$\doteq$ means literal ("graphical") equality of two strings of symbols.
Category $\Re = \ (Ob\Re, Mor\ \Re)$.

$$H(A,B) \subset Mor\Re \qquad f: A \to B,\ A \xrightarrow{f} B,$$

In $f: A \to B$, A is the *source*, B is the *target* ("from A to B").

$$g \cdot f:\ A \to C \text{ for } A \xrightarrow{f} B \xrightarrow{g} C,$$

$$f \cdot (g \cdot h) = \ (f \cdot g) \cdot h,$$

$$1 = 1_A : A \to A,\ 1f = f1 = f$$

The leading example. A category corresponding to a formal system $\mathcal{S}$.
Objects: formulas of $\mathcal{S}$.
Morphisms $f : A \to B$; deductions $A \vdash_{\mathcal{S}} B$.
1_A: the deduction $A \vdash A$.
The composition fg: superimposing f over G:

$$\begin{array}{c} A \\ f \\ B \\ g \\ C. \end{array}$$

The System HCC

Formulas: constructed from the propositional variables and I by $\&, \supset$.

Derivable objects: $f : A \to B$ the same as $A \xrightarrow{f} B$.

Axioms $1_A : A \to A$

$$O_A : A \to I; \qquad \ell = \ell_{AB} : A\&B \to A; \ \mathbf{r} = \mathbf{r}_{AB} : A\&B \to B;$$

$$\varepsilon = \varepsilon_{AB} : (A \supset B)\&A \to B.$$

Inference rules

$$\frac{A \xrightarrow{b} B \quad B \xrightarrow{c} C}{A \xrightarrow{cb} C}$$

$$\frac{A \xrightarrow{b} B \quad A \xrightarrow{c} C}{A \xrightarrow{\langle b,c \rangle} B\&C} \qquad \frac{A \xrightarrow{c} C}{A \xrightarrow{c^+} B \supset C}$$

Canonical maps = derivations in HCC= *combinators* constructed from $1, O, \ell, \mathbf{r}, \varepsilon$ by pairing and composition $\langle a, b \rangle$, ab.

Axioms DA for equality of maps in CCC

1. $b1_A \equiv 1_A b \equiv b$ for $b : A \to C$,
2. $d(cb) \equiv (dc)b$
3. $O_A \equiv f$ for all $f : A \to I$,
4. $\ell_{BC}\langle b, c\rangle \equiv b,\ \mathbf{r}_{bc}\langle b, c\rangle \equiv c$ for $A \xrightarrow{b} B, A \xrightarrow{c} C$,
5. $\langle \ell f, \mathbf{r} f\rangle \equiv f$ for $A \xrightarrow{f} B\&C$,
6. $\varepsilon\langle c^{+}\ell_{AB}, \mathbf{r}_{AB}\rangle \equiv c$ for $A\&B \xrightarrow{c} C$,
7. $(\varepsilon\langle f\ell_{AB}, \mathbf{r}_{AB}\rangle \equiv f$ for $A \xrightarrow{f} b \supset c$.

Categorical Equivalence Relation $\equiv_{HCC}$

A relation $\equiv$ for combinators is the least congruence relation turning HCC into a cartesian closed category.
More precisely $\equiv$ is defined by axioms DA and $a \equiv a$ plus the rules:

$$\frac{a \equiv b \quad a \equiv c}{b \equiv c} \qquad \frac{a \equiv a' \quad b \equiv b'}{\langle a, b \rangle \equiv \langle a', b' \rangle} \qquad \frac{a \equiv a' \quad b \equiv b'}{(ab) \equiv (a'b')}$$

$$\frac{a \equiv b}{a^{+} \equiv b^{+}} \qquad \frac{a \equiv b}{\ell a \equiv \ell b} \qquad \frac{a \equiv b}{\mathbf{r} a \equiv \mathbf{r} b}$$

A sequent $A \to B$ is valid (for CCC) if for any CCC $\Re$ and any substitution ζ of objects from $\Re$ for propositional variables of $A \to B$ one has a CCC map $\zeta A \xrightarrow{b} \zeta B \in Mor\Re$.
Two canonical maps are considered *equal* iff their realizations are equivalent in any CCC.
We consider recognizing validity *word problem* and recognizing equality of canonical maps *coherence problem*.

