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Preliminaries

iff for "if and only if",

$:=$ denotes equality by definition.

The symbol $\dashv$ indicates the end of a proof.

Expressions that differ only in the names of bound variables are regarded as identical.

Substitution of expressions involves a systematic renaming operation for bound variables, thereby avoiding clashes.

$FV(e)$: the set of free variables of e.

Natural Deduction for Propositional Logic

Formulas: from *propositional variables* and a constant $\top$ (True) by $\&, \rightarrow$.
We often drop outermost parentheses as well as parentheses dividing terms in a conjunction or a disjunction.
Finite multisets are finite sequences modulo permutation.
The number of occurrences of each formula is important.

$\alpha, \Gamma = \{\alpha\} \cup \Gamma$.
Γ, Σ is the multiset union of Γ and Σ:
$\{\alpha, \alpha\}, \{\alpha, \alpha, \alpha\}$ is $\{\alpha, \alpha, \alpha, \alpha, \alpha\}$.
The $[\Gamma, \Delta]$ a multiset union of Γ, Δ plus possible identification of some formulas in Δ with identical formulas in Γ. For example:

$$[\{\alpha, \alpha, \beta, \beta\}, \{\alpha, \alpha, \alpha, \gamma, \gamma\}]$$

can be any of:

$\{\alpha, \alpha, \alpha, \alpha, \alpha, \beta, \beta, \gamma, \gamma\}, \{\alpha, \alpha, \alpha, \alpha, \beta, \beta, \gamma, \gamma\}, \{\alpha, \alpha, \alpha, \beta, \beta, \gamma, \gamma\}$,

but not $\{\alpha, \alpha, \beta, \beta, \gamma, \gamma\}$.

Intuitionistic Propositional System NJp

$$\text{Sequents:} \qquad \alpha_1, \ldots, \alpha_n \Rightarrow \alpha$$

read "*assumptions* $\alpha_1, \ldots, \alpha_n$ imply α".
α is the *succedent*, $\alpha_1, \ldots, \alpha_n$ constitute the *antecedent.*
Axioms:

$$\alpha, \Gamma \Rightarrow \alpha, \qquad \Gamma \Rightarrow \top.$$

Inference rules (I, E stand for introduction, elimination):

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{[\Gamma, \Delta] \Rightarrow \alpha \& \beta}\ \&I \qquad \frac{\Gamma \Rightarrow \alpha \& \beta}{\Gamma \Rightarrow \alpha}\ \&E \qquad \frac{\Gamma \Rightarrow \alpha \& \beta}{\Gamma \Rightarrow \beta}\ \&E$$

$$\frac{\Gamma \Rightarrow (\alpha \to \beta) \quad \Delta \Rightarrow \alpha}{[\Gamma, \Delta] \Rightarrow \beta} \to E \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow (\alpha \to \beta)} \to I$$

Sequents written above the line are *premises* of the rule, and the sequent written under the line is the *conclusion*.

Classical Propositional Calculus

: add $\alpha \vee \neg\alpha$.

A *natural deduction* or a *proof* in NJp
is a *tree* beginning with axioms and proceeding by the inference rules of the system. A sequent is *deducible* or *provable* if it is a last sequent of a deduction.

A formula α is *deducible* or *provable* if the sequent $\Rightarrow \alpha$ is provable.

$d : \Gamma \Rightarrow \alpha$: d is a natural deduction of $\Gamma \Rightarrow \alpha$.
$\Gamma \vdash \alpha$: $\Gamma \Rightarrow \alpha$ is derivable in NJp.
$\rightarrow I$ *discharges* the assumption α.
$\rightarrow E$ is called also *detachment* or *modus ponens* .

Ax α means the axiom $\alpha \Rightarrow \alpha$.

$$\dfrac{\dfrac{p \Rightarrow p}{p \Rightarrow q \to p} \to I}{\Rightarrow p \to (q \to p)} \to I$$

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{Ax\ p \to (q \to r) \quad p \Rightarrow p}{p \to (q \to r), p \Rightarrow q \to r} \to E \quad \cfrac{Ax\ p \to q \quad p \Rightarrow p}{p \to q, p \Rightarrow q} \to E}{p \to (q \to r), p \to q, p \Rightarrow r} \to E}{p \to (q \to r), p \to q \Rightarrow p \to r} \to I}{p \to (q \to r) \Rightarrow (p \to q) \to (p \to r)} \to I}{d : \Rightarrow (p \to (q \to r)) \to ((p \to q) \to (p \to r))} \to I
$$

In the $\to E$-inference above

$$
\frac{p \to (q \to r), p \Rightarrow q \to r \quad p \to q, p \Rightarrow q}{p \to (q \to r), p \to q, p \Rightarrow r} \to E
$$

the assumption p is used twice, one time in each of the premises.

Implicit application of the *contraction* rule:

$$\frac{\Gamma, \alpha, \alpha \Rightarrow \gamma}{\Gamma, \alpha \Rightarrow \gamma}$$

Another rule derivable in NJp is weakening:

$$\frac{\Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma} \; \textit{weak}$$

for example a $\rightarrow I$ rule in the form:

$$\frac{\Gamma \Rightarrow \beta}{\Gamma, \Delta \Rightarrow (\alpha \rightarrow \beta).}$$

Substitution Rule

$$\frac{\Gamma(p, q, \ldots) \Rightarrow \gamma(p, q, \ldots)}{\Gamma(\alpha, \beta, \ldots) \Rightarrow \gamma(\alpha, \beta, \ldots)}$$

$\dashv$

Traditional notation for natural deduction.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \to (q \to r) \quad p}{q \to r} \quad \dfrac{p \to q \quad p}{q}}{r}}{p \to r}}{(p \to q) \to (p \to r)}}{d^{-} : \; (p \to (q \to r)) \to ((p \to q) \to (p \to r))}$$

$$\dfrac{\dfrac{p\&q}{p}}{(p\&q) \rightarrow p}$$

$$\dfrac{\dfrac{\dfrac{p \quad q}{p\&q}}{q \rightarrow (p\&q)}}{p \rightarrow (q \rightarrow (p\&q))}$$

Exercises

1. Prove

$$\alpha \to (\beta \to \alpha)$$

$$(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$$

$$\alpha \to (\beta \to (\alpha \& \beta))$$

$$\alpha \& \beta \to \alpha$$

$$\alpha \& \beta \to \beta$$

2.Prove ("translations" of all postulates of the system HCC below)
$((A \to B) \& A) \to B$,
$A \to B, A \to C \Rightarrow A \to B \& C$,
$A \& B \to C \Rightarrow A \to (B \to C)$

Derivable Rules

A series of inference rules treated as one rule.

