Construction of initial algebras and final coalgebras

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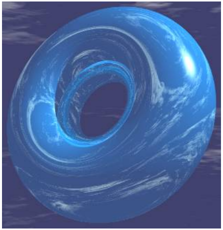

set with algebraic operations	set with transitions and observations
algebra for a functor	coalgebra for a functor
initial algebra	final coalgebra
useful in syntax	useful in semantics

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In some ways, the mathematics of transitions and observations is less familiar than that of sets and operations.

Coalgebra is trying to be the general mathematics of transitions and observations.

Final coalgebras are like the most abstract collections of "transitions" or "observations".
I know that this is very vague,
and so perhaps the examples throughout this talk will help.

The main questions that this talk seeks to address are:
Given F, does the initial algebra exist?
Does the final coalgebra?
How can we get our hands on them?

Kleene's Theorem

Let $(A, \leq)$ be poset with a least element 0 and with the property that every countable chain $C \subseteq A$ has a least upper bound $\bigvee C$.

Let $F : A \to A$ be monotone and ω-continuous.

Then there is a least μF such that $F(\mu F) \leq \mu F$,
and any such element is a least fixed point.

Given F, how can we get our hands on an initial algebra for F?

Answer: generalize Kleene's Theorem

Kleene's Theorem

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Then there is a least μF such that $F(\mu F) \leq \mu F$,
and any such element is a least fixed point.

Note that

$$0 \leq F(0) \leq F^2(0) \leq \cdots \leq F^n(0) \leq \cdots$$

so we have a chain. Let $\mu F = \bigvee_{n=0}^{\infty} F^n(0)$.
By continuity,

$$F(\mu F) = F(\bigvee_{n=0}^{\infty} F^n(0)) = \bigvee_{n=1}^{\infty} F^n(0) \leq \mu F.$$

If $Fx \leq x$, then we show by induction on n that $F^n(0) \leq x$; hence $\mu F \leq x$ as well.

Given F, how can we get our hands on an initial algebra for F?

Answer: generalize Kleene's Theorem

Kleene's Theorem

Let $(A, \leq)$ be poset with a least element 0 and with the property that every countable chain $C \subseteq A$ has a least upper bound $\bigvee C$.

Let $F : A \to A$ be monotone and ω-continuous.

Then there is a least μF such that $F(\mu F) \leq \mu F$,
and any such element is a least fixed point.

We still need to see that μF is also a fixed point.

As we know, $F(\mu F) \leq \mu F$. So also $FF(\mu F) \leq F(\mu F)$.

Thus by what we have seen, $\mu F \leq F(\mu F)$.

order-theoretic concept	category-theoretic generalization
preorder $(A, \leq)$	category $\mathcal{A}$
$x \leq y$ and $y \leq x$	A and B are isomorphic objects
least element 0	initial object 0
monotone $F : A \to A$	functor $F : \mathcal{A} \to \mathcal{A}$
pre-fixed point: $Fx \leq x$	F-algebra: $f : FA \to A$
countable chain	functor from $(\omega, \leq)$ to $\mathcal{A}$
F is ω-continuous	F preserves ω-colimits
least pre-fixed point: $Fx \leq x$	initial F-algebra: $f : FA \to A$

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Category: a structure with
objects, morphisms, composition, and identity morphisms;
and some minimal requirements.
Example: sets and functions.
Example: a preorder $(A, \leq)$ with a morphism from x to y iff $x \leq y$

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$A \cong B$: there are $f : A \to B$ and $g : B \to A$
such that $f \cdot g = id_B$ and $g \cdot f = id_A$.

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An initial object is an object 0 such that for every object A, there is a unique morphism $!_A : 0 \to A$.

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A functor $F : \mathcal{A} \to \mathcal{A}$ takes objects to objects
and morphisms to morphisms,
preserving identity morphisms and composition.

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An F-algebra is a pair (A, a), where A is an object and $a : FA \to A$.