Lemma

$A \to B$ is derivable in HCC iff it is valid.
Canonical maps a, b are equal iff $a \equiv b$.

Proof. For difficult directions consider HCC. □

&-maps

&-*map* is a canonical map without any use of $\varepsilon, ^+$, only $\ell, \mathbf{r}$ pairing and composition.
Non-uniqueness:

$$\langle \mathbf{r}, \ell \rangle : p\&p \to p\&p; \qquad 1 : p\&p \to p\&p.$$

$f : A \to B$ is an *isomorphism* if there is an *inverse* $g : B \to A$ such that $fg = 1_B,\ gf = 1_A$. To achieve uniqueness: do not make (unnecessary) identifications in the source.

Theorem

Let A, B be conjunctions constructed from I and variables, no variable occurs twice in A and each variable from B is in A. Then there exists an &-map

$$\alpha_{A,B} : A \to B$$

which is unique (among &-maps) modulo $\equiv$. In particular if the same variables occur in A and B and no variable occurs twice in either of A, B then $\alpha_{A,B}$ is an isomorphism:

$$\alpha_{AB}\alpha_{BA} = 1.$$

$$\alpha_{A,B} : A \to B$$

Proof. The passage to projections:

$$a \equiv b \text{ iff } \qquad \ell a \equiv \ell b \text{ and } \mathbf{r}a \equiv \mathbf{r}b.$$

Indeed by DA5 $a \equiv \langle \ell a, \mathbf{r}a\rangle \equiv \langle \ell b, \mathbf{r}b\rangle \equiv b$. In particular

$$\langle a, b\rangle c \equiv \langle ac, bc\rangle$$

since

$$\ell(\langle a, b\rangle c) \equiv (\ell\langle a, b\rangle)c \equiv \langle a, c\rangle \tag{1}$$

and similarly for $\mathbf{r}$.
Let's construct αAB by induction on B.
Induction step. Given $\alpha_{AB} : A \to B$, $\alpha_{AC} : A \to C$
define $\alpha_{A(B\&C)} : A \to B\&C$ by $\alpha_{A(B\&C)} := \langle \alpha_{AB}, \alpha_{AC}\rangle$.
Uniqueness follow by passing to projections.

Induction base: the target B in $A \to B$ is atomic.
If $B \doteq I$ then $\alpha_{AB} := O_A$. The uniqueness is the O_A-axiom.
If B is a variable then B is contained in A, since otherwise $A \supset B$ is not even a tautology.
In this case α_{AB} is a combination of ℓ, rb corresponding to the position of B in A. For example

$$A \doteq (C\&(D\&(E\&B)), \qquad \alpha_{AB} := \mathbf{rrr},$$

$$A \doteq (((B\&C)\&D)\&E, \qquad \alpha_{AB} := \ell\ell\ell$$

$$A \doteq (((D\&C)\&B)\&E, \qquad \alpha_{AB} := \ell\mathbf{r}$$

It remains to prove uniqueness when B is a variable.

Uniqueness Proof

Lemma

(normal form for &-maps) For any &-map a there is an &-map $a' \equiv a$ a' contains no part of the form

$$\ell\langle c, d\rangle, \mathbf{r}\langle c, d\rangle, 1c, c1,$$

marginpar$\langle a, b\rangle c$? Proof. Use DA1,DA2,DA4to shorten a. □
Assume now $b \equiv \alpha_{AB}$ in this normal form. If $b \doteq \mathrm{pb}'$ for $p \in \{\ell, \mathbf{r}\}$, use IH for b'. [Why the source of b' is a conjunction of variables?] Otherwise $b \doteq 1$, also OK. Indeed the case

$$b \doteq (\langle c, d\rangle)e$$

is impossible. If $c : X \to C,\ d : X \to D, e : Y \to X$ then the target of b is $C\&D$, not a variable. □