Definition

A deduction *of S* from $S_1, \ldots, S_n$ *is a tree beginning with axioms or sequents $S_1, \ldots, S_n$ and proceeding by inference rules.*

A rule

$$\frac{S_1, \ldots, S_n}{S}$$

is derivable *if there is a deduction of S from $S_1, \ldots, S_n$.*

The cut rule:

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \beta}\ cut$$

$$\frac{\dfrac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta} \quad \Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \beta}$$

Uniform versions of two-premise rules. A rule:

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \phi}$$

is derivable iff the rule

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \rightarrow \phi} \; R$$

is derivable too.

Some tautologies including

$$((p \to q) \to p) \to p$$

are not derivable in NJp.

Hilbert-style system HJp

$$\frac{\alpha \quad \alpha \to \beta}{\beta} \textit{ modus ponens}$$

$$\alpha \to (\beta \to \alpha)$$
$$(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$$
$$\alpha \to (\beta \to (\alpha \& \beta))$$
$$\alpha \& \beta \to \alpha$$
$$\alpha \& \beta \to \beta$$
$$\alpha \to (\alpha \vee \beta)$$
$$\beta \to (\alpha \vee \beta)$$
$$(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$$
$$\bot \to \alpha$$

BHK-interpretation

We write $c\mathbf{r}\alpha$ for "c realizes α" or "c is a construction for α":

$c\mathbf{r}(\alpha_0 \wedge \alpha_1)$ iff c is a pair $c = \mathbf{p}(a_0, a_1)$ and $a_0\mathbf{r}\alpha_0$ and $a_1\mathbf{r}a_1$,

$c\mathbf{r}(\alpha_0 \vee \alpha_1)$ iff ($c = \mathbf{k}_0 a$ and $a\mathbf{r}\alpha_0$) or ($c = \mathbf{k}_1 a$ and $a\mathbf{r}\alpha_1$),

$c\mathbf{r}(\alpha \to \beta)$ iff c is a function and for every d, if $d\mathbf{r}\alpha$ then $c(d)\mathbf{r}\beta$,

not $c\mathbf{r}\perp$.

Realization t of $(\alpha_0 \& \alpha_1) \to (\beta_0 \& \beta_1)$ is a program that for every pair $x = \mathbf{p}(a_0, a_1)$ such that a_i realizes α_i, produces a pair $t(x) = \mathbf{p}(b_0, b_1)$ such that b_j realizes β_j.

Exercise

What are realizations of formulas of the following form:

$$\alpha \to (\beta \& \gamma); \qquad (\alpha \to (\beta \to \gamma)) \qquad ((\alpha \to \beta) \to \gamma).$$

Assignment $\mathcal{T}$ of Deductive Terms

A $(\&, \rightarrow)$ language for writing realizations of formulas derivable in NJp.
Pairing $\mathbf{p}$ with *projections* $\mathbf{p}_0, \mathbf{p}_1$ satisfying:

$$\mathbf{p}_i(\mathbf{p}(t_0, t_1)) = t_i, \qquad i = 0, 1 \tag{1}$$

and lambda abstraction providing explicit definitions:

$$(\lambda x.t)(u) = t[x/u], \tag{2}$$

$t[x]$, $t[u]$.
Application of a function t to an argument u is denoted by $t(u)$.

For every formula ϕ: $x^\phi, y^\phi, z^\phi, \ldots$

A term $\mathcal{T}(d)$ for a natural deduction d

$$\Gamma \Rightarrow \phi. \tag{3}$$

We sometimes write t^{ϕ} to stress this.
Every assignment depends on a *context*

$$z^{\phi_1} : \phi_1, \ldots, z^{\phi_n} : \phi_n$$

with distinct typed variables for formulas in
$\Gamma \equiv \phi_1, \ldots, \phi_n$
written as $\mathbf{z} : \Gamma$.
(3) is transformed into a statement:

$$z^{\phi_1} : \phi_1, \ldots, z^{\phi_n} : \phi_n \Rightarrow \; u : \phi \qquad or \qquad \mathbf{z} : \Gamma \Rightarrow u : \phi$$

Contexts are treated as multisets. In particular $\mathbf{z} : \Gamma, \mathbf{z}' : \Delta$ stands for the union of multisets.

Deductive terms and the assignment of a term to a deduction is defined inductively. Assignments for axioms are given explicitly, and for every logical inference rule, there is an operation that transforms assignments for the premises into an assignment for the conclusion of the rule.

Assignment Rules

Axioms: $\mathbf{z} : \Gamma, x : \phi \Rightarrow x : \phi, \qquad \Rightarrow \ I : \top$

Inference rules:

$$\frac{\mathbf{z} : \Gamma \Rightarrow t : \phi \quad \mathbf{z}' : \Delta \Rightarrow u : \psi}{\mathbf{z} : \Gamma, \mathbf{z}' : \Delta \Rightarrow \mathbf{p}(t, u) : (\phi \& \psi)} \ \&I \qquad \frac{\mathbf{z} : \Gamma \Rightarrow t : \phi_0 \& \phi_1}{\mathbf{z} : \Gamma \Rightarrow \mathbf{p}_i t : \phi_i} \ \&E \qquad i = 0, 1$$

$$\frac{\mathbf{z} : \Gamma \Rightarrow t : (\phi \to \psi) \quad \mathbf{z}' : \Delta \Rightarrow u : \phi}{\mathbf{z} : \Gamma, \mathbf{z}' : \Delta \Rightarrow t(u) : \psi} \to E \qquad \frac{x : \phi, \mathbf{z} : \Gamma \Rightarrow t : \psi}{\mathbf{z} : \Gamma \Rightarrow \lambda x.t : (\phi \to \psi)} \to$$

Term assignment $\mathcal{T}(d)$ to a natural deduction d
$\Gamma \Rightarrow\ t : \alpha$ or $\mathbf{z} : \Gamma \Rightarrow\ t : \alpha$ means that $t = \mathcal{T}(d)$ for some natural deduction $d : \Gamma \Rightarrow \alpha$.

$$\frac{\mathbf{z} : \Gamma \Rightarrow u : \phi}{x : \psi, \mathbf{z} : \Gamma \Rightarrow u : \phi}\ \textit{weak} \qquad \frac{x : \psi, y : \psi, \mathbf{z} : \Gamma \Rightarrow u : \phi}{x : \psi, \mathbf{z} : \Gamma \Rightarrow u[y/x] : \phi}\ \textit{contr}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{q \to r \Rightarrow q \to r \qquad \dfrac{p \to q \Rightarrow p \to q \qquad p \Rightarrow p}{p, p \to q \Rightarrow q}}{p \to q, q \to r, p \Rightarrow r}}{p \to q, q \to r \Rightarrow p \to r}}{p \to q \Rightarrow (q \to r) \to (p \to r)}}{d : \Rightarrow (p \to q) \to ((q \to r) \to (p \to r))}$$