The category-theoretic generalization

"Preorders are the poor person's category"

order-theoretic concept	category-theoretic generalization
preorder $(A, \leq)$	category $\mathcal{A}$
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I don't want to define colimits in general.
The initial sequence of F is

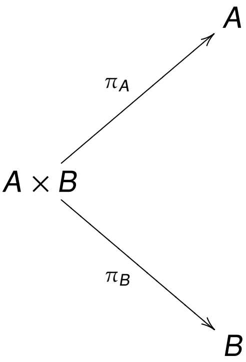
$$0 \xrightarrow{\;!\;} F0 \xrightarrow{\;F!\;} F^2 0 \qquad \cdots \qquad F^n 0 \xrightarrow{\;F^n!\;} F^{n+1} 0 \qquad \cdots$$

This is a diagram.

$$A$$

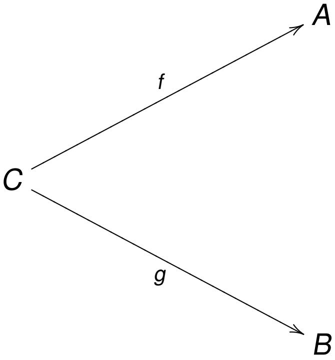
$$B$$

A limit is an object $A \times B$ together with morphisms π_A and π_B.



A limit is an object $A \times B$ together with morphisms π_A and π_B subject to the following requirement:

Given any C, f, and g



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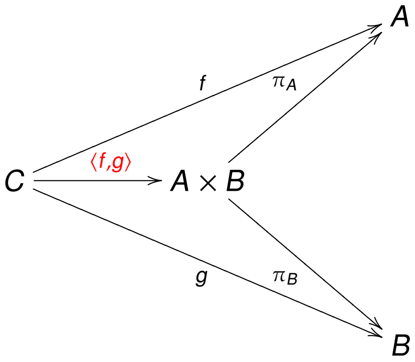


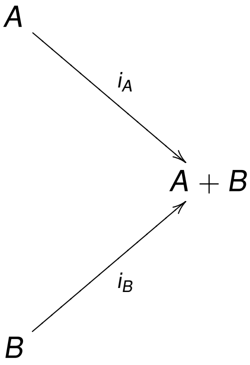
there is a unique $\langle f, g \rangle$ making the triangles commute.

This is the same diagram again.

A

B

A colimit is an object $A + B$ together with morphisms i_A and i_B



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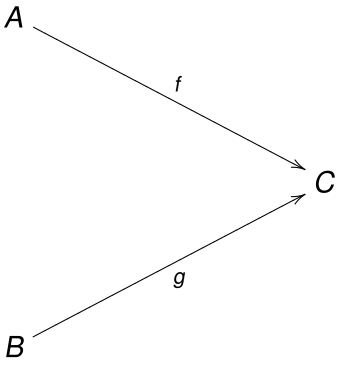

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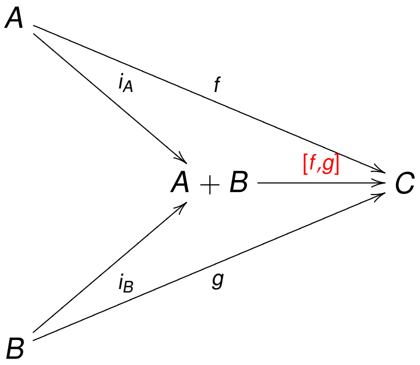


there is a unique $[f, g]$ making the triangles commute.

In the category of *Sets*, if we start with the diagram
we always can find the limit:
It's the product with the usual projections.

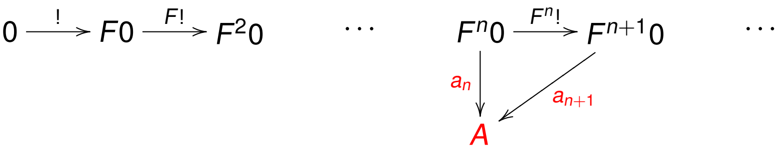
And we can always find the colimit:
the disjoint union
with the usual injections.

Actually, if we start with any diagram whatsoever,
we can again find a limit and a colimit
(when we generalize the definitions appropriately).

