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Preliminaries

iff for "if and only if",

$:=$ denotes equality by definition.

The symbol $\dashv$ indicates the end of a proof.

Expressions that differ only in the name of bound variables are regarded as identical.

Substitution of expressions involves a systematic renaming operation for bound variables, thereby avoiding clashes.

$FV(e)$: the set of free variables of e.

Natural Deduction for Propositional Logic

Formalization of intuitionistic logic was obtained by dropping some axioms of *classical propositional logic* or *classical propositional calculus* abbreviated CPC . Basic properties of CPC are assumed.
Formulas of propositional logic are constructed in the standard way from *propositional variables* (called also *propositional letters* and a constant $\top$ (True) by means of *logical connectives* $\&, \rightarrow$.
We often drop outermost parentheses as well as parentheses dividing terms in a conjunction or a disjunction.
finite multisets are finite sequences modulo the ordering:
Permutation
$\alpha_{i_1}, \ldots, \alpha_{i_n}$ is identified with $\alpha_1, \ldots, \alpha_n$,
but the number of occurrences of each α_i is important. Multisets of formulas are denoted by $\Gamma, \Gamma_1, \Delta, \Delta_1, \ldots$.

$\alpha, \Gamma = \{\alpha\} \cup \Gamma$.
Γ, Σ is the multiset union of Γ and Σ:
$\{\alpha, \alpha\}, \{\alpha, \alpha, \alpha\}$ is $\{\alpha, \alpha, \alpha, \alpha, \alpha\}$.
The $[\Gamma, \Delta]$ represents the multiset union of Γ, Δ and a possible identification of some formulas in Δ with identical formulas in Γ.
For example:

$$[\{\alpha, \alpha, \beta, \beta\}, \{\alpha, \alpha, \alpha, \gamma, \gamma\}]$$

can be any of:

$$\{\alpha, \alpha, \alpha, \alpha, \alpha, \beta, \beta, \gamma, \gamma\}, \{\alpha, \alpha, \alpha, \alpha, \beta, \beta, \gamma, \gamma\}, \{\alpha, \alpha, \alpha, \beta, \beta, \gamma, \gamma\},$$

but not $\{\alpha, \alpha, \beta, \beta, \gamma, \gamma\}$.

Intuitionistic Propositional System NJp

$$\text{Sequents:} \qquad \alpha_1, \ldots, \alpha_n \Rightarrow \alpha$$

read "*assumptions* $\alpha_1, \ldots, \alpha_n$ imply α".
α is the *succedent*, $\alpha_1, \ldots, \alpha_n$ constitute the *antecedent* that is treated as a *multiset*.
Axioms:

$$\alpha, \Gamma \Rightarrow \alpha, \qquad \Gamma \Rightarrow I.$$

Inference rules (I, E stand for introduction, elimination):

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{[\Gamma, \Delta] \Rightarrow \alpha \& \beta}\ \&I \qquad \frac{\Gamma \Rightarrow \alpha \& \beta}{\Gamma \Rightarrow \alpha}\ \&E \qquad \frac{\Gamma \Rightarrow \alpha \& \beta}{\Gamma \Rightarrow \beta}\ \&E$$

$$\frac{\Gamma \Rightarrow (\alpha \to \beta) \quad \Delta \Rightarrow \alpha}{\Gamma, \Delta \Rightarrow \beta} \to E \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow (\alpha \to \beta)} \to I$$

Each of the sequents written above the line in an inference rule is a *premise* of the rule, and the sequent written under the line is the *conclusion*.

Classical Propositional Calculus is obtained by adding the classical negation rule $\bot_c$ for proofs *ad absurdum* to the system NJp:

$$\frac{\neg\alpha, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow \alpha}\ \bot_c$$

A *natural deduction* or a *proof* in the system NJp is defined in a standard way:

It is a *tree* beginning with axioms and proceeding by the inference rules of the system. A sequent is *deducible* or *provable* if it is a last sequent of a deduction.

A formula α is *deducible* or *provable* if the sequent $\Rightarrow \alpha$ is provable.

Notation $d : \Gamma \Rightarrow \alpha$ indicates that d is a natural deduction of $\Gamma \Rightarrow \alpha$, and $\Gamma \vdash \alpha$ means that the sequent $\Gamma \Rightarrow \alpha$ is derivable in NJp.

Comments. Axiom $\alpha, \Gamma \Rightarrow \alpha$ introduces an *assumption* α. Applications of inference rules (or *inferences*) transform goal formulas written to the right of $\Rightarrow$ and leave assumptions intact except for $\rightarrow I$ and $\vee E$-inferences, which *discharge* the assumption α ($\vee E$ discharges also assumption β). Every connective $\odot$ has two rules: An introduction rule $\odot I$ for proving a formula beginning with the connective and an elimination rule $\odot E$ for deriving consequences from proved formula beginning with this connective. For example $\rightarrow$-elimination rule $\rightarrow E$ called also *detachment* or *modus ponens* uses proved formula $\alpha \rightarrow \beta$ to derive β provided α is proved too.

Unless stated otherwise, we are interested in derivability in the system NJp.

$Ax\ \alpha$ means the axiom $\alpha \Rightarrow \alpha$.
Example. The formula $p \rightarrow (q \rightarrow p)$ is derived in NJp by two $\rightarrow I$-inferences from the axiom $p \Rightarrow p$:

$$\dfrac{\dfrac{p \Rightarrow p}{p \Rightarrow q \rightarrow p} \rightarrow I}{\Rightarrow p \rightarrow (q \rightarrow p)} \rightarrow I$$

Example.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Ax\ p \rightarrow (q \rightarrow r) \quad p \Rightarrow p}{p \rightarrow (q \rightarrow r), p \Rightarrow q \rightarrow r} \rightarrow E \quad \dfrac{Ax\ p \rightarrow q \quad p \Rightarrow p}{p \rightarrow q, p \Rightarrow q} \rightarrow E}{p \rightarrow (q \rightarrow r), p \rightarrow q, p \Rightarrow r} \rightarrow E}{p \rightarrow (q \rightarrow r), p \rightarrow q \Rightarrow p \rightarrow r} \rightarrow I}{p \rightarrow (q \rightarrow r) \Rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)} \rightarrow I}{d : \Rightarrow (p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))} \rightarrow I$$

In the $\rightarrow E$-inference above

$$\frac{p \rightarrow (q \rightarrow r), p \Rightarrow q \rightarrow r \quad p \rightarrow q, p \Rightarrow q}{p \rightarrow (q \rightarrow r), p \rightarrow q, p \Rightarrow r} \rightarrow E$$

the assumption p is used twice, one time in each of the premises. Implicit application of the *contraction* rule:

$$\frac{\Gamma, \alpha, \alpha \Rightarrow \gamma}{\Gamma, \alpha \Rightarrow \gamma}$$

Another rule derivable in NJp is weakening (cf. Example 14):

$$\frac{\Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma} \textit{ weak}$$

We sometimes use weakening implicitly, for example applying a $\rightarrow I$ rule in the form:

$$\frac{\Gamma \Rightarrow \beta}{\Gamma, \Delta \Rightarrow (\alpha \rightarrow \beta).}$$

Substitution Rule

Lemma

If a sequent $\Gamma(p, q, \ldots) \Rightarrow \gamma(p, q, \ldots)$ *is derivable then the sequent* $\Gamma(\alpha, \beta, \ldots) \Rightarrow \gamma(\alpha, \beta, \ldots)$ *is derivable too.*

$\dashv$

Traditional notation for natural deduction.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \to (q \to r) \quad p}{q \to r} \quad \dfrac{p \to q \quad p}{q}}{r}}{p \to r}}{(p \to q) \to (p \to r)}}{d^- : \ (p \to (q \to r)) \to ((p \to q) \to (p \to r))}$$

Example. To derive $(p\&q) \rightarrow p$, assume the premise and apply $\&E$ followed by $\rightarrow I$:

$$\dfrac{\dfrac{p\&q}{p}}{(p\&q) \rightarrow p}$$

Example. To derive $p \rightarrow (q \rightarrow (p\&q))$, assume both premises and apply $\&I$ twice :

$$\dfrac{\dfrac{\dfrac{p \quad q}{p\&q}}{q \rightarrow (p\&q)}}{p \rightarrow (q \rightarrow (p\&q))}$$

Exercises

1. Prove all $(\&, \to)$ axioms of the systen HJp below.
2. Prove "translations" of all postulates of the system HCC below, for example $((A \to B)\&A) \to B$,
3. $A \to B, A \to C \Rightarrow A \to B\&C$,
4. $A\&B \to C \Rightarrow A \to (B \to C)$,
5.