Assign terms:

$$\dfrac{\dfrac{\dfrac{\dfrac{y : q \to r \Rightarrow y : q \to r \qquad \dfrac{x : p \to q \Rightarrow x : p \to q \qquad z : p \Rightarrow z : p}{z : p, x : p \to q \Rightarrow x(z) : q}}{x : p \to q, y : q \to r, z : p \Rightarrow y(x(z)) : r}}{x : p \to q, y : q \to r \Rightarrow \lambda z^{p}.y(x(z)) : p \to r}}{x : p \to q \Rightarrow \lambda y^{q \to r} \lambda z^{p}.y(x(z)) : (q \to r) \to (p \to r)}}{\Rightarrow \lambda x^{p \to q} \lambda y^{q \to r} \lambda z^{p}.y(x(z)) : (p \to q) \to ((q \to r) \to (p \to r))}$$

Hence $\mathcal{T}(d) \equiv \lambda x \lambda y \lambda z.y(x(z))$.

Several more examples are

$$\Rightarrow \lambda x^{p}.x : (p \rightarrow p)$$

$$\Rightarrow \lambda x^{p}\lambda y^{q}.x : p \rightarrow (q \rightarrow p)$$

$$\Rightarrow \lambda x^{p_i}.\mathbf{k}_i x : (p_i \rightarrow p_0 \vee p_1)$$

Exercise

Confirm the preceding realizations and find realizations for the following formulas using deductions in NJp:

$$p\&q \to p,\ \ p\&q \to q,\ \ p \to (q \to p\&q),$$

$$(p \to q) \to ((p \to (q \to r)) \to (p \to r)).$$

Properties of Term Assignment $\mathcal{T}$

The $\mathcal{T}(d)$ is defined up to renaming of free variables assigned to axioms.
If $d : \Gamma \Rightarrow \phi$ then $\mathbf{z} : \Gamma \Rightarrow \mathcal{T}(d) : \phi$.
The operation $\mathcal{T}$ is an isomorphism: It has an inverse operation $\mathcal{D}$ preserving both syntactic identity and more important relation of $\beta\eta$-equality

Operation $\mathcal{D}$

For t^ϕ with $FV(t) = \mathbf{z}^\Gamma$ we define a deduction:

$$\mathcal{D}(t) : \ \Gamma \Rightarrow \phi \tag{4}$$

If $t^\phi \equiv x^\phi$ then $\mathcal{D}(t) := \ \phi \Rightarrow \phi \quad$ (Axiom)
If $t^{\phi_i} \equiv \mathbf{p}_i u^{\phi_0 \& \phi_1}$ then $\mathcal{D}(t)$ is obtained from $\mathcal{D}(u)$ by $\&E$:

$$\frac{\mathcal{D}(u) : \ \Gamma \Rightarrow \phi_0 \& \phi_1}{\mathcal{D}(\mathbf{p}_i u) : \ \Gamma \Rightarrow \phi_i} \ \&E$$

If $t^{\phi \& \psi} \equiv \mathbf{p}(u^\phi, v^\psi)$ with $FV(u) \equiv \mathbf{z}^\Sigma$, $FV(v) \equiv \mathbf{z}'^\Delta$, then:

$$\frac{\mathcal{D}(u) : \ \Sigma \Rightarrow \phi \quad \mathcal{D}(v) : \ \Delta \Rightarrow \psi}{\mathcal{D}(\mathbf{p}(u, v)) : \ [\Sigma, \Delta] \Rightarrow \phi \& \psi} \ \&I,$$

where occurrences of identical assumptions in Σ and Δ are identified in $[\Sigma, \Delta]$ exactly when these occurrences are assigned the same variable in the contexts $\mathbf{z} : \Sigma$ and $\mathbf{z}' : \Delta$.

If $t^{\psi} \equiv u^{\phi \to \psi}(v^{\phi})$, then $\mathcal{D}(t)$ is obtained from $\mathcal{D}(u), \mathcal{D}(v)$ by $\to E$ with the same identification of assumptions as in the previous case.
If $t^{\phi \to \psi} \equiv \lambda x^{\phi}.u^{\psi}$, then:

$$\frac{\mathcal{D}(u) : \ \phi, \Gamma \Rightarrow \psi}{\mathcal{D}(\lambda x^{\phi}.u) : \ \Gamma \Rightarrow \phi \to \psi} \to I \quad ,$$

Exercise. Write down a term assignment for $\to E$.

Lemma

Up to renaming of free and bound variables,
(a) $\mathcal{D}(\mathcal{T}(d)) \equiv d$ *for every deduction* d
(b) $\mathcal{T}(\mathcal{D}(t)) \equiv t$ *for every deductive term* t

Computations with Deductions

Conversions and Reductions of Deductive Terms

Conversion relations = computation rules simplifying the l.h.s. into r.h.s.

An operational semantics for the language of terms.

$$(\lambda x.t)(t') \quad conv \quad t[x/t'] \tag{5}$$

$$\mathbf{p}_i(\mathbf{p}(t_0, t_1)) \quad conv \quad t_i \quad i = 0, 1 \tag{6}$$

$$t^{\top} \quad conv \quad I \tag{7}$$

β-*conversions*. [Originally the term referred only to (5)].

One-step reduction red_1 is a conversion of a subterm:

$$\text{if } u \; conv \; u' \;\; \text{then} \;\; t[x/u] \; red_1 \; t[x/u']. \tag{8}$$

Here u is a *redex* and u' is a *reductum*.

The relation *red* is a transitive reflexive closure of red_1:
$t\ red\ t'$ if there is a *reduction sequence*:

$$t \equiv t_0, \ldots, t_n \equiv t' \quad (n \geq 0)$$

such that $t_i\ red_1\ t_{i+1}$ for every $i < n$.
A term t is *in normal form* or *t is normal* if it does not contain a redex; *t has a normal form* if there is a normal s such that $t\ red\ s$.
Reduction sequence is an analog of a computation, and a normal form is an analog of a value.