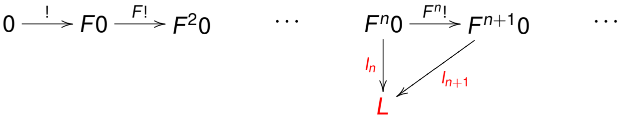
The initial sequence of F is

$$0 \xrightarrow{!} F0 \xrightarrow{F!} F^2 0 \qquad \cdots \qquad F^n 0 \xrightarrow{F^n !} F^{n+1} 0 \qquad \cdots$$

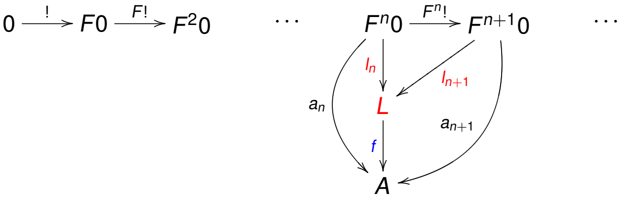
A cocone over the initial sequence is an object A of $\mathcal{A}$ and a family of morphisms $a_n : F^n 0 \to A$ such that $a_n = a_{n+1} \cdot F^n!$ for all n:



A colimit of the initial sequence is
a cocone over it $(L, l_n : F^n 1 \to L)$



with the universal property that if we have any cocone



$$(A, a_n : F^n 1 \to A)$$

then there is a unique factorizing morphism
$f : L \to A$ such that for all n, $a_n = f \cdot l_n$.

In place of ω-continuity, we have the condition that
F preserves colimits.
In our setting, this means that if we take a colimit

$$
\begin{array}{ccccccccc}
0 & \xrightarrow{!} & F0 & \xrightarrow{F!} & F^2 0 & \cdots & F^n 0 & \xrightarrow{F^n !} & F^{n+1} 0 \quad \cdots \\
 & & & & & & {\scriptstyle l_n}\downarrow & \swarrow{\scriptstyle l_{n+1}} & \\
 & & & & & & L & &
\end{array}
$$

and apply F throughout, we get another colimit

$$
\begin{array}{ccccccccc}
F0 & \xrightarrow{F!} & F^2 0 & \xrightarrow{F!} & F^3 0 & \cdots & F^{n+1} 0 & \xrightarrow{F^{n+1} !} & F^{n+2} 0 \quad \cdots \\
 & & & & & & {\scriptstyle Fl_n}\downarrow & \swarrow{\scriptstyle Fl_{n+1}} & \\
 & & & & & & FL & &
\end{array}
$$

Not every F preserves colimits.

But let's assume that we are working with one that does
preserve
the colimit of the initial sequence.

We just saw the colimit cocone

$$\begin{array}{ccccccccc}
F0 \xrightarrow{F!} F^2 0 \xrightarrow{F!} F^3 0 & \cdots & F^{n+1}0 & \xrightarrow{F^{n+1}!} & F^{n+2}0 & \cdots \\
& & {\scriptstyle Fl_n}\downarrow & \swarrow {\scriptstyle Fl_{n+1}} & & \\
& & FL & & &
\end{array}$$

and we of course have a similar cocone to L

$$\begin{array}{ccccccccc}
\text{forget } 0 & F0 \xrightarrow{F!} F^2 0 & \cdots & F^{n}0 & \xrightarrow{F^{n}!} & F^{n+1}0 & \cdots \\
& & & {\scriptstyle l_n}\downarrow & \swarrow {\scriptstyle l_{n+1}} & & \\
& & & L & & &
\end{array}$$

So we get a unique $m : FL \to L$ so that for all n, $l_n = m \cdot Fl_n$
Now we have an F-algebra $(L, m : FL \to L)$.

KLEENE'S THEOREM

Let $(A, \leq)$ be poset with a least element 0 and with the property that every countable chain $C \subseteq A$ has a least upper bound $\bigvee C$.

Let $F : A \to A$ be monotone and ω-continuous.

Let $\mu F = \bigvee F^n(0)$.
Then μF is the least fixed point of F.