Derivable Rules

It is often convenient to treat a series of inference rules as one rule.

Definition

A deduction *of a sequent S* from *sequents $S_1, \ldots, S_n$ is a tree beginning with axioms or sequents $S_1, \ldots, S_n$ and proceeding by inference rules.*
A rule

$$\frac{S_1, \ldots, S_n}{S}$$

is derivable *if there is a deduction of S from $S_1, \ldots, S_n$.*

Example. The cut rule:

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \beta} \; cut$$

$$\frac{\dfrac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta} \quad \Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \beta}$$

The weakening rule is derived as follows:

$$\dfrac{\dfrac{\Gamma \Rightarrow \phi}{\Gamma \Rightarrow \alpha \rightarrow \phi} \quad \alpha \Rightarrow \alpha}{\alpha, \Gamma \Rightarrow \phi}$$

Example. Uniform versions of two-premise rules. A rule:

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \phi}$$

is derivable iff the rule

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \rightarrow \phi}\ R$$

is derivable too. Indeed, the second rule is an instance of the first one. The other direction is justified as follows:

$$\frac{\dfrac{\Gamma \Rightarrow \alpha}{\Gamma, \Delta \Rightarrow \alpha}\ \mathit{weak} \quad \dfrac{\Gamma \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \beta}\ \mathit{weak}}{\Gamma, \Delta \Rightarrow \phi}$$

Hence it is possible to assume the antecedents Γ to be the same in all premises of the rule.

Direct Chaining and Analysis into Subgoals

The problem of deriving given sequent (goal) S can often be reduced to deriving simpler sequents (subgoals), say, S_1, S_2, if the rule

$$\frac{S_1 \quad S_2}{S}$$

is derivable. Let us list some of these rules.

Lemma

The following rules are derivable in NJp: $\rightarrow I, \& I$,

$$\frac{\alpha, \Gamma \Rightarrow \beta \quad \beta, \Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \leftrightarrow \beta} \leftrightarrow I$$

Proof. $\leftrightarrow I$: expand abbreviation $\alpha \leftrightarrow \beta$ into $(\alpha \rightarrow \beta)\&(\beta \rightarrow \alpha)$ and use $\rightarrow I, \& I$.

Rules for direct chaining

Lemma

The following rules are derivable in NJp: $\to E, \& E$, *Trans*,

$$\frac{\Gamma \Rightarrow \alpha \leftrightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta} \leftrightarrow E \qquad \frac{\Gamma \Rightarrow \alpha \leftrightarrow \beta}{\Gamma \Rightarrow \beta \to \alpha} \leftrightarrow E$$

Proof. $\leftrightarrow E$: expand $\alpha \leftrightarrow \beta$ and use $\& E$. $\dashv$

Definition

A deduction using only elimination rules and structural rules is called direct chaining

Note

A good heuristic for deducing $\Gamma \Rightarrow \phi$ by direct chaining is to take Γ as the initial set of data and saturate it by adding conclusions of all elimination rules.

Exercise

Derive $p_0, p_0 \rightarrow p_1 \& p_2, p_1 \rightarrow p_3, p_2 \rightarrow p_4, p_3 \rightarrow (p_4 \rightarrow p_5) \Rightarrow p_5$ *by direct chaining.*

ADC method of establishing deducibility

When the goal is an implication $\phi \equiv \alpha_1 \to (\ldots \to (\alpha_n \to \psi)\ldots)$, it is reduced to a sequent $\alpha_1, \ldots, \alpha_n \Rightarrow \psi$.

$$\dfrac{\dfrac{\begin{array}{c}\phi_n \equiv \alpha_1, \ldots, \alpha_n \Rightarrow \psi \\ \vdots\end{array}}{\phi_1 \equiv \alpha_1 \Rightarrow \alpha_2 \to (\ldots \to (\alpha_n \to \psi)\ldots)}}{\phi \equiv \alpha_1 \to (\ldots \to (\alpha_n \to \psi)\ldots)}$$

One of the most straightforward methods of establishing deducibility of a sequent $\Gamma \Rightarrow \alpha$ consists in its analysis into subgoals $\Gamma_1 \Rightarrow \alpha_1, \ldots, \Gamma_n \Rightarrow \alpha_n$ and establishing each subgoal by direct chaining. We say that a sequent $\Delta \Rightarrow \alpha \vee \beta$ is established when one of $\Delta \Rightarrow \alpha,\ \Delta \Rightarrow \beta$ is established. The combination of Analysis and Direct Chaining described above will be called ADC-*method* or simply ADC. It is not complete: some valid formulas are not deducible by ADC.

Heuristics for Natural Deduction

$\Gamma \vdash \phi$ means that the sequent $\Gamma \Rightarrow \phi$ is derivable in NJp.

Lemma

Every sequent derivable in NKp (and hence in NJp) is a tautology according to classical truth tables.

Proof. For a given truth value assignment v define

$$v(\alpha_1, \ldots, \alpha_n \Rightarrow \alpha) := v(\&_{i \leq n}\alpha_i \rightarrow \alpha) :=$$

$$\max(1 - \min(v(\alpha_1), \ldots, v(\alpha_n)), v(\alpha))$$

Check that every rule preserves truth (that is, value 1) under every assignment v. $\dashv$

Some tautologies including

$$((p \to q) \to p) \to p$$

are not derivable in NJp.

Lemma

$\Gamma \vdash \alpha \to \beta$ *iff* $\Gamma, \alpha \vdash \beta$

One direction is $\to E$, the other direction is $\to I$:

$$\frac{\Gamma \Rightarrow \alpha \to \beta \quad \alpha \Rightarrow \alpha}{\Gamma, \alpha \Rightarrow \beta} \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}$$

$\dashv$

Consider more complicated derivation.
Example. Deduce $\alpha \to q \Rightarrow q$ where $\alpha \equiv ((p \to q) \to p) \to p$.
Preparing application of $\to E$ we prove first $\alpha \to q \Rightarrow \alpha$.
By Lemma 5 it is sufficient to deduce

$$\alpha \to q, (p \to q) \to p \Rightarrow p,$$

which would follow by $\to E$ if we had $\alpha \to q \Rightarrow p \to q$.
The latter is reduced to $\alpha \to q, p \Rightarrow q$, then to $p \Rightarrow \alpha$, then to $p, (p \to q) \to p \Rightarrow p$, which is an axiom. Let us write the deduction in traditional form:

$$\dfrac{\alpha \to q \qquad \dfrac{\dfrac{(p \to q) \to p \qquad \dfrac{\dfrac{\alpha \to q \qquad \dfrac{p}{\alpha}}{q}}{p \to q}}{p}}{\alpha}}{q}$$

Hilbert-style system HJp

The systems of natural deduction have few axioms (just two in our case) and many inference rules.
Before that logical systems had many axioms (each usually accounting for some feature of the logical connectives or other basic notions) and very few inference rules, one and the same for many systems. Such axiomatizations are often called Hilbert systems after D. Hilbert who made that kind of formalization popular.
The following axioms together with the rule of modus ponens

$$\frac{\alpha \quad \alpha \to \beta}{\beta} \textit{ modus ponens}$$

constitute a complete axiomatization of the intuitionistic propositional calculus NJp.