Conversions and Reductions of Natural Deductions

Remove f an introduction rule immediately followed by an elimination of the introduced connective
called *cut* here, a kind of detour to be rectified by a conversion.
There is a connection with the *cut rule*.
The &-conversion corresponding to the pairing conversion (6):

$$\dfrac{\dfrac{d_0 : \ \Gamma \Rightarrow \phi_0 \quad d_1 : \ \Delta \Rightarrow \phi_1}{[\Gamma, \Delta] \Rightarrow \phi_0 \& \phi_1} \ \& I}{d : \ [\Gamma, \Delta] \rightarrow \phi_i} \ \& E \qquad\qquad \mathit{conv} \ d_i : \ \Gamma \Rightarrow \phi_i$$

A conversion can change the set of assumptions.

A Substitution Operation for Natural Deductions

.

Every undischarged assumption in a premise goes into the same formula in the conclusion,
its *immediate descendant*.
The chain of such descendants stops at discharged assumptions.
Ancestors of a given formula are occurrences that have it as a descendant.
A given antecedent formula is *traceable* to any of its ancestors (including itself).
Important: ancestors of a given (occurrence of) assumption are assigned one and the same variable in the assignment of deductive terms to deductions.

Example. $d: \ \delta \to q \Rightarrow q$, where $\delta \equiv \ p \vee (p \to q)$.

$$\dfrac{\underline{\delta \to q} \Rightarrow \delta \to q \qquad \dfrac{\dfrac{\dfrac{\underline{\delta \to q} \Rightarrow \delta \to q \qquad \dfrac{p \Rightarrow p}{p \Rightarrow \delta}}{\underline{\delta \to q}, p \Rightarrow q}}{\underline{\delta \to q} \Rightarrow p \to q}}{\underline{\delta \to q} \Rightarrow \delta}}{\underline{\delta \to q} \Rightarrow q}$$

Underlined occurrences of the assumption $\delta \to q$ are ancestors of the lowermost occurrence of this formula.

$d :\ \alpha, \Gamma \Rightarrow \beta$, and $d' :\ \Delta \Rightarrow s : \alpha$.

$$\begin{array}{cc}
x : \alpha \Rightarrow x : \alpha & \mathbf{y} : \Delta \Rightarrow s : \alpha \\
\nwarrow \uparrow \nearrow & \nwarrow \uparrow \nearrow \\
x : \alpha, \mathbf{z}' : \Gamma' \Rightarrow t' : \beta' & \mathbf{z}', \mathbf{y} : [\Gamma', \Delta] \Rightarrow t'[x/s] : \beta' \\
\nwarrow \uparrow \nearrow & \nwarrow \uparrow \nearrow \\
d :\ x : \alpha, \mathbf{z} : \Gamma \Rightarrow t : \beta & \mathbf{z}, \mathbf{y} : [\Gamma, \Delta] \Rightarrow t[x/s] : \beta
\end{array}$$

The arrows $\nwarrow \uparrow \nearrow$ show possible branching of the deduction at the binary rules $(\& I, \rightarrow E)$.

Lemma

(a) All inference rules are preserved by substitution.
(b) Operations $\mathcal{T}, \mathcal{D}$ commute with substitution
(b$\mathcal{T}$) If a deduction e is the result of substituting a deduction $d' : \ \Delta \Rightarrow \alpha$ for the assumption (occurrence) $x : \alpha$ into a deduction $d : \ x : \alpha, \Gamma \Rightarrow \beta$, then:

$$\mathcal{T}(e) \equiv \mathcal{T}(d)[x/\mathcal{T}(d')] \tag{9}$$

(b$\mathcal{D}$) The deduction $\mathcal{D}(t^{\beta}[x^{\alpha}/s^{\alpha}])$ is the result of substituting a deduction $\mathcal{D}(s) : \Delta \Rightarrow \alpha$ for the assumption (occurrence) $x : \alpha$ into a deduction
$\mathcal{D}(t) : \ x : \alpha, \Gamma \Rightarrow \beta$.

Proof. Check the statement for each rule of NJp and apply induction on the length of deduction. $\dashv$

The $\to$-*conversion* is now defined as follows:

$$
\dfrac{\dfrac{\begin{array}{c} u:\alpha \Rightarrow u:\alpha \\ \nwarrow\uparrow\nearrow \\ \mathbf{z}':\Gamma', u:\alpha \Rightarrow t':\beta' \\ \nwarrow\uparrow\nearrow \\ \mathbf{z}:\Gamma, u:\alpha \Rightarrow t:\beta \end{array}}{\mathbf{z}:\Gamma \Rightarrow \lambda u.t:\alpha\to\beta} \quad \mathbf{y}:\Delta \Rightarrow s:\alpha}{\mathbf{z},\mathbf{y}:[\Gamma,\Delta] \Rightarrow (\lambda u.t)s:\beta} \quad \mathit{conv} \quad \begin{array}{c} \mathbf{y}:\Delta \Rightarrow u:\alpha \\ \nwarrow\uparrow\nearrow \\ \mathbf{z}',\mathbf{y}:[\Gamma',\Delta] \Rightarrow t'[u/s]:\beta' \\ \nwarrow\uparrow\nearrow \\ \mathbf{z},\mathbf{y}:[\Gamma,\Delta] \Rightarrow t[u/s]:\beta \end{array}
$$

The result of conversion is obtained from the derivation of the premise of $\to I$ in the original derivation by substitution. If there is no dependence on the assumption α in the $\to$-introduction, then the result of conversion is just the given derivation of $\Gamma \Rightarrow \beta$.

Curry-Howard Isomorphism

Terminology related to reduction and normalization is transferred to natural deduction. In particular a deduction is *normal* if no reduction is applicable to it.
In the case of $\&, \rightarrow$-derivations (that contain only $\&, \rightarrow$-inferences) there is a perfect match between natural deductions and deductive terms.

Theorem (Curry–Howard isomorphism between terms and natural deductions)

(a) Every natural deduction d in NJp uniquely defines $\mathcal{T}(d)$ and vice versa: Every term t uniquely defines a natural deduction $\mathcal{D}(t)$.
(b) Cuts in d uniquely correspond to redexes in $\mathcal{T}(d)$, and vice versa.
(c) Every conversion in d uniquely corresponds to a conversion in $\mathcal{T}(d)$, and reduction sequences for d uniquely correspond to reduction sequences for $\mathcal{T}(d)$, and vice versa.
(d) The derivation d is normal iff the term $\mathcal{T}(d)$ is normal.