ADÁMEK 1974

Let $\mathcal{A}$ be a category with initial object 0
and with the property that every ω-chain in $\mathcal{A}$ has a colimit.

Let $F : \mathcal{A} \to \mathcal{A}$ preserve ω-colimits,

let μF be the colimit of the initial sequence of F, and let
$m : F(\mu F) \to F$ be the factorizing morphism.

Then $(\mu F, m)$ is an initial F-algebra.

Let (A, a) be any F-algebra, so $a : FA \to A$.
We get a cocone as follows:

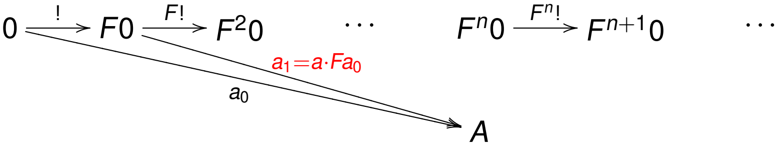
$$0 \xrightarrow{!} F0 \xrightarrow{F!} F^2 0 \quad \cdots \quad F^n 0 \xrightarrow{F^n !} F^{n+1} 0 \quad \cdots$$

$$A$$

Let (A, a) be any F-algebra, so $a : FA \to A$.

Initiality of (L, m)

Let (A, a) be any F-algebra, so $a : FA \to A$.

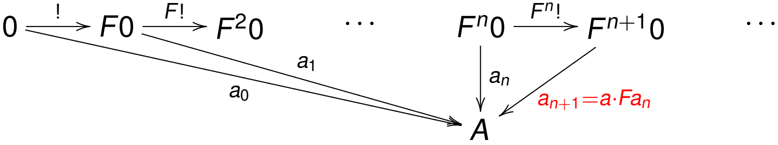


$a_0 : 0 \to A$,

$F0 \xrightarrow{Fa_0} FA \xrightarrow{a} A$

INITIALITY OF (L, m)

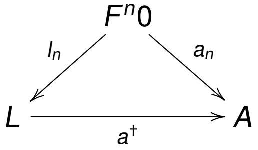
Let (A, a) be any F-algebra, so $a : FA \to A$.



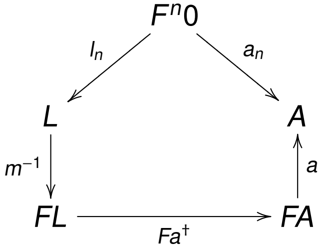
$a_n : F^n 0 \to A$,

$F(F^n 0) \xrightarrow{Fa_n} FA \xrightarrow{a} A$

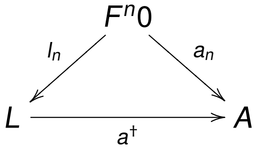
Let (A, a) be any F-algebra, so $a : FA \to A$.
With this cocone, we get $a^\dagger : L \to A$ such that for all n,



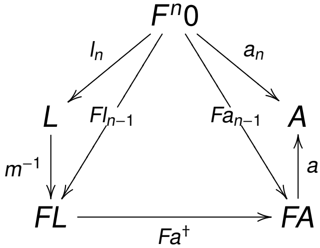
We'll show that this property of $a^\dagger$ is also shared by $a \cdot Fa^\dagger \cdot m^{-1}$.
That is, we'll show that for all n, the diagram below commutes:



Let (A, a) be any F-algebra, so $a : FA \to A$.
We have $a^\dagger : L \to A$ such that for all n,



We'll show by induction on n that the diagram below commutes:



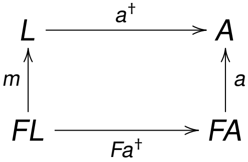
For $n = 0$ it's trivial. For $n > 0$ we use the preceding definitions.

Let (A, a) be any F-algebra, so $a : FA \to A$.
We have $a^\dagger : L \to A$ so that

$$a^\dagger \quad = \quad a \cdot Fa^\dagger \cdot m^{-1}$$

That is,

$$a^\dagger \cdot m \quad = \quad a \cdot Fa^\dagger$$



So $a^\dagger$ is an algebra morphism.
The uniqueness of $a^\dagger$ is a similar argument.