Axioms of HJp

$$
\begin{array}{l}
\alpha \to (\beta \to \alpha) \\
(\alpha \to (\beta \to \gamma)) \to \ ((\alpha \to \beta) \to \ (\alpha \to \gamma)) \\
\alpha \to (\beta \to (\alpha \& \beta)) \\
\alpha \& \beta \to \ \alpha \\
\alpha \& \beta \to \ \beta \\
\alpha \to (\alpha \vee \beta) \\
\beta \to (\alpha \vee \beta) \\
(\alpha \to \gamma) \to \ ((\beta \to \gamma) \to \ ((\alpha \vee \beta) \to \gamma)) \\
\perp \to \alpha
\end{array}
$$

Exercise

Prove that each of the $(\&, \to)$ *axioms of HJp is deducible in NJp.*

BHK-interpretation

We write $c\mathbf{r}\alpha$ for "c realizes α" or "c is a construction for α":
$c\mathbf{r}(\alpha_0 \wedge \alpha_1)$ iff c is a pair $c = \mathbf{p}(a_0, a_1)$ and $a_0\mathbf{r}\alpha_0$ and $a_1\mathbf{r}a_1$,
$c\mathbf{r}(\alpha_0 \vee \alpha_1)$ iff $(c = \mathbf{k}_0 a$ and $a\mathbf{r}\alpha_0)$ or $(c = \mathbf{k}_1 a$ and $a\mathbf{r}\alpha_1)$,
$c\mathbf{r}(\alpha \rightarrow \beta)$ iff c is a function and for every d, if $d\mathbf{r}\alpha$ then $c(d)\mathbf{r}\beta$,
not $c\mathbf{r}\bot$.
A language for such constructions should include pairing, projections (components of pairs), forming functions, and applying functions to arguments.
Example. Realization t of $(\alpha_0 \& \alpha_1) \rightarrow (\beta_0 \& \beta_1)$ is a program that for every pair $x = \mathbf{p}(a_0, a_1)$ such that a_i realizes α_i, produces a pair $t(x) = \mathbf{p}(b_0, b_1)$ such that b_j realizes β_j.

Exercise

What are realizations of formulas of the following form:

$$\alpha \to (\beta \& \gamma); \qquad (\alpha \to (\beta \to \gamma)) \qquad ((\alpha \to \beta) \to \gamma).$$

Assignment $\mathcal{T}$ of Deductive Terms

A $(\&, \to)$ language for writing realizations of formulas derivable in NJp.

Pairing $\mathbf{p}$ with *projections* $\mathbf{p}_0, \mathbf{p}_1$ satisfying:

$$\mathbf{p}_i(\mathbf{p}(t_0, t_1)) = t_i, \qquad i = 0, 1 \tag{1}$$

and lambda abstraction providing explicit definitions:

$$(\lambda x.t)(u) = t[x/u], \tag{2}$$

where $t[x/u]$ stands for the result of substituting a term u for all free occurrences of a variable x (of the same type) in t

In other words, if $t[x]$ is an expression containing a variable x, it is possible to define a function (denoted by $\lambda x.t[x]$) that outputs the value $t[u]$ when the value of x is set to u. Often we omit a dot and write λxt.

Application of a function t to an argument u is denoted by $t(u)$.

We assume for every formula ϕ of the language under consideration a countably infinite supply of variables of type ϕ. For distinct formulas ϕ, the corresponding sets of variables are disjoint. We use x^ϕ, y^ϕ, z^ϕ for arbitrary variables of type ϕ; we omit the type superscript when clear from the context.

Now we assign a term $\mathcal{T}(d)$ to every natural deduction d deriving a sequent:

$$\Gamma \Rightarrow \phi. \tag{3}$$

A term u assigned to a derivable sequent (3) is supposed to realize a formula ϕ according to the BHK-interpretation under assumptions Γ. We sometimes write t^{ϕ} to stress this. To reflect dependence of assumptions, every assignment depends on a *context* that is itself an assignment:

$$z^{\phi_1} : \phi_1, \ldots, z^{\phi_n} : \phi_n$$

of distinct typed variables to formulas in $\Gamma \equiv \phi_1, \ldots, \phi_n$ written sometimes as $\mathbf{z} : \Gamma$. These typed variables stand for hypothetical realizations of the assumptions Γ. When a term u is assigned to ϕ, the sequent (3) is transformed into a statement:

$$z^{\phi_1} : \phi_1, \ldots, z^{\phi_n} : \phi_n \Rightarrow \ u : \phi \qquad or \qquad \mathbf{z} : \Gamma \Rightarrow u : \phi$$

Contexts are treated as multisets. In particular $\mathbf{z} : \Gamma, \mathbf{z}' : \Delta$ stands for the union of multisets.

Deductive terms and the assignment of a term to a deduction is defined inductively. Assignments for axioms are given explicitly, and for every logical inference rule, there is an operation that transforms assignments for the premises into an assignment for the conclusion of the rule.

Assignment Rules

Axioms: $\mathbf{z} : \Gamma, x : \phi \Rightarrow x : \phi, \qquad \Rightarrow \ I : I$

Inference rules:

$$\frac{\mathbf{z} : \Gamma \Rightarrow t : \phi \quad \mathbf{z}' : \Delta \Rightarrow u : \psi}{\mathbf{z} : \Gamma, \mathbf{z}' : \Delta \Rightarrow \mathbf{p}(t, u) : (\phi \& \psi)} \ \& I \qquad \frac{\mathbf{z} : \Gamma \Rightarrow t : \phi_0 \& \phi_1}{\mathbf{z} : \Gamma \Rightarrow \mathbf{p}_i t : \phi_i} \ \& E \qquad i = 0, 1$$

$$\frac{\mathbf{z} : \Gamma \Rightarrow t : (\phi \rightarrow \psi) \quad \mathbf{z}' : \Delta \Rightarrow u : \phi}{\mathbf{z} : \Gamma, \mathbf{z}' : \Delta \Rightarrow t(u) : \psi} \rightarrow E \qquad \frac{(x : \phi)^0, \mathbf{z} : \Gamma \Rightarrow t : \psi}{\mathbf{z} : \Gamma \Rightarrow \lambda x.t : (\phi \rightarrow \psi)} \rightarrow$$

Term assignment $\mathcal{T}(d)$ to a natural deduction d is defined inductively by application of the term assignment rules.
Notation $\Gamma \Rightarrow\ t : \alpha$ or $\mathbf{z} : \Gamma \Rightarrow\ t : \alpha$ means that $t = \mathcal{T}(d)$ for some natural deduction $d : \Gamma \Rightarrow \alpha$.
The symbol λx binds variable x, *Free variables* of a term are defined in a familiar way:

$$FV(x) := x; \quad FV(\mathbf{p}_i t) := FV(t)$$

$$FV(\mathbf{p}(t, u)) := FV(t(u)) := FV(t) \cup FV(u); \quad FV(\lambda x t) := FV(t) - \{x\}$$

$$\frac{\mathbf{z} : \Gamma \Rightarrow u : \phi}{x : \psi, \mathbf{z} : \Gamma \Rightarrow u : \phi}\ \textit{weak} \qquad \frac{x : \psi, y : \psi, \mathbf{z} : \Gamma \Rightarrow u : \phi}{x : \psi, \mathbf{z} : \Gamma \Rightarrow u[y/x] : \phi}\ \textit{contr}$$

One simple and sufficiently general way of finding a realization of a formula is to derive it in NJp and compute the assigned term.