$$\begin{array}{ccccc} & & & & B \\ & b & \nearrow & & \uparrow \ell \\ A & & \stackrel{f}{\rightarrow} & & B\&C \\ & c & \searrow & & \downarrow \mathbf{r} \\ & & & & C \end{array}$$

$$\begin{array}{ccc} A\&B & & \\ \langle f\ell_{AB}, \mathbf{r}_{AB}\rangle \downarrow & \searrow c & \\ (B \supset C)\&B & \stackrel{\varepsilon}{\rightarrow} & C \end{array}$$

Translation τ of Combinators into Deductive Terms

Recall that a term I of the type $\top$ belongs to the set of deductive terms. We identify the constant $\top$ with the constant I of HCC.
Example. $\varepsilon_{AB} : (A \supset B)\&A \to B$.

$$\dfrac{\dfrac{Ax\ (A \supset B)\&A}{A} \quad \dfrac{Ax\ (A \to B)\&A}{A \supset B}}{(A \supset B)\&A \Rightarrow B}$$

$$\dfrac{\dfrac{x : (A \supset B)\&A}{\mathbf{p}_1 x : A} \quad \dfrac{x : (A \to B)\&A}{\mathbf{p}_0 x : A \supset B}}{\mathbf{p}_0 x(\mathbf{p}_1 x) : (A \supset B)\&A \Rightarrow B}$$

for $x = x^{(A \supset B)\&A}$.

$$b : A \to B; \qquad \tau(b) : B \qquad \text{with } FV(\tau(b)) \subset \{x^A\}.$$

$$\tau(1_a) = x^A; \qquad \tau(O_A) = I_A;$$

$$\tau(\ell_{AB}) = \mathbf{p}_0 x^{A\&B}; \qquad \tau(\mathbf{r}_{AB}) = \mathbf{p}_1 x^{A\&B};$$

$$\tau(\langle b, c\rangle) = \mathbf{p}(\tau(b), \tau(c)) \text{ for } A \xrightarrow{b} B, A \xrightarrow{c} C;$$

$$\tau(\varepsilon_{AB}) = \mathbf{p}_0 x(\mathbf{p}_1 x) \text{ for } x = x^{(A \supset B)\&A};$$

$$\tau(c^+) = \lambda x^B.(\lambda x^{A\&B} \tau(c))\mathbf{p}(\tau(a), \tau(b)) \text{ for } C : A\&B \to C;$$

$$\tau(bc) = (\lambda x^C \tau(b))(\tau(c)) \text{ for } A \xrightarrow{c} C \xrightarrow{b} B$$

Note that last two clauses can be simplified:

$$\tau[c^+) \textit{ red } \lambda x^B.\tau(c))[x^{A\&B}/\mathbf{p}(\tau(a),\tau(b))],$$

$$\tau(bc) \textit{ red } \tau(b))[x^C/\tau(c)].$$

Translation **c** of Deductive Terms into Combinators

$((A\&B)\&A)\&A \to A$ with the 2-nd A matching the target A.
$A, B, A, A \Rightarrow A$ with the 2-nd A matching the target A

$$x : A, y : B, z : A, u : A \Rightarrow z : A$$

or by the map

$$\mathbf{r}\ell : (((I\&A)\&B)\&A)\&A \to A$$

The general case.

$$\Gamma = x_1^{A_1}, \ldots, x_n^{A_n}, \qquad \Delta = x_{n+1}^{A_{n+1}}, \ldots, x_m^{A_m}$$

$$K'(\Gamma) := (\ldots(p_1\&p_2)\&\ldots\&p_n);$$

$$K'(\Delta) := (\ldots(p'_{n+1}1\&p'_{n+2})\&\ldots\&p'_m)$$

where $p'_j = p_{i_j}$ iff $x_j^{A_j} = x_{i_j}^{A_{i_j}}$ for $n+1 \leq j \leq m,\ 1 \leq i_j \leq n$.