An Example

Let $\alpha :\equiv \ ((p \to q) \to p) \to p$. Ax$\varphi$ denotes here and below axiom $\varphi \Rightarrow \varphi$.

$$
\dfrac{
\dfrac{\dfrac{\dfrac{Ax\ p}{p \Rightarrow \alpha} \quad Ax\ \alpha \to q}{\alpha \to q, p \Rightarrow q}}{\alpha \to q \Rightarrow p \to q}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{Ax\ p \to q \quad Ax\ (p \to q) \to p}{p \to q, (p \to q) \to p \Rightarrow p}}{p \to q \Rightarrow \alpha} \quad Ax\ \alpha \to q}{\alpha \to q, p \to q \Rightarrow q}}{\alpha \to q \Rightarrow (p \to q) \to q}\ \to I
}{\alpha \to q \Rightarrow q}\ \to E
$$

Let's commpute the term assignment for this derivation.

$$
\dfrac{
\dfrac{\dfrac{\dfrac{z^p}{\lambda w^{(p\to q)\to q}.z^p} \quad x^{\alpha\to q}}{x^{\alpha\to q}(\lambda w^{(p\to q)\to q}.z^p)}}{\lambda z^p.x^{\alpha\to q}(\lambda w^{(p\to q)\to q}.z^p)}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{v^{p\to q} \quad u^{(p\to q)\to p}}{u^{(p\to q)\to p}(v^{p\to q})}}{\lambda u^{(p\to q)\to p}.u^{(p\to q)\to p}(v^{p\to q})} \, x^{\alpha\to q}}{x^{\alpha\to q}(\lambda u^{(p\to q)\to p}.u^{(p\to q)\to p}(v^{p\to q}))}}{\lambda v^{p\to q}.x^{\alpha\to q}(\lambda u^{(p\to q)\to p}.u^{(p\to q)\to p}(v^{p\to q}))}}{}
}{(\lambda z^p.x^{\alpha\to q}(\lambda w^{(p\to q)\to q}.z^p))(\lambda v^{p\to q}.x^{\alpha\to q}(\lambda u^{(p\to q)\to p}.u^{(p\to q)\to p}(v^{p\to q}))}
$$

In computation below we drop the superscript of variables to simplify notation.
The pair of inferences $\to I$, $\to E$ explicitly shown at the end of the derivation constitutes a cut. Conversion of this cut leads to the following deduction.

$$\dfrac{\alpha \to q \qquad \dfrac{\dfrac{(p \to q) \to p \qquad \dfrac{\dfrac{\alpha \to q \quad \dfrac{p}{\alpha}}{q}}{p \to q}}{p}}{\alpha}}{q}$$

The term

$$(\lambda z.x(\lambda w.z))(\lambda v.x(\lambda u.u(v)))$$

assigned to our derivation admits a conversion with the result

$$x(\lambda w.\lambda v.x(\lambda u.u(v))).$$

Normalization for $(\&, \rightarrow)$-derivations

Let us measure complexity of a formula by its *length*, that is, the number of occurrences of logical connectives: $lth(p) = 0$;

$$lth(\phi \& \psi) = lth(\phi \vee \psi) = lth(\phi \rightarrow \psi) := lth(\phi) + lth(\psi) + 1$$

The complexity or *cutrank* of a cut in a deduction is the length of its cut formula. In the language of deductive terms:

$$cutrank((\lambda x^{\phi}.t^{\psi})u^{\phi}) = cutrank(\mathbf{p}_i\mathbf{p}(t^{\phi}, s^{\psi})) := lth(\phi) + lth(\psi) + 1$$

Let $maxrank(t)$ be the maximal complexity of redeces in a term t (and 0 if t is normal).

Lemma

(a) If t, s are deductive terms, $t \neq x^{\phi}$, and $t[x^{\phi}/s^{\phi}]$ is a redex, then either t is a redex (and $cutrank(t) = cutrank(t[x/s])$) or one of the following conditions is satisfied:

$$\begin{array}{lcl} t \equiv x(t') & & t \equiv \mathbf{p}_i x \\ s \equiv \lambda y.s' & \text{or} & s \equiv \mathbf{p}(s_0, s_1) \end{array} \tag{10}$$

and $cutrank(t[x/s]) = lth(\phi)$.

(b) If t^{ϕ} conv t' and $cutrank(t) > cutrank(s)$ for every proper subterm s of t, then $maxrank(t) = cutrank(t) > maxrank(t')$.

Proof. Part (a) says that really new redeces in a term can arise after a substitution only where an elimination rule was applied to a variable substituted by an introduction term. Indeed it is easy to see by inspection that every other non redex goes into a non redex. A complete proof is done by induction on the construction of t.
To prove (b), note that:

$$maxrank(t) = cutrank(t) > maxrank(s) \tag{11}$$

for every proper subterm s by the assumption, and consider possible cases. If $t \equiv \mathbf{p}_i\mathbf{p}(t_0, t_1)$ *conv* t_i, then $maxrank(t) > maxrank(t_i)$ by (11). If $t \equiv (\lambda x^{\phi}.t_0)(s)$ *conv* $t_0[x/s] \equiv t'$, then by Part (a) every redex in t' either has the same cutrank as some redex in t_0 [which is less than $cutrank(t)$ by the assumption] or has cutrank $length(\phi) < cutrank(t)$.

⊣

Theorem (normalization theorem)

For the $(\&, \to)$*-fragment,*

(a) Every deductive term t *can be normalized.*

(b) Every natural deduction d *can be normalized.*

Proof. Part (b) follows from Part (a) by the Curry–Howard isomorphism. For Part (a) we use a main induction on $n = maxrank(t)$ with a subinduction on m, the number of redeces of cutrank n.

The induction base is obvious for both inductions.

For the induction step on m, choose in t the rightmost redex ρ of the cutrank n and convert it into its reductum ρ'. Since ρ is the rightmost, it does not have proper subterms of cutrank n. By Lemma 3(b) $maxrank(\rho) = n > maxrank(\rho')$. Write $t \equiv t'[y/\rho]$ to indicate the unique occurrence of ρ in t: The variable y has exactly one occurrence in t', term t' has exactly $m - 1$ redeces of cutrank n, and

$$t \equiv t'[y^{\phi}/\rho^{\phi}] \; conv \; t'[y/\rho']$$

Applying Lemma 3(a) to $t'[y/\rho']$, new redeces have cutranks equal to $lth(\phi) < maxrank(\rho') < n$, and old redeces preserve their cutranks. Since the redex ρ of cutrank n disappeared, the m decreased by one. $\dashv$

Exercises

Prove following reductions.

$$\dfrac{\dfrac{b : A \Rightarrow B \quad c : A \Rightarrow C}{A \Rightarrow B\&C}\,\&^+ \qquad \dfrac{Ax\ \ B\&C}{B\&C \Rightarrow B}\,\&^-}{A \Rightarrow B}\,cut \qquad red\ \ b$$

Let $\beta :=\ (B \to C)\&B$,

$$c^+ = \dfrac{\dfrac{A, B \Rightarrow A\&B \quad c :\ A\&B \Rightarrow C}{A, B \to C}\,cut}{A \Rightarrow (B \to C)}$$

Note cuts on A and β in the following derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{Ax\ \ A}{A\&B \Rightarrow A} \quad c^+\ : A \Rightarrow B \to C}{A\&B \Rightarrow (B \to C)} \qquad \dfrac{Ax\ \ A\&B}{A\&B \Rightarrow B}}{A\&B \Rightarrow \beta} \qquad \dfrac{\dfrac{Ax\ \ \beta}{\beta \Rightarrow B} \qquad \dfrac{Ax\ \ \beta}{\beta \Rightarrow B \to C}}{\beta \Rightarrow C}}{A\&B \Rightarrow C}$$

and reduce it to c.