Example. Find a term t realizing $(p \to q) \to ((q \to r) \to (p \to r))$. Consider the following natural deduction d:

$$\dfrac{\dfrac{\dfrac{\dfrac{q \to r \Rightarrow q \to r \qquad \dfrac{p \to q \Rightarrow p \to q \quad p \Rightarrow p}{p, p \to q \Rightarrow q}}{p \to q, q \to r, p \Rightarrow r}}{p \to q, q \to r \Rightarrow p \to r}}{p \to q \Rightarrow (q \to r) \to (p \to r)}}{\Rightarrow (p \to q) \to ((q \to r) \to (p \to r))}$$

Assign terms:

$$\dfrac{\dfrac{\dfrac{\dfrac{y : q \to r \Rightarrow y : q \to r \qquad \dfrac{x : p \to q \Rightarrow x : p \to q \quad z : p \Rightarrow z : p}{z : p, x : p \to q \Rightarrow x(z) : q}}{x : p \to q, y : q \to r, z : p \Rightarrow y(x(z)) : r}}{x : p \to q, y : q \to r \Rightarrow \lambda z^{p}.y(x(z)) : p \to r}}{x : p \to q \Rightarrow \lambda y^{q \to r} \lambda z^{p}.y(x(z)) : (q \to r) \to (p \to r)}}{\Rightarrow \lambda x^{p \to q} \lambda y^{q \to r} \lambda z^{p}.y(x(z)) : (p \to q) \to ((q \to r) \to (p \to r))}$$

Hence $\mathcal{T}(d) \equiv \lambda x \lambda y \lambda z.y(x(z))$.

Several more examples are

$$\Rightarrow \lambda x^{p}.x : (p \to p)$$

$$\Rightarrow \lambda x^{p}\lambda y^{q}.x : p \to (q \to p)$$

$$\Rightarrow \lambda x^{p_i}.\mathbf{k}_i x : (p_i \to p_0 \vee p_1)$$

Exercise

Confirm the preceding realizations and find realizations for the following formulas using deductions in NJp:

$$p\&q \to p,\ \ p\&q \to q,\ \ p \to (q \to p\&q),$$

$$(p \to q) \to ((p \to (q \to r)) \to (p \to r)).$$

Properties of Term Assignment $\mathcal{T}$

The $\mathcal{T}(d)$ is defined up to renaming of free variables assigned to axioms. If $d : \Gamma \Rightarrow \phi$, that is, d is a deduction of $\Gamma \Rightarrow \phi$, then $\mathbf{z} : \Gamma \Rightarrow \mathcal{T}(d) : \phi$.
The operation $\mathcal{T}$ is an isomorphism: It has an inverse operation $\mathcal{D}$ preserving both syntactic identity and more important relation of $\beta\eta$-equality

operation $\mathcal{D}$

For every deductive term t^ϕ with $FV(t) = \mathbf{z}^\Gamma$ we define a deduction:

$$\mathcal{D}(t): \ \Gamma \Rightarrow \phi \tag{4}$$

If $t^\phi \equiv x^\phi$ then $\mathcal{D}(t) := \ \phi \Rightarrow \phi$ (Axiom)
If $t^{\phi_i} \equiv \mathbf{p}_i u^{\phi_0 \& \phi_1}$ then $\mathcal{D}(t)$ is obtained from $\mathcal{D}(u)$ by $\&E$:

$$\frac{\mathcal{D}(u): \ \Gamma \Rightarrow \phi_0 \& \phi_1}{\mathcal{D}(\mathbf{p}_i u): \ \Gamma \Rightarrow \phi_i} \ \&E$$

If $t^{\phi \& \psi} \equiv \mathbf{p}(u^\phi, v^\psi)$ with $FV(u) \equiv \mathbf{z}^\Sigma$, $FV(v) \equiv \mathbf{z}'^\Delta$, then:

$$\frac{\mathcal{D}(u): \ \Sigma \Rightarrow \phi \quad \mathcal{D}(v): \ \Delta \Rightarrow \psi}{\mathcal{D}(\mathbf{p}(u, v)): \ [\Sigma, \Delta] \Rightarrow \phi \& \psi} \ \&I,$$

where occurrences of identical assumptions in Σ and Δ are identified in $[\Sigma, \Delta]$ exactly when these occurrences are assigned the same variable in the contexts $\mathbf{z} : \Sigma$ and $\mathbf{z}' : \Delta$.

If $t^{\psi} \equiv u^{\phi \to \psi}(v^{\phi})$, then $\mathcal{D}(t)$ is obtained from $\mathcal{D}(u), \mathcal{D}(v)$ by $\to E$ with the same identification of assumptions as in the previous case.
If $t^{\phi \to \psi} \equiv \lambda x^{\phi}.u^{\psi}$, then:

$$\frac{\mathcal{D}(u) : \ \phi, \Gamma \Rightarrow \psi}{\mathcal{D}(\lambda x^{\phi}.u) : \ \Gamma \Rightarrow \phi \to \psi} \to I,$$

Computations with Deductions

Lemma

Up to renaming of free and bound variables,
(a) $\mathcal{D}(\mathcal{T}(d)) \equiv d$ *for every deduction* d
(b) $\mathcal{T}(\mathcal{D}(t)) \equiv t$ *for every deductive term* t

Conversions and Reductions of Deductive Terms

Conversion relations are naturally treated as computation rules that simplify the left-hand side into the right-hand side. In other words, an operational semantics for the language of terms is given by the following term *conversion* (rewriting) rules:

$$(\lambda x.t)(t') \quad conv \quad t[x/t'] \tag{5}$$

$$\mathbf{p}_i(\mathbf{p}(t_0, t_1)) \quad conv \quad t_i \quad i = 0, 1 \tag{6}$$

$$D_{x_0,x_1}(\mathbf{k}_i t, t_0, t_1) \quad conv \quad t_i[x_i/t], \ i = 0, 1 \tag{7}$$

These relations are called β-*conversions*. [Originally the term referred only to (5)]. *One-step reduction* red_1 is a conversion of a subterm:

$$\text{if } u \ conv \ u' \ \text{ then } \ t[x/u] \ red_1 \ t[x/u']. \tag{8}$$

The relation *red* is a transitive reflexive closure of red_1: $t \; red \; t'$ if there is a *reduction sequence*:

$$t \equiv t_0, \ldots, t_n \equiv t' \quad (n \geq 0)$$

such that $t_i \; red_1 \; t_{i+1}$ for every $i < n$.
A term t is *in normal form* or *t is normal* if it does not contain a redex; *t has a normal form* if there is a normal s such that $t \; red \; s$.
Reduction sequence is an analog of a computation, and a normal form is an analog of a value.

Conversions and Reductions of Natural Deductions

Let us describe transformations of natural deductions corresponding to a reduction of terms. Each of these transformations converts an occurrence of an introduction rule immediately followed by an elimination of the introduced connective. Such a pair of inferences is called a *cut* in this Section. It is possible to treat such a cut as a kind of detour to be rectified by a conversion. There is a connection with the *cut rule* of Example 2.14 but we do not elaborate on it here.
The &-conversion corresponding to the pairing conversion (6):

$$\dfrac{\dfrac{d_0 : \ \Gamma \Rightarrow \phi_0 \quad d_1 : \ \Delta \Rightarrow \phi_1}{[\Gamma, \Delta] \Rightarrow \phi_0 \& \phi_1}\ \&I}{d : \ [\Gamma, \Delta] \to \phi_i}\ \&E \qquad conv \qquad d_i : \ \Gamma \Rightarrow \phi_i$$

Note that conversion can change the set of assumptions.

A description of remaining conversions uses a *substitution operation for natural deductions*. Let us recall that in every inference (application of an inference rule), each occurrence of an undischarged assumption formula in a premise is represented by an occurrence of the same formula in the conclusion, which is called its *immediate descendant* in the conclusion. For any natural deduction, starting with an assumption formula in the antecedent of one of the sequents, we are led successively to unique *descendants* of this occurrence in the sequents below it. The chain of such descendants stops at discharged assumptions. *Ancestors* of a given formula (occurrence) are occurrences that have it as a descendant. We say that a given antecedent formula (occurrence) is *traceable* to any of its ancestors (including itself). Each occurrence has at most one descendant in a given sequent. It is important to note that all ancestors of a given (occurrence of) assumption are assigned one and the same variable in the assignment of deductive terms to deductions.