$$K(\Gamma) := (\ldots(I\&A_1)\&\ldots\&A_n).$$

$$\alpha'_{\Gamma,\Delta} := \alpha_{K'(\Gamma),K'(\Delta)} : K'(\Gamma) \to K'(\Delta); \qquad \alpha_{\Gamma,\Delta} := \alpha'_{\Gamma,\Delta}[p_i/A_i].$$

$\Gamma \sim \Gamma'$: Γ' is a permutation of Γ. Then $\alpha_{\Gamma,\Gamma'}$ is an isomorphism:

$$\alpha_{\Gamma\Gamma'}\alpha_{\Gamma',\Gamma} \equiv 1, \qquad \alpha_{\Gamma\Gamma} \equiv 1,$$

$$\alpha\Gamma'\Sigma \equiv \alpha_{\Gamma\Sigma}\alpha_{\Gamma',\Gamma} \equiv \alpha_{\Sigma\Gamma'}\alpha_{\Sigma\Gamma},$$

$$\alpha_{\Gamma x,\Delta x} \equiv \langle \alpha_{\Gamma\Delta}\ell, \mathbf{r} \rangle \text{ if } x \notin \Gamma, \Delta$$

$$\alpha_{\Gamma\Sigma,\Delta} \equiv \alpha_{\Gamma,\Delta}\ell^{\Sigma} \qquad \ell^{\Sigma}\alpha_{\Gamma,\Delta\Sigma} \equiv \alpha_{\Gamma,\Delta}$$

where $\ell^{A,B,C} = \ell^A \ell^B \ell^C$.

$$\mathbf{c}_\Gamma(t^A) : K(\Gamma) \to A \qquad \Gamma \supset FV(T^A)$$

1. $\mathbf{c}_{\Pi x^A \Sigma}(x^A) := \alpha_{x^I \Pi x^A \Sigma, x^A} \equiv \mathbf{r}\ell^\Sigma$.
2. $\mathbf{c}_\Gamma((b(a)) := \varepsilon\langle \mathbf{c}_\Gamma(b), \mathbf{c}_\Gamma(a)\rangle$.
3. $\mathbf{c}_\Gamma(\mathbf{p}(a, b)) := \langle \mathbf{c}_\Gamma(a), \mathbf{c}_\Gamma(b)\rangle$.
4. $\mathbf{c}_\Gamma(\lambda x^A b) := (\mathbf{c}_{\Gamma x^A}(b))^+$ if $x^A \notin \Gamma$. Otherwise
   $\mathbf{c}_\Gamma(\lambda x^A b) := (\mathbf{c}_{\Gamma y^A}(b_{x^A}[y^A]))^+$ for a fresh variable y^A.
5. $\mathbf{c}_\Gamma(ft) := \tilde{f}\mathbf{c}_g a(t)$ for $f \in \{\ell, \mathbf{r}\}$, with $\tilde{\ell} := \mathbf{p}_0$, $\tilde{\mathbf{r}} := \mathbf{p}_1$.
6. $\mathbf{c}_\Gamma(I) := O_{K_\Gamma}$.

Lemma

HCC proves $A \to B$ iff there is a deductive term t^B containing free no variables except x^A.

Proof. If $A \xrightarrow{b} B$ in HCC, take $\tau(b)$.
If t^B contains free at most x^A then $\mathbf{c}_{x^A}(t^B) : I\&A \to B$, hence $\mathbf{c}_{x^A}(t^B)\langle O_A, 1_A\rangle : A \to B$ as required. □

The Relation between Translations τ and $\mathbf{c}$

Recall that $\equiv$ for deductive terms means $\beta - \eta$ interconversion. We would like to have $\mathbf{c}(\tau(b)) \equiv b, \; \tau(\mathbf{c}(t)) \equiv t$.

Lemma

$\mathbf{c}_{x^A}(\tau(b)) \equiv b\mathbf{r}$ for $A \xrightarrow{b} B$.