We have already seen examples with three functors
$F : \text{Set} \to \text{Set}$:

	initial algebra
$FX = 1 + (X \times X)$	finite binary trees
$FX = \mathcal{P}_{fin}X$	hereditarily finite sets
$FX = 1 + X$	natural numbers
$FX = 1 + \text{Bag}(X)$	unordered finitely-branching trees
$FX = \mathcal{P}_{ctbl}(X)$	countably-branching well-founded trees

Adámek's Theorem is not the only way to get an initial algebra, but it is the most common way.

Adámek 1974

Let $\mathcal{A}$ be a category with initial object 0
and with the property that every ω-chain in $\mathcal{A}$ has a colimit.

Let $F : \mathcal{A} \to \mathcal{A}$ preserve ω-colimits,

let μF be the colimit of the initial sequence of F, and let
$m : F(\mu F) \to F$ be the factorizing morphism.

Then $(\mu F, m)$ is an initial F-algebra.

Barr 1993

Let $\mathcal{A}$ be a category with final object 1
and with the property that every ω^{op}-chain in $\mathcal{A}$ has a limit.

Let $F : \mathcal{A} \to \mathcal{A}$ preserve ω^{op}-limits,

let νF be the limit of the final sequence of F, and let
$m : F \to F(\nu F)$ be the factorizing morphism.

Then $(\nu F, m)$ is a final F-coalgebra.

The initial sequence of an endofunctor F is

$$0 \xrightarrow{!} F0 \xrightarrow{F!} F^2 0 \qquad \cdots \qquad F^n 0 \xrightarrow{F^n !} F^{n+1} 0 \qquad \cdots$$

The final sequence goes the other way

$$1 \xleftarrow{!} F1 \xleftarrow{F!} F^2 1 \qquad \cdots \qquad F^n 1 \xleftarrow{F^n !} F^{n+1} 1 \qquad \cdots$$

and it also starts with the final object.

Example: streams over $2 = \{0, 1\}$

Here our functor is $FX = 2 \times X$.

1 is any one point set, say $\{*\}$.

So $F1 = 2 \times 1 = \{(0, *), (1, *)\}$.

$F^2 1 = 2 \times F1 = \{(0, (0, *)), (0, (1, *)), (1, (0, *)), (1, (1, *))\}$.

$$
\begin{array}{ccccccccc}
1 & \xleftarrow{\;!\;} & F1 & \xleftarrow{\;F!\;} & F^2 1 & \cdots & F^n 1 & \xleftarrow{\;F^n !\;} & F^{n+1} 1 \quad \cdots \\
 & & & & & & \uparrow {\scriptstyle l_n} & & \\
 & & & & & & L & &
\end{array}
$$

There are several natural descriptions of the limit.

Example: streams over $2 = \{0, 1\}$

Here our functor is $FX = 2 \times X$.

1 is any one point set, say $\{*\}$.

So $F1 = \{0, 1\}$.

$F^2 1 = \{0, 1]^2\}$. The map $F!$ drops the last element.

$$
\begin{array}{ccccccccc}
1 & \xleftarrow{\;!\;} & F1 & \xleftarrow{\;F!\;} & F^2 1 & \cdots & F^n 1 & \xleftarrow{\;F^n!\;} & F^{n+1} 1 & \cdots \\
 & & & & & & \uparrow {\color{red} l_n} & & & \\
 & & & & & & L & & &
\end{array}
$$

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\begin{array}{ccccccccc}
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 & & & & & & \uparrow {\scriptstyle l_n} & & \\
 & & & & & & L & &
\end{array}
$$

From a general construction, it is the set L of functions f such that for all n, $f(n) \in F^n 1$, and

$$F^n!(f(n+1)) = f(n).$$

Then the cone maps $l_n : L \to F^n 1$ are given by $l_n(f) = f(n)$.