Example. Consider the following deduction of a sequent $\delta \to q \Rightarrow q$, where $\delta \equiv \ p \vee (p \to q)$.

$$\dfrac{\underline{\delta \to q} \Rightarrow \delta \to q \qquad \dfrac{\dfrac{\dfrac{\underline{\delta \to q} \Rightarrow \delta \to q \quad \dfrac{p \Rightarrow p}{p \Rightarrow \delta}}{\underline{\delta \to q}, p \Rightarrow q}}{\underline{\delta \to q} \Rightarrow p \to q}}{\underline{\delta \to q} \Rightarrow \delta}}{\underline{\delta \to q} \Rightarrow q}$$

Underlined occurrences of the assumption $\delta \to q$ are ancestors of the lowermost occurrence of this formula.

For given deductions d and d' of sequents $\alpha, \Gamma \Rightarrow \beta$, and $\Delta \Rightarrow \alpha$, the *result of substituting d' for α into d* is obtained by replacing all ancestors of α in d by Δ and writing d' over former axioms $\alpha \Rightarrow \alpha$. Taking deductive terms into consideration and writing the result of substitution at the right, yield the following:

$$
\begin{array}{lcc}
 & x : \alpha \Rightarrow x : \alpha & \mathbf{y} : \Delta \Rightarrow s : \alpha \\
 & \nwarrow\uparrow\nearrow & \nwarrow\uparrow\nearrow \\
 & x : \alpha, \mathbf{z}' : \Gamma' \Rightarrow t' : \beta' & \mathbf{z}', \mathbf{y} : [\Gamma', \Delta] \Rightarrow t'[x/s] : \beta' \\
 & \nwarrow\uparrow\nearrow & \nwarrow\uparrow\nearrow \\
d' :\ \mathbf{y} : \Delta \Rightarrow s : \alpha & d :\ x : \alpha, \mathbf{z} : \Gamma \Rightarrow t : \beta & \mathbf{z}, \mathbf{y} : [\Gamma, \Delta] \Rightarrow t[x/s] : \beta
\end{array}
$$

The arrows $\nwarrow\uparrow\nearrow$ show possible branching of the deduction at the binary and ternary rules $(\&I, \rightarrow E, \vee E)$.

Lemma

(a) All inference rules are preserved by substitution.
(b) Operations $\mathcal{T}, \mathcal{D}$ commute with substitution under suitable proviso to avoid collision of bound variables.
(b$\mathcal{T}$) If a deduction e is the result of substituting a deduction $d' : \ \Delta \Rightarrow \alpha$ for the assumption (occurrence) $x : \alpha$ into a deduction $d : \ x : \alpha, \Gamma \Rightarrow \beta$, then:

$$\mathcal{T}(e) \equiv \mathcal{T}(d)[x/\mathcal{T}(d')] \tag{9}$$

(b$\mathcal{D}$) The deduction $\mathcal{D}(t^{\beta}[x^{\alpha}/s^{\alpha}])$ is the result of substituting a deduction $\mathcal{D}(s) : \Delta \Rightarrow \alpha$ for the assumption (occurrence) $x : \alpha$ into a deduction
$\mathcal{D}(t) : \ x : \alpha, \Gamma \Rightarrow \beta$.

Proof. Check the statement for each rule of NJp and apply induction on the length of deduction. $\dashv$

The $\rightarrow$-*conversion* is now defined as follows:

$$
\dfrac{\dfrac{\begin{array}{c} u : \alpha \Rightarrow u : \alpha \\ \nwarrow \uparrow \nearrow \\ \mathbf{z}' : \Gamma', u : \alpha \Rightarrow t' : \beta' \\ \nwarrow \uparrow \nearrow \\ \mathbf{z} : \Gamma, u : \alpha \Rightarrow t : \beta \end{array}}{\mathbf{z} : \Gamma \Rightarrow \lambda u.t : \alpha \rightarrow \beta} \quad \mathbf{y} : \Delta \Rightarrow s : \alpha}{\mathbf{z}, \mathbf{y} : [\Gamma, \Delta] \Rightarrow (\lambda u.t)s : \beta} \quad conv \quad \begin{array}{c} \mathbf{y} : \Delta \Rightarrow u : \alpha \\ \nwarrow \uparrow \nearrow \\ \mathbf{z}', \mathbf{y} : [\Gamma', \Delta] \Rightarrow t'[u/s] : \beta' \\ \nwarrow \uparrow \nearrow \\ \mathbf{z}, \mathbf{y} : [\Gamma, \Delta] \Rightarrow t[u/s] : \beta \end{array}
$$

The result of conversion is obtained from the derivation of the premise of $\rightarrow I$ in the original derivation by substitution. If there is no dependence on the assumption α in the $\rightarrow$-introduction, then the result of conversion is just the given derivation of $\Gamma \Rightarrow \beta$.

Curry-Howard Isomorphism

Terminology related to reduction and normalization is transferred to natural deduction. In particular a deduction is *normal* if no reduction is applicable to it.
In the case of $\&, \rightarrow$-derivations (that contain only $\&, \rightarrow$-inferences) there is a perfect match between natural deductions and deductive terms.

Theorem (Curry–Howard isomorphism between terms and natural deductions)

(a) Every natural deduction d in NJp uniquely defines $\mathcal{T}(d)$ and vice versa: Every term t uniquely defines a natural deduction $\mathcal{D}(t)$.
(b) Cuts in d uniquely correspond to redexes in $\mathcal{T}(d)$, and vice versa.
(c) Every conversion in d uniquely corresponds to a conversion in $\mathcal{T}(d)$, and reduction sequences for d uniquely correspond to reduction sequences for $\mathcal{T}(d)$, and vice versa.
(d) The derivation d is normal iff the term $\mathcal{T}(d)$ is normal.

An Example

Let $\alpha :\equiv\ ((p \to q) \to p) \to p$. Ax$\varphi$ denotes here and below axiom $\varphi \Rightarrow \varphi$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{Ax\ p}{p \Rightarrow \alpha} \quad Ax\ \alpha \to q}{\alpha \to q, p \Rightarrow q}}{\alpha \to q \Rightarrow p \to q} \qquad \dfrac{\dfrac{\dfrac{\dfrac{Ax\ p \to q \quad Ax\ (p \to q) \to p}{p \to q, (p \to q) \to p \Rightarrow p}}{p \to q \Rightarrow \alpha} \quad Ax\ \alpha \to q}{\alpha \to q, p \to q \Rightarrow q}\ \to I}{\alpha \to q \Rightarrow (p \to q) \to q}\ \to E}{\alpha \to q \Rightarrow q}
$$

Let's commpute the term assignment for this derivation.

$$
\dfrac{\dfrac{\dfrac{\dfrac{z^p}{\lambda w^{(p\to q)\to q}.z^p} \quad x^{\alpha\to q}}{x^{\alpha\to q}(\lambda w^{(p\to q)\to q}.z^p)}}{\lambda z^p.x^{\alpha\to q}(\lambda w^{(p\to q)\to q}.z^p)} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{v^{p\to q} \quad u^{(p\to q)\to p}}{u^{(p\to q)\to p}(v^{p\to q})}}{\lambda u^{(p\to q)\to p}.u^{(p\to q)\to p}(v^{p\to q})} \quad x^{\alpha\to q}}{x^{\alpha\to q}(\lambda u^{(p\to q)\to p}.u^{(p\to q)\to p}(v^{p\to q}))}}{\lambda v^{p\to q}.x^{\alpha\to q}(\lambda u^{(p\to q)\to p}.u^{(p\to q)\to p}(v^{p\to q}))}}{(\lambda z^p.x^{\alpha\to q}(\lambda w^{(p\to q)\to q}.z^p))(\lambda v^{p\to q}.x^{\alpha\to q}(\lambda u^{(p\to q)\to p}.u^{(p\to q)\to p}(v^{p\to q}))}
$$