Proof. Induction on b with some computations. Cf. G. Mints, Proof Theory and Category Theory, in: Selected papers in Proof Theory, North Holland/Bibliopolis p. 168,169. □

Lemma

$a \equiv b$ implies $\tau(a) \equiv \tau(b)$
for all suitable combinators a, b, if the free variables for construction of $\tau(a), \tau(b)$ are chosen in the same way.

Proof. Induction on demostration of $a \equiv b$. (Exercise) □

Lemma

If $\Gamma \subset \Gamma'$ *are lists of typed variables without repetitions and* $FV(t) \subset \Gamma$ *then*

$$\mathbf{c}_{\Gamma'}(t) \equiv \mathbf{c}_{\Gamma}(t)\alpha_{x^{l}\Gamma', x^{l}\Gamma}$$

Proof. Induction on t. A half-page of computations □

Lemma

$\mathbf{c}_{\Gamma}((\lambda x A t^{B})(a^{A})) \equiv \mathbf{c}_{\Gamma}(t^{B}_{x^{B}}[a])$.

Proof. Induction on t. A page of computations. □

Lemma

$t \equiv t'$ *implies* $\mathbf{c}_{\Gamma}(t) \equiv \mathbf{c}_{\Gamma}(t')$.

Proof. . Go through the definition of $t \equiv t'$. Use previous Lemmata. □

Theorem

For any combinators a, b,

$$a \equiv b \text{ iff } \tau(a) \equiv \tau(b).$$

Free CCC

Let $[f]$ means the equivalence class of f modulo $\equiv$.
Ob CCC: the set of all $\&, \supset, I$ propositional formulas.
Mor CCC: the set of all equivalence classes $[f]$ of combinators f *modulo* $\equiv$.

$$CCC = (Ob\ CCC, Mor\ CCC).$$

$f = g$ means $f \equiv g$.
This is expressed by saying that maps are combinators f considered modulo $\equiv$.

Lemma

CCC is a cartesian closed category.

Proof. All category and cc properties are explicitly postulated. □
In fact *CCC* is the free (minimal) ccc category in a suitable sense.

Exercise. Write down natural deductions for all combinators.
Exercise. If $a \equiv b$ then a, b have the same source and same target.
Exercise. Prove that if $A \to B$ is provable in HCC then $A \supset B$ is a tautology.
*Prove that HCC is in fact contained in intuitionistic propositional calculus. This is not too difficult, but takes time.
Prove that If A is a conjunction of atoms (variables and I) and B is a variable not contained in A then $A \supset B$ is not a tautology.
Exercise. Write down and prove (1) for $\mathbf{r}$.
Prove uniqueness for $\alpha_{A(B\&C)}$ from one for α_{AB}, α_{AC} by passing to projections.
Exercise. Write down the types of all formulas in the definition of $\mathbf{c}_\Gamma$.
Exercise. Do several steps in the proof of Lemma 5.

Provable Isomorphism of Types

after D.Bruce,R. Di Cosmo, Guiseppe Longo.
Propositional Formulas $\&, \supset, I$. A deductive term $M : A \to B$ is an isomorphism iff there is an $M : B \to A$ such that

$$M(N) \equiv 1, \;\; N(M) \equiv 1$$

Implicational Formulas

$$A \supset (B \supset (C \supset D)) \sim C \supset (B \supset (A \supset D))$$

Finite hereditary permutation: $A \sim A$,

$$A \supset (B \supset (C \supset D)) \sim C' \supset (B' \supset (A' \supset D'))$$

Theorem (essentialy Dezani, 1976)

An implicational term M is $\lambda\beta\eta$-invertible iff M is a f.h.p.

□

Elimination of I

$$A\&I \sim A; \qquad A \supset I \sim I$$

Both of these isomorphisms are natural, therefore I can be eliminated.

Elimination of $\&$

$$(A\&B) \supset C \ \sim \ A \supset (B \supset C)$$

$$A \supset (B\&C) \ \sim \ (A \supset B)\&(A \supset C)$$

This reduces the problem to the &-free case:

$$A\&B\&C \sim C'\&A'\&B'$$

up to permutation in the outermost &.