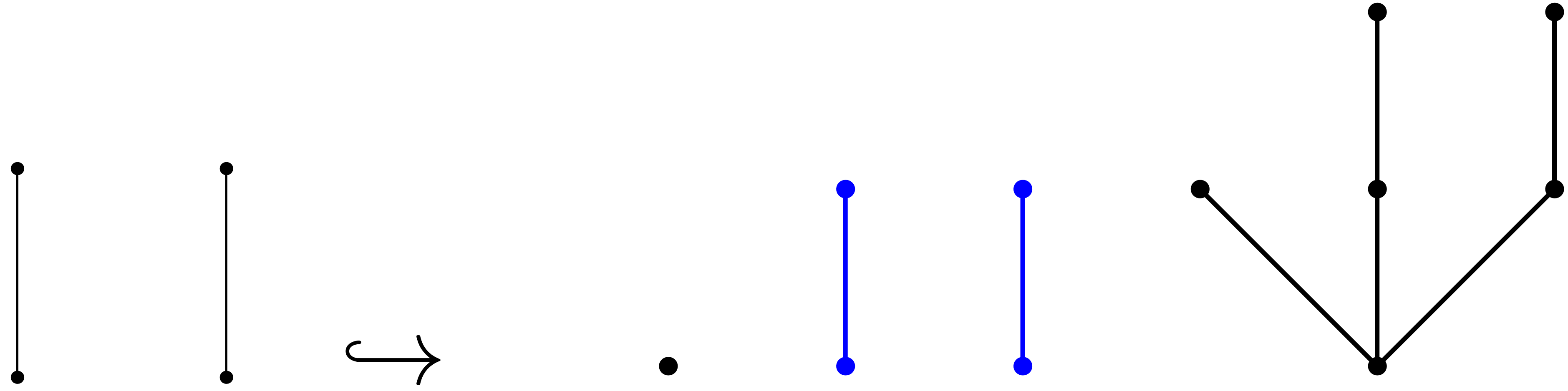
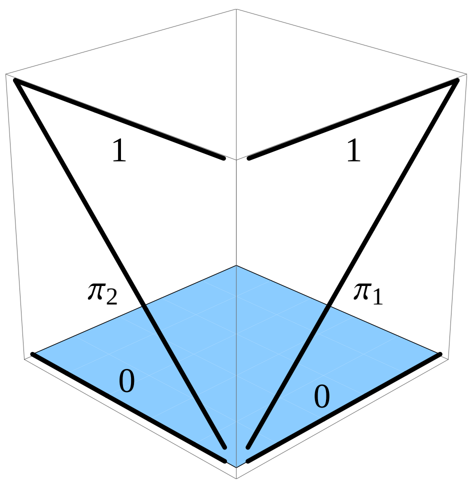
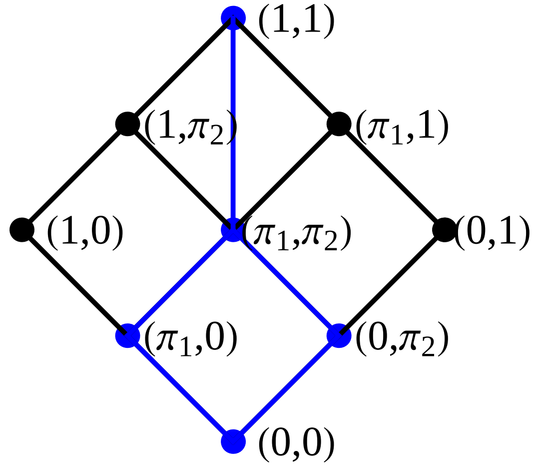




# States of a finite Heyting algebra

$H$ is a finite Heyting algebra.

$\iota\colon U(H) \to U(G)$ is the Gödelian hull of $H$.

Dually, $G \cong \mathrm{Sub\,Path\,Spec}\, H$.

A state $s\colon H \to [0,1]$ is the restriction to $H$ of a state $s'\colon G \to [0,1]$.

- $s(\bot) = 0$, $s(\top) = 1$;
- $s(a) + s(b) = s(a \vee b) + s(a \wedge b)$;
- If $a \leq b$ then $s(a) \leq s(b)$;
- If $a \leq b$ and $s(b) = s(a)$ then $s(b \to a) = 1$.

States of $H$ coincide with integrals of functions of $G$.

For each $s$ there is a (regolar Borel probability) measure $\mu$ (and viceversa) s. th.

$$s(a) = \int_{[0,1]^n} a \, d\mu \qquad \text{for all } a \in H$$

# States of a finite Heyting algebra labeled posets

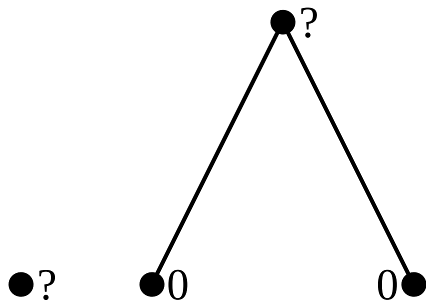
$H$ is a finite Heyting algebra.

States of $H$ are in bijection with labelings $l$ of $\mathrm{Spec}\,H$:

$l\colon \mathrm{Spec}\,H \to [0,1]$ such that:

- $\sum_{\mathfrak{p}\in \mathrm{Spec}\,H} l(\mathfrak{p}) = 1$;
- For any $\mathfrak{p} \in \mathrm{Spec}\,H$: if each $\mathfrak{q}$ covered by $\mathfrak{p}$ is such that $l(\mathfrak{q}) = 0$ then $l(\mathfrak{p}) = 0$, too.

For each $a \in H$ , $s(a) = \sum\{l(b) \mid b \in \mathrm{J.I.}(A),\, b \le a\}$.



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