In computation below we drop the superscript of variables to simplify notation.
The pair of inferences $\to I$, $\to E$ explicitly shown at the end of the derivation constitutes a cut. Conversion of this cut leads to the following deduction.

$$\dfrac{\alpha \to q \qquad \dfrac{\dfrac{(p \to q) \to p \qquad \dfrac{\dfrac{\alpha \to q \qquad \dfrac{p}{\alpha}}{q}}{p \to q}}{p}}{\alpha}}{q}$$

The term

$$(\lambda z.x(\lambda w.z))(\lambda v.x(\lambda u.u(v)))$$

assigned to our derivation admits a conversion with the result

$$x(\lambda w.\lambda v.x(\lambda u.u(v))).$$

Normalization for $(\&, \to)$-derivations

Let us measure complexity of a formula by its *length*, that is, the number of occurrences of logical connectives: $lth(p) = 0$;

$$lth(\phi \& \psi) = lth(\phi \vee \psi) = lth(\phi \to \psi) := lth(\phi) + lth(\psi) + 1$$

The complexity or *cutrank* of a cut in a deduction is the length of its cut formula. In the language of deductive terms:

$$cutrank((\lambda x^{\phi}.t^{\psi})u^{\phi}) = cutrank(\mathbf{p}_i\mathbf{p}(t^{\phi}, s^{\psi})) := lth(\phi) + lth(\psi) + 1$$

Let $maxrank(t)$ be the maximal complexity of redeces in a term t (and 0 if t is normal).

Lemma

(a) If t, s *are deductive terms,* $t \neq x^{\phi}$*, and* $t[x^{\phi}/s^{\phi}]$ *is a redex, then either* t *is a redex (and* $cutrank(t) = cutrank(t[x/s])$*) or one of the following conditions is satisfied:*

$$\begin{array}{ccc} t \equiv x(t') & & t \equiv \mathbf{p}_i x \\ s \equiv \lambda y.s' & or & s \equiv \mathbf{p}(s_0, s_1) \end{array} \qquad (10)$$

and $cutrank(t[x/s]) = lth(\phi)$*.*

(b) If t^{ϕ} *conv* t' *and* $cutrank(t) > cutrank(s)$ *for every proper subterm* s *of* t*, then* $maxrank(t) = cutrank(t) > maxrank(t')$*.*

Proof. Part (a) says that really new redeces in a term can arise after a substitution only where an elimination rule was applied to a variable substituted by an introduction term. Indeed it is easy to see by inspection that every other non redex goes into a non redex. A complete proof is done by induction on the construction of t.
To prove (b), note that:

$$maxrank(t) = cutrank(t) > maxrank(s) \tag{11}$$

for every proper subterm s by the assumption, and consider possible cases. If $t \equiv \mathbf{p}_i\mathbf{p}(t_0, t_1)$ *conv* t_i, then $maxrank(t) > maxrank(t_i)$ by (11). If $t \equiv (\lambda x^\phi . t_0)(s)$ *conv* $t_0[x/s] \equiv t'$, then by Part (a) every redex in t' either has the same cutrank as some redex in t_0 [which is less than $cutrank(t)$ by the assumption] or has cutrank $length(\phi) < cutrank(t)$.

$\dashv$

Theorem (normalization theorem)

For the $(\&, \to)$*-fragment,*
(a) Every deductive term t *can be normalized.*
(b) Every natural deduction d *can be normalized.*

Proof. Part (b) follows from Part (a) by the Curry–Howard isomorphism. For Part (a) we use a main induction on $n = maxrank(t)$ with a subinduction on m, the number of redeces of cutrank n.
The induction base is obvious for both inductions.

For the induction step on m, choose in t the rightmost redex ρ of the cutrank n and convert it into its reductum ρ'. Since ρ is the rightmost, it does not have proper subterms of cutrank n. By Lemma 8(b) $maxrank(\rho) = n > maxrank(\rho')$. Write $t \equiv t'[y/\rho]$ to indicate the unique occurrence of ρ in t: The variable y has exactly one occurrence in t', term t' has exactly $m - 1$ redeces of cutrank n, and

$$t \equiv t'[y^{\phi}/\rho^{\phi}] \; conv \; t'[y/\rho']$$

Applying Lemma 8(a) to $t'[y/\rho']$, new redeces have cutranks equal to $lth(\phi) < maxrank(\rho') < n$, and old redeces preserve their cutranks. Since the redex ρ of cutrank n disappeared, the m decreased by one. $\dashv$

Equality of derivations, Isomorphisms

Recall

$$\frac{b : \alpha \Rightarrow \beta \quad g : \beta \Rightarrow \gamma}{g[b] : \alpha \Rightarrow \gamma}\ cut$$

Cut is interpreted as a substitution.

Definition

Two derivations $d, e : \Gamma \to \gamma$ *are* equal *if they are* $\beta\eta$*-convertible to one and the same derivation.*
$b : \alpha \Rightarrow \beta$ *is an* isomorphism *iff there is a* g *such that*

$$g[b] = Ax\alpha, \qquad b[g] = Ax\beta$$

up to $\beta\eta$*-conversion. In this case* α *is isomorphic to* β

Exercises

Prove that the following derivations are equal.

$$\dfrac{\dfrac{b : A \Rightarrow B \quad c : A \Rightarrow C}{A \Rightarrow B\&C}\,\&^{+} \qquad \dfrac{Ax\ \ B\&C}{B\&C \Rightarrow B}\,\&^{-}}{A \Rightarrow B}\,cut \quad = b$$

Let $\beta :=\ (B \to C)\&B$.

$$
\dfrac{\dfrac{\dfrac{A\&B \Rightarrow A \quad c : A \Rightarrow B \to C}{A\&B \Rightarrow (B \to C)}\ \mathit{cut} \qquad \dfrac{Ax\ A\&B}{A\&B \Rightarrow B}\ \&^-}{A\&B \Rightarrow \beta}\ \&^+ \qquad \dfrac{Ax\ \beta}{\beta \Rightarrow B}\ \&^-}{A\&B \Rightarrow C}\ \mathit{cut}
$$

Prove the following isomorphisms.

1. $A\&B \;\sim\; B\&A$
2. $(A\&B) \to C \;\sim\; A \to (B \to C)$

Consequences of Normalization

Recall that the *principal formula* of an elimination rule is the succedent formula explicitly shown in the rule and containing eliminated connective: $\alpha\&\beta$ in $\&E$, and so on. The *principal premise* contains the principal formula.

An occurrence of a subformula is *positive* in a formula if it is in the premise of an even number (maybe 0) of occurrences of implication. An occurrence is *strictly positive* if it is not in the premise of any implication. An occurrence is *negative* if it is not positive, that is, it is inside an odd number of premises of implication. The sign of an occurrence in a sequent $\Gamma \Rightarrow \alpha$ is the same as in the formula $\&\Gamma \rightarrow \alpha$.

The *main branch* of a deduction is a branch ending in the final sequent and containing principal premises of elimination rules with conclusions in the main branch. Hence the main branch of a deduction ending in an introduction rule contains only the final sequent.