Example: streams over $2 = \{0, 1\}$

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$$
\begin{array}{ccccccccc}
1 & \xleftarrow{\;!\;} & F1 & \xleftarrow{\;F!\;} & F^2 1 & \cdots & F^n 1 & \xleftarrow{\;F^n !\;} & F^{n+1} 1 & \cdots \\
 & & & & & & \uparrow{\scriptstyle l_n} & & & \\
 & & & & & & L & & &
\end{array}
$$

This amounts to taking the infinite sequences of 0s and 1s,
with $l_n : L \to \{0, 1\}^n$
taking a sequence to its first n terms.
In this case, $m : L \to 2 \times L$ is the obvious $\langle head, tail \rangle$.

Example: streams over $2 = \{0, 1\}$

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$$
\begin{array}{ccccccccc}
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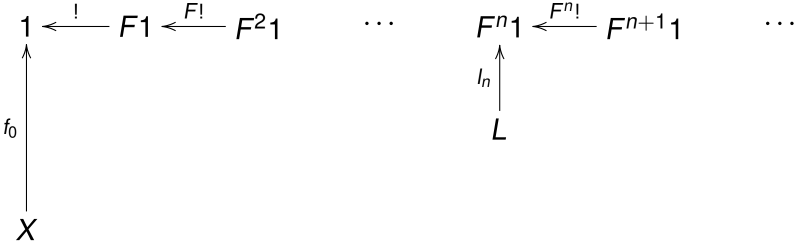
 & & & & & & \uparrow {\scriptstyle l_n} & & \\
 & & & & & & L & &
\end{array}
$$

Another way, special to this F:
take L to be the set 2^N
of functions from natural numbers to 2.
$l_n : L \to F^n 1$ is $f \mapsto (f(0), (f(1), (f(2), \ldots f(n)))))$.

The coalgebra structure $m : L \to 2 \times L$ is a little easier:
$m(f) = (f(0), n \mapsto f(n+1))$.

Suppose we are given $\xi : X \to 2 \times X$, say $X = \{x, y, z\}$

$$\begin{aligned} \xi(x) &= \langle 0, y \rangle \\ \xi(y) &= \langle 0, z \rangle \\ \xi(z) &= \langle 1, x \rangle \end{aligned}$$

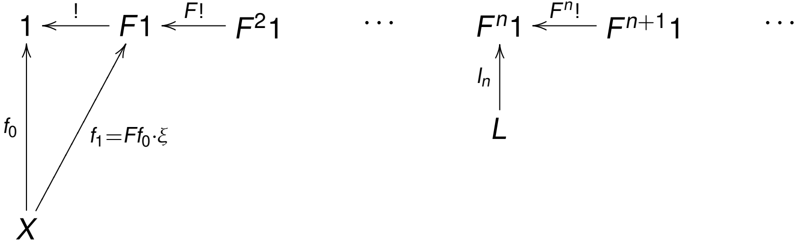


No choice here: 1 is a final object.

Let's try to understand how this works for a given coalgebra

Suppose we are given $\xi : X \to 2 \times X$, say $X = \{x, y, z\}$

$$\begin{aligned} \xi(x) &= \langle 0, y \rangle \\ \xi(y) &= \langle 0, z \rangle \\ \xi(z) &= \langle 1, x \rangle \end{aligned}$$



$f_1(x) = 0$, $f_1(y) = 0$, $f_1(z) = 1$.

Suppose we are given $\xi : X \to 2 \times X$, say $X = \{x, y, z\}$

$$
\begin{aligned}
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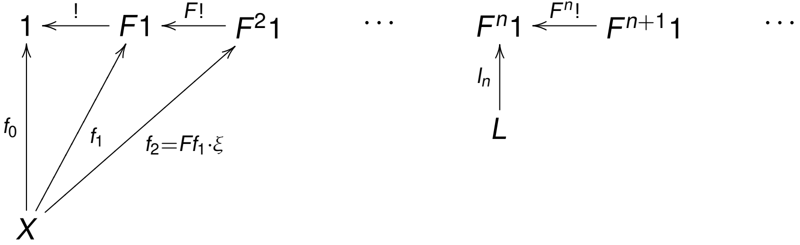