Consequences of Normalization

The *principal premise* of an elimination rule contains the principal formula.

A subformula is *positive* in a formula if it is in the premise of an even number (maybe 0) of implications.

It is *strictly positive* if it is not in the premise of any implication.

An occurrence is *negative* if it is not positive.

A sequent $\Gamma \Rightarrow \alpha$: $\qquad \&\Gamma \rightarrow \alpha$.

The *main branch* of a deduction ends in the final sequent and containing principal premises of elimination rules with conclusions in the main branch.

Theorem (properties of normal deductions)

Let $d : \Gamma \Rightarrow \gamma$ be a normal deduction in NJp.
(a) If d ends in an elimination rule, then the main branch contains only elimination rules, begins with an axiom, and every sequent in it is of the form $\Gamma' \Rightarrow \alpha$, where $\Gamma' \subset \Gamma$ and α is some formula.
(a1) In particular the axiom at the top of the main branch is of the form $\alpha \Rightarrow \alpha$ where $\alpha \in \Gamma$ and every succedent in the main branch is a strictly positive subformula of Γ.
(b) If $\Gamma = \emptyset$, then d ends in an introduction rule
(c) Every formula in d is a subformula of the endsequent.

$$\begin{array}{c} \alpha \Rightarrow \alpha \\ \vdots \\ \alpha, \Gamma' \Rightarrow g+ \\ \vdots \\ \alpha, \Gamma \Rightarrow g \end{array}$$

Proof. Part (a): If d ends in an elimination rule, then the main branch does not contain an introduction rule:
Conclusion of such a rule would be a redex.
Now Part (a) is proved by induction on the number of rules in the main branch using an *observation*: An antecedent of the principal premise of an elimination rule is contained in the antecedent of the conclusion.
Part (a1) immediately follows from (a) by induction on the length of the branch.
Part (b): Otherwise the main branch of d cannot begin with an axiom by (a).

Part (c). Induction on the derivation d. Induction base (axiom) is obvious.
For induction step consider the last rule L of d. If L is an introduction rule, apply IH since all introduction rule have the subformula property.
If L is an elimination rule, the subformula property seems to be lacking. However the principal formula is a (strictly positive) subformula of Γ by (a1). Now apply IH. $\dashv$

Pierce Formula is not Provable

Assume $(p \to q) \to p \vdash p$.
Then the main branch begins with $Ax\ (p \to q) \to p$ followed by an elimination rule, which can be only $\to E$:

$$\frac{\overset{\dots}{(p \to q) \to p \Rightarrow (p \to q)} \quad Ax\ (p \to q) \to p}{(p \to q) \to p \Rightarrow p}$$

But the minor premise $(p \to q) \to p \Rightarrow (p \to q)$ is not even a tautology: a contradiction.

η-reduction

For applications to category theory, we require a stronger reduction relation than β-reduction. The η-*conversion* for deductive terms corresponding to deductions in the language $\{\&, \to\}$ is defined as follows:

$$\mathbf{p}(\mathbf{p}_0(t), \mathbf{p}_1(t)) \ \mathit{conv} \ t,$$

$$\lambda x.(tx) \ \mathit{conv} \ t \quad \text{provided } x \notin FV(t).$$

Corresponding conversions for deductions are as follows:

$$\dfrac{\dfrac{d : \Gamma \Rightarrow \phi_0 \& \phi_1}{\Gamma \Rightarrow \phi_0} \quad \dfrac{d : \Gamma \Rightarrow \phi_0 \& \phi_1}{\Gamma \Rightarrow \phi_1}}{\Gamma \Rightarrow \phi_0 \& \phi_1} \qquad \mathit{conv} \quad d : \Gamma \Rightarrow \phi_0 \& \phi_1$$

$$\dfrac{\dfrac{d : \ \Gamma \Rightarrow \alpha \to \beta \quad \alpha \Rightarrow \alpha}{\Gamma, \alpha \Rightarrow \beta}}{\Gamma \Rightarrow \alpha \to \beta} \qquad \mathit{conv} \quad d : \ \Gamma \Rightarrow \alpha \to \beta$$

Hence the Curry–Howard isomorphism (Theorem 5.1) is preserved.

The $\beta\eta$-conversion is a combination of these conversions and β-conversions (5),(6). The η-reduction, $\beta\eta$-reduction, and corresponding normal forms $|t|_\eta, |t|_{\beta\eta}$ are defined as for β-conversion. These normal forms are unique, but we shall not prove it here.

Lemma

(a) Every η-reduction sequence terminates.
(b) Every deductive term and every deduction has a $\beta\eta$-normal form.

Proof. Part (a): Every η-conversion reduces the size of the term. Part (b): A β-normal form $|t|_\beta$ exists by Theorem 5.2, and its η-normal form [see Part (a)] is $\beta\eta$-normal, since η-conversions preserve β-normal form. $\dashv$

Equality of Derivations, Isomorphisms

Recall

$$\frac{b : \alpha \Rightarrow \beta \quad g : \beta \Rightarrow \gamma}{g[b] : \alpha \Rightarrow \gamma} \; cut$$

Cut is interpreted as a substitution.

Definition

Two derivations $d, e : \Gamma \rightarrow \gamma$ are equal *if they are $\beta\eta$-convertible to one and the same derivation.*

$b : \alpha \Rightarrow \beta$ is an isomorphism *iff there is a g such that*

$$g[b] = Ax\alpha, \qquad b[g] = Ax\beta$$

up to $\beta\eta$-conversion. In this case we say that α is isomorphic to β.

Prove the following isomorphisms.