Theorem (properties of normal deductions)

Let $d : \Gamma \Rightarrow \gamma$ *be a normal deduction in NJp.*
(a) If d *ends in an elimination rule, then the main branch contains only elimination rules, begins with an axiom, and every sequent in it is of the form* $\Gamma' \Rightarrow \alpha$, *where* $\Gamma' \subset \Gamma$ *and* α *is some formula.*
(a1) In particular the axiom at the top of the main branch is of the form $\alpha \Rightarrow \alpha$ *where* $\alpha \in \Gamma$ *and every succedent in the main branch is a strictly positive subformula of* Γ.
(b) If $\Gamma = \emptyset$, *then* d *ends in an introduction rule*
(c) Every formula in d *is a subformula of the endsequent.*

Proof. Part (a): If d ends in an elimination rule, then the main branch does not contain an introduction rule: Conclusion of such a rule would be a redex. Now Part (a) is proved by induction on the number of rules in the main branch using an *observation*: An antecedent of the principal premise of an elimination rule is contained in the antecedent of the conclusion.

Part (a1) immediately follows from (a) by induction on the length of the branch.

Part (b): Otherwise the main branch of d cannot begin with an axiom by (a).

Part (c). Induction on the derivation d. Induction base (axiom) is obvious. For induction step consider the last rule L of d. If L is an introduction rule, apply IH. since all introduction rule have the subformula property. If L is an elimination rule, the subformula property seems to be lacking. However the principal formula is a (strictly positive) subformula of Γ by (a1). Now apply IH. $\dashv$

Structure of a Normal Deduction

Theorem (subformula property)

.
Let $d : \Gamma \Rightarrow \gamma$ *be a normal* $\vee$*-free deduction.*
(a) If d *ends in an elimination rule, then the main branch begins with an axiom* $\alpha \Rightarrow \alpha$ *for* $\alpha \in \Gamma$ *and all succedents in the main branch are strictly positive subformulas of* α *(and hence of* Γ*).*
(b) All formulas in d *are subformulas of the last sequent.*

Proof. Part (a) is proved by an easy induction on the length of d. The induction base and the case when d ends in an introduction rule are trivial. If d ends in an elimination rule L, the major premise of L takes the form $\Gamma' \Rightarrow \gamma'$, with $\Gamma' \subset \Gamma$ and γ' strictly positive in α by IH. Since the succedent in the conclusion of $\& E, \rightarrow E$ is strictly positive in the major formula γ', this succedent is strictly positive in α as required.

Part(b): Induction on the deduction d. The induction base (axiom) is trivial. In the induction step, consider cases depending of the last rule L:

Case 1. The L is an introduction rule. Then all formulas in premises are subformulas of the conclusion, and the subformula property follows from IH.

Case 2. The L is an elimination rule, say:

$$\frac{\Gamma \Rightarrow \alpha \to \beta \quad \Delta \Rightarrow \alpha}{[\Gamma, \Delta] \Rightarrow \beta}$$

By part (a) $\alpha \to \beta$ is a subformula of the last sequent. By IH all subformulas in subdeductions are subformulas of $\Gamma, \alpha \to \beta, \Delta, \alpha$, and hence of the last sequent. $\dashv$

refute Pierce

η-reduction

For applications to category theory, we require a stronger reduction relation than β-reduction. The η-*conversion* for deductive terms corresponding to deductions in the language $\{\&, \to\}$ is defined as follows:

$$\mathbf{p}(\mathbf{p}_0(t), \mathbf{p}_1(t)) \ \mathit{conv} \ t,$$

$$\lambda x.(tx) \ \mathit{conv} \ t \qquad \text{provided } x \notin FV(t).$$

Corresponding conversions for deductions are as follows:

$$\dfrac{\dfrac{d : \Gamma \Rightarrow \phi_0 \& \phi_1}{\Gamma \Rightarrow \phi_0} \qquad \dfrac{d : \Gamma \Rightarrow \phi_0 \& \phi_1}{\Gamma \Rightarrow \phi_1}}{\Gamma \Rightarrow \phi_0 \& \phi_1} \qquad \mathit{conv} \quad d : \Gamma \Rightarrow \phi_0 \& \phi_1$$

$$\dfrac{\dfrac{d : \ \Gamma \Rightarrow \alpha \to \beta \quad \alpha \Rightarrow \alpha}{\Gamma, \alpha \Rightarrow \beta}}{\Gamma \Rightarrow \alpha \to \beta} \qquad \mathit{conv} \quad d : \ \Gamma \Rightarrow \alpha \to \beta$$

Hence the Curry–Howard isomorphism (Theorem 5.1) is preserved.

The $\beta\eta$-conversion is a combination of these conversions and (5),(6). The η-reduction, $\beta\eta$-reduction, and corresponding normal forms $|t|_\eta, |t|_{\beta\eta}$ are defined as for β-conversion. These normal forms are unique, but we shall not prove it here.

Lemma

(a) Every η-reduction sequence terminates.
(b) Every deductive term and every deduction has a $\beta\eta$-normal form.

Proof. Part (a): Every η-conversion reduces the size of the term. Part (b): A β-normal form $|t|_\beta$ exists by Theorem 5.2, and its η-normal form [see Part (a)] is $\beta\eta$-normal, since η-conversions preserve β-normal form. $\dashv$

Coherence Theorem

In this section we consider $NJp_{\rightarrow}$-deductions of implicative formulas and corresponding deductive terms modulo $\beta\eta$-conversion: The $d = d'$ stands for $|d|_{\beta\eta} = |d'|_{\beta\eta}$ and similarly for $t = t'$.
A sequent is *balanced* if every propositional variable occurs there at most twice and at most once with a given sign (positively or negatively.
Example. $p \rightarrow (q \rightarrow r) \Rightarrow q \rightarrow (p \rightarrow r)$ and $(p \rightarrow q) \rightarrow r \Rightarrow q \rightarrow r$ are balanced, but $p, p \rightarrow p \Rightarrow p$ is not.
We prove that a balanced sequent has unique deduction up to $\beta\eta$-equality. For non-balanced sequents that is false: The sequent $p, p \rightarrow p \Rightarrow p$ has infinitely many different normal proofs:

$$\frac{p \to p \Rightarrow p \to p \quad p \Rightarrow p}{d_1 :\ p, p \to p \Rightarrow p} \qquad \frac{p \to p \Rightarrow p \to p \quad d_n :\ p, p \to p \Rightarrow p}{d_{n+1} :\ p, p \to p \Rightarrow p}$$

The d_n can be described as a "component" of the unique proof of the balanced sequent $p_1, p_1 \to p_2, \ldots, p_n \to p_{n+1} \Rightarrow p_{n+1}$ obtained by identifying all variables with p.

Note

Formulas of *NJp*$_\rightarrow$ as objects and the normal *NJp*-deductions as morphisms form a $\rightarrow$-part of a *Cartesian closed category*. Theorem 5 below shows that a morphism $d : \alpha \Rightarrow \beta$ with a balanced $\alpha \rightarrow \beta$ is unique. In fact Theorem 5 extends to the language $\{\&, \rightarrow\}$ ([**?**], [**?**]).

Abbreviation: $(\alpha_1 \ldots \alpha_n \rightarrow \beta) := (\alpha_1 \rightarrow \ldots \rightarrow (\alpha_n \rightarrow \beta) \ldots)$.

The next Lemma shows that some of the redundant assumptions are pruned by normalization. Notation $\delta^0, \Gamma \Rightarrow \alpha$ means that δ may be present or absent.

Lemma (pruning lemma)

.

(a) Assume that Σ, α *are implicative formulas, propositional variable* q *does not occur positively in* $\Sigma \Rightarrow \alpha$*, and a deduction* $d : (\Delta \rightarrow q)^0, \Sigma \Rightarrow \alpha$ *is normal; then* $d : \Sigma \Rightarrow \alpha$*.*
(b) If $NJp_{\rightarrow} \vdash (\alpha_1, \ldots, \alpha_n \rightarrow q)$*, then one of* α_i *contains* q *positively.*

Proof. For Part (a) use induction on d. Induction base and the case when d ends in an introduction rule are obvious. Let d end in an $\rightarrow E$. Consider the main branch of d.

$$\dfrac{\dfrac{\mathcal{A} \Rightarrow (\alpha_1 \dots \alpha_n \to \alpha)}{\vdots}}{[\mathcal{A}, \Gamma_1, \dots, \Gamma_{i-1}] \Rightarrow (\alpha_i \dots \alpha_n \to \alpha) \quad (\Delta \to q)', \Gamma_i \Rightarrow \alpha_i}$$

$$\dfrac{[\mathcal{A}, (\Delta \to q)^{**}, \Gamma_1, \dots, \Gamma_i] \Rightarrow (\alpha_{i+1} \dots \alpha_n \to \alpha)}{\vdots}$$

$$[\mathcal{A}, (\Delta \to q)^0, \Gamma_1, \dots, \Gamma_n] \Rightarrow \alpha$$

Since q is not positive in α, the formula $\mathcal{A}$ is distinct from $(\Delta \to q)$. Since $\mathcal{A}$ occurs in the antecedent of the last sequent, q is not negative in $\mathcal{A}$, and hence it is not positive in α_i, since $(\alpha_i, \dots, \alpha_n \to \alpha)$ is strictly positive in $\mathcal{A}$. All other formulas in the minor premises $(\Delta \to q)', \Gamma_i \Rightarrow \alpha_i$ have the same sign in the last sequent. Hence IH is applicable to all minor premises, and $(\Delta \to q)$ is not present in the antecedent.

Part (b): Assign $q := 0,\ p := 1$ for all $p \not\equiv q$ and compute by truth tables. If all α_i are of the form $\Pi \to p$, and hence true, then $\alpha_1, \ldots, \alpha_n \to q$ is false under our assignment. Thus it is not even a tautology. Alternatively, apply (a). $\dashv$

Theorem (coherence theorem)

(a) Let $d, d' : \Rightarrow \alpha$ for a balanced implicative formula α; then $d = d'$.
(b)Let $[\Gamma, \Gamma'] \Rightarrow \alpha$ be balanced, $d : \Gamma \Rightarrow \alpha,\ d' : \Gamma' \Rightarrow \alpha$; then $d = d'$.

Proof. Part (a) follows from Part (b), which claims that Γ and Γ' are pruned during normalization into one and the same set of formulas. Since $[\Gamma, \Gamma']$ is balanced, each of Γ, Γ' is balanced. To prove Part (b), we apply induction on the length of $[\Gamma, \Gamma'] \Rightarrow \alpha$. Assume $d :\ \Gamma \Rightarrow t : \alpha$, $d :\ \Gamma' \Rightarrow t' : \alpha$ and recall that $d = d'$ iff $t = t'$.

Case 1. $\alpha \equiv (\beta \to \gamma)$; then
$[(\beta, \Gamma), \Gamma'] \Rightarrow \gamma \ \equiv \ [(\beta, \Gamma), (\beta, \Gamma')] \Rightarrow \gamma$ is balanced, and IH is applicable to sequents obtained by applying $\to E$-rule with the minor premise $\beta \Rightarrow \beta$ to d, d'. This corresponds to applying a new variable x^β to deductive terms $\mathcal{T}(d), \mathcal{T}(d')$. We have $(\mathcal{T}(d), x^\beta) = (\mathcal{T}(d'), x^\beta)$; hence
$\mathcal{T}(d) = \lambda x^\beta(\mathcal{T}(d), x^\beta) = \lambda x^\beta(\mathcal{T}(d'), x^\beta) = \mathcal{T}(d')$ and $d = d'$.
Case 2. The α is a propositional variable; then each of the $\beta\eta$-normal forms $|d|, |d'|$ is an axiom or ends in $\to E$.
Case 2.1. The $|d|$ is an axiom $\alpha \Rightarrow \alpha$; then no member of $[\Gamma, \Gamma']$ different from α contains α positively, and by the Lemma 10 (a), we have $|d'| : \alpha \Rightarrow \alpha$; that is, $d = d'$.

Case 2.2. Both $|d|$ and $|d'|$ end in $\to E$. Consider the main branch of each of these deductions. Since α is strictly positive in the axiom formula of the main branch (Theorem 4), and $[\Gamma, \Gamma'] \Rightarrow \alpha$ is balanced, this axiom formula $\mathcal{A} \equiv \alpha_1 \ldots \alpha_n \to \alpha$ is one and the same in $|d|$ and $|d'|$ and the number of $\to E$-inferences in the main branch is the same:

$$\dfrac{\begin{array}{c}\mathcal{A} \Rightarrow \alpha_1 \ldots \alpha_n \to \alpha \\ \vdots \\ [\mathcal{A}, \Gamma_1, \ldots, \Gamma_{i-1}] \Rightarrow \alpha_i \ldots \alpha_n \to \alpha \end{array} \quad d_i : \ \Gamma_i \Rightarrow \alpha_i}{\begin{array}{c}[\mathcal{A}, \Gamma_1, \ldots, \Gamma_i] \Rightarrow \alpha_{i+1} \ldots \alpha_n \to \alpha \\ \vdots \\ |d| : \ [\mathcal{A}, \Gamma_1, \ldots, \Gamma_n] \Rightarrow \alpha\end{array}}$$

$$\dfrac{\begin{array}{c}\mathcal{A} \Rightarrow \alpha_1 \ldots \alpha_n \to \alpha \\ \vdots \\ [\mathcal{A}, \Gamma'_1, \ldots, \Gamma'_{i-1}] \Rightarrow \alpha_i \ldots \alpha_n \to \alpha \end{array} \quad d'_i : \ \Gamma'_i \Rightarrow \alpha_i}{\begin{array}{c}[\mathcal{A}, \Gamma'_1, \ldots, \Gamma'_i] \Rightarrow \alpha_{i+1} \ldots \alpha_n \to \alpha \\ \vdots \\ |d'| : \ [\mathcal{A}, \Gamma'_1, \ldots, \Gamma'_n] \Rightarrow \alpha\end{array}}$$

The only positive occurrence of the propositional variable α in a balanced sequent:

$$[\mathcal{A}, \Gamma_1, \ldots, \Gamma_n] \Rightarrow \alpha$$

is the succedent, and the same is true for $[\mathcal{A}, \Gamma'_1, \ldots, \Gamma'_n] \Rightarrow \alpha$. In particular α is not negative in $\Gamma_1, \ldots, \Gamma_n$, $\Gamma'_1, \ldots, \Gamma'_n$ and in $\mathcal{A}$; hence α is not positive in $\alpha_1, \ldots, \alpha_n$. By the Lemma 10 (a) the formula $\mathcal{A}$ is not a member of $\Gamma_1, \ldots, \Gamma_n, \Gamma'_1, \ldots, \Gamma'_n$; hence each of $[\Gamma_i, \Gamma'_i] \Rightarrow \alpha_i$ is balanced. Indeed compare the following:

$$[\Gamma_i, \Gamma'_i] \Rightarrow \alpha_i \text{ and } (\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n \to \alpha), [\Gamma, \Gamma'] \Rightarrow \alpha.$$

Every occurrence in $[\Gamma_i, \Gamma'_i]$ is uniquely matched with an occurrence of the same sign in:

$$[\Gamma, \Gamma'] \equiv [(\Gamma_1, \ldots, \Gamma_i, \ldots, \Gamma_n), (\Gamma'_1, \ldots, \Gamma'_i, \ldots, \Gamma'_n)].$$

Every occurrence in α_i is uniquely matched with an occurrence of the same sign generated by an occurrence of α_i in $\mathcal{A} \equiv (\alpha_1 \ldots, \alpha_i, \ldots, \alpha_n \to \alpha)$. Applying IH to deductions of $\Gamma_i \Rightarrow A_i$ and $\Gamma'_i \Rightarrow \alpha_i$ yields $d_i = d'_i$; hence $|d| = |d'|$ as required. $\dashv$