1. $A \& B \ \sim \ B \& A$
2. $(A \& B) \rightarrow C \ \sim \ A \rightarrow (B \rightarrow C)$

Coherence Theorem

Consider $NJp_{\to}$-deductions of implicative formulas and corresponding deductive terms modulo $\beta\eta$-conversion:
The $d = d'$ stands for $|d|_{\beta\eta} = |d'|_{\beta\eta}$ and similarly for $t = t'$.
A sequent is *balanced* if every propositional variable occurs there at most twice and at most once with a given sign (positively or negatively).
Example. $p \to (q \to r) \Rightarrow q \to (p \to r)$ and
$(p \to q) \to r \Rightarrow q \to r$ are balanced,
but $p, p \to p \Rightarrow p$ is not.
We prove that a balanced sequent has unique deduction up to $\beta\eta$-equality.
For non-balanced sequents that is false: The sequent
$p, p \to p \Rightarrow p$ has infinitely many different normal proofs:

$$\frac{p \to p \Rightarrow p \to p \quad p \Rightarrow p}{d_1 :\ p, p \to p \Rightarrow p} \qquad \frac{p \to p \Rightarrow p \to p \quad d_n :\ p, p \to p \Rightarrow p}{d_{n+1} :\ p, p \to p \Rightarrow p}$$

The d_n can be described as a “component” of the unique proof of the balanced sequent $p_1, p_1 \to p_2, \ldots, p_n \to p_{n+1} \Rightarrow p_{n+1}$ obtained by identifying all variables with p.

Note

Formulas of $NJp_{\to}$ as objects and the normal NJp-deductions as morphisms form a $\to$-part of a *Cartesian closed category*. Theorem 4 below shows that a morphism $d : \alpha \Rightarrow \beta$ with a balanced $\alpha \to \beta$ is unique. In fact Theorem 4 extends to the language $\{\&, \to\}$ [A. Babaev,S. Solovjev].

Abbreviation: $(\alpha_1 \ldots \alpha_n \to \beta) := (\alpha_1 \to \ldots \to (\alpha_n \to \beta) \ldots)$.

The next Lemma shows that some of the redundant assumptions are pruned by normalization.

Notation $\delta^0, \Gamma \Rightarrow \alpha$ means that δ may be present or absent.

Lemma (pruning lemma)

(a) Assume that Σ, α *are implicative formulas, propositional variable* q *does not occur positively in* $\Sigma \Rightarrow \alpha$*, and a deduction* $d : (\Delta \to q)^0, \Sigma \Rightarrow \alpha$ *is normal; then* $d : \Sigma \Rightarrow \alpha$*.*
(b) If $NJp_{\to} \vdash (\alpha_1, \ldots, \alpha_n \to q)$*, then one of* α_i *contains* q *positively.*

Proof. For Part (a) use induction on d. Induction base and the case when d ends in an introduction rule are obvious. Let d end in an $\to E$. Consider the main branch of d.

$$\dfrac{\begin{array}{c}\mathcal{A} \Rightarrow (\alpha_1 \ldots \alpha_n \to \alpha) \\ \vdots \\ [\mathcal{A}, \Gamma_1, \ldots, \Gamma_{i-1}] \Rightarrow (\alpha_i \ldots \alpha_n \to \alpha) \quad (\Delta \to q)', \Gamma_i \Rightarrow \alpha_i\end{array}}{\begin{array}{c}[\mathcal{A}, (\Delta \to q)^{**}, \Gamma_1, \ldots, \Gamma_i] \Rightarrow (\alpha_{i+1} \ldots \alpha_n \to \alpha) \\ \vdots \\ [\mathcal{A}, (\Delta \to q)^0, \Gamma_1, \ldots, \Gamma_n] \Rightarrow \alpha\end{array}}$$

Since q is not positive in α, the formula $\mathcal{A}$ is distinct from $(\Delta \to q)$. Since $\mathcal{A}$ occurs in the antecedent of the endsequent, q is not negative in $\mathcal{A}$, and hence it is not positive in α_i, since $(\alpha_i, \ldots, \alpha_n \to \alpha)$ is strictly positive in $\mathcal{A}$. All other formulas in the minor premises $(\Delta \to q)', \Gamma_i \Rightarrow \alpha_i$ have the same sign in the last sequent. Hence IH is applicable to all minor premises, and $(\Delta \to q)$ is not present in the antecedent.

(b) If $NJp_{\to} \vdash (\alpha_1, \ldots, \alpha_n \to q)$, then one of α_i contains q positively.

Part (b): Assign $q := 0$, $p := 1$ for all $p \not\equiv q$ and compute by truth tables. If all α_i are of the form $\Pi \to p$, and hence true, then $\alpha_1, \ldots, \alpha_n \to q$ is false under our assignment. Thus it is not even a tautology. Alternatively, apply (a). $\dashv$

Theorem (coherence theorem)

(a) Let $d, d' : \Rightarrow \alpha$ *for a balanced implicative formula* α*; then* $d = d'$.
(b) Let $[\Gamma, \Gamma'] \Rightarrow \alpha$ *be balanced,* $d : \Gamma \Rightarrow \alpha,\ d' : \Gamma' \Rightarrow \alpha$*; then* $d = d'$.

Proof. Part (a) follows from Part (b), which claims that Γ and Γ' are pruned during normalization into one and the same set of formulas. Since $[\Gamma, \Gamma']$ is balanced, each of Γ, Γ' is balanced. To prove Part (b), we apply induction on the length of $[\Gamma, \Gamma'] \Rightarrow \alpha$. Assume $d : \ \Gamma \Rightarrow t : \alpha$, $d : \ \Gamma' \Rightarrow t' : \alpha$ and recall that $d = d'$ iff $t = t'$.

Case 1. $\alpha \equiv (\beta \to \gamma)$; then $[(\beta, \Gamma), \Gamma'] \Rightarrow \gamma \ \equiv \ [(\beta, \Gamma), (\beta, \Gamma')] \Rightarrow \gamma$ is balanced, and IH is applicable to sequents obtained by applying $\to E$-rule with the minor premise $\beta \Rightarrow \beta$ to d, d'. This corresponds to applying a new variable x^β to deductive terms $\mathcal{T}(d), \mathcal{T}(d')$. We have $(\mathcal{T}(d), x^\beta) = (\mathcal{T}(d'), x^\beta)$; hence $\mathcal{T}(d) = \lambda x^\beta(\mathcal{T}(d), x^\beta) = \lambda x^\beta(\mathcal{T}(d'), x^\beta) = \mathcal{T}(d')$ and $d = d'$.

Case 2. The α is a propositional variable; then each of the $\beta\eta$-normal forms $|d|, |d'|$ is an axiom or ends in $\to E$.

Case 2.1. The $|d|$ is an axiom $\alpha \Rightarrow \alpha$; then no member of $[\Gamma, \Gamma']$ different from α contains α positively, and by the Lemma 5 (a), we have $|d'| : \alpha \Rightarrow \alpha$; that is, $d = d'$.

Case 2.2. Both $|d|$ and $|d'|$ end in $\to E$. Consider the main branch of each of these deductions. Since α is strictly positive in the axiom formula of the main branch, and $[\Gamma, \Gamma'] \Rightarrow \alpha$ is balanced, this axiom formula $\mathcal{A} \equiv \alpha_1 \ldots \alpha_n \to \alpha$ is one and the same in $|d|$ and $|d'|$ and the number of $\to E$-inferences in the main branch is the same:

$$\dfrac{\begin{array}{c}\mathcal{A} \Rightarrow \alpha_1 \ldots \alpha_n \to \alpha \\ \vdots \\ [\mathcal{A}, \Gamma_1, \ldots, \Gamma_{i-1}] \Rightarrow \alpha_i \ldots \alpha_n \to \alpha \quad d_i : \Gamma_i \Rightarrow \alpha_i\end{array}}{\begin{array}{c}[\mathcal{A}, \Gamma_1, \ldots, \Gamma_i] \Rightarrow \alpha_{i+1} \ldots \alpha_n \to \alpha \\ \vdots \\ |d| : [\mathcal{A}, \Gamma_1, \ldots, \Gamma_n] \Rightarrow \alpha\end{array}}$$

$$\dfrac{\begin{array}{c}\mathcal{A} \Rightarrow \alpha_1 \ldots \alpha_n \to \alpha \\ \vdots \\ [\mathcal{A}, \Gamma'_1, \ldots, \Gamma'_{i-1}] \Rightarrow \alpha_i \ldots \alpha_n \to \alpha \quad d'_i : \Gamma'_i \Rightarrow \alpha_i\end{array}}{\begin{array}{c}[\mathcal{A}, \Gamma'_1, \ldots, \Gamma'_i] \Rightarrow \alpha_{i+1} \ldots \alpha_n \to \alpha \\ \vdots \\ |d'| : [\mathcal{A}, \Gamma'_1, \ldots, \Gamma'_n] \Rightarrow \alpha\end{array}}$$

The only positive occurrence of variable α in

$$[\mathcal{A}, \Gamma_1, \dots, \Gamma_n] \Rightarrow \alpha$$

is the succedent, and the same is true for $[\mathcal{A}, \Gamma'_1, \dots, \Gamma'_n] \Rightarrow \alpha$. In particular α is not negative in $\Gamma_1, \dots, \Gamma_n$, $\Gamma'_1, \dots, \Gamma'_n$ and in $\mathcal{A}$; hence α is not positive in $\alpha_1, \dots, \alpha_n$. By the Lemma 5 (a) the formula $\mathcal{A}$ is not a member of $\Gamma_1, \dots, \Gamma_n, \Gamma'_1, \dots, \Gamma'_n$; hence each of $[\Gamma_i, \Gamma'_i] \Rightarrow \alpha_i$ is balanced. Indeed compare the following:

$$[\Gamma_i, \Gamma'_i] \Rightarrow \alpha_i \quad \text{and} \quad (\alpha_1, \dots, \alpha_i, \dots, \alpha_n \to \alpha), [\Gamma, \Gamma'] \Rightarrow \alpha.$$

Every occurrence in $[\Gamma_i, \Gamma'_i]$ is uniquely matched with an occurrence of the same sign in:

$$[\Gamma, \Gamma'] \equiv [(\Gamma_1, \dots, \Gamma_i, \dots, \Gamma_n), (\Gamma'_1, \dots, \Gamma'_i, \dots, \Gamma'_n)].$$

Every occurrence in α_i is uniquely matched with an occurrence of the same sign generated by an occurrence of α_i in $\mathcal{A} \equiv (\alpha_1 \dots, \alpha_i, \dots, \alpha_n \to \alpha)$. Applying IH to deductions of $\Gamma_i \Rightarrow A_i$ and $\Gamma'_i \Rightarrow \alpha_i$ yields $d_i = d'_i$; hence $|d| = |d'|$ as required.
$\dashv$

Elimination of I

We use the following isomorphisms:

$$\alpha \& I \ \sim \alpha$$

$$\alpha \rightarrow I \ \sim I$$

In more detail:

$$\frac{Ax\ \alpha\& I}{\mathbf{p}_0 x^{\alpha\& I} :\ \alpha\& I \Rightarrow \alpha} \qquad \frac{x^{\alpha} :\ \alpha \Rightarrow \alpha \quad I :\ \alpha \rightarrow I}{\mathbf{p}(x^{\alpha}, I) :\ \alpha \Rightarrow \alpha\& I}$$

$$I :\ \alpha \rightarrow I \Rightarrow I \qquad \frac{I :\ I, \alpha \rightarrow I}{\lambda x^{\alpha}.I :\ I \Rightarrow \alpha \rightarrow I}$$

Let's check isomorphism relations.

$$\frac{I : \ (\alpha \to I) \Rightarrow I \quad \dfrac{I, \alpha \Rightarrow I}{I \Rightarrow \alpha \to I}}{\alpha \to I \Rightarrow \alpha \to I} \quad = \quad \frac{\alpha \Rightarrow I}{\lambda x^{\alpha}.I : \Rightarrow \alpha \to I}$$

$$1_{\alpha \to I} = \frac{I : \ \alpha \to I, \alpha \Rightarrow I}{\lambda x^{\alpha}.I : \ \alpha \Rightarrow I \Rightarrow \alpha \to I}$$

$$\frac{\alpha \to I \Rightarrow I \quad I \Rightarrow \alpha \to I}{(\alpha \to I) \Rightarrow (\alpha \to I)} \quad = 1_{\alpha \to I}$$

The opposite direction $I \Rightarrow (\alpha \to I) \Rightarrow I$ is obvious since all maps targeting I are equal to I.

Generally if there are isomorphisms

$$g : A \to A^*, \qquad h : B \to B^*$$

we can define for every $f : A \to B$

$$f^* := hfg^{-1} : \; A^* \xrightarrow{g^{-1}} A \xrightarrow{f} B \xrightarrow{h} B^*.$$

Then for $f, k : A \to B$

$$f = k \leftrightarrow f^* = k^*$$

Indeed

$$f^* = k^* \leftrightarrow hfg^{-1} = ghkg^{-1} \leftrightarrow hfg^{-1}g = hkg^{-1}g$$

$$\leftrightarrow hf = hk \leftrightarrow f = k.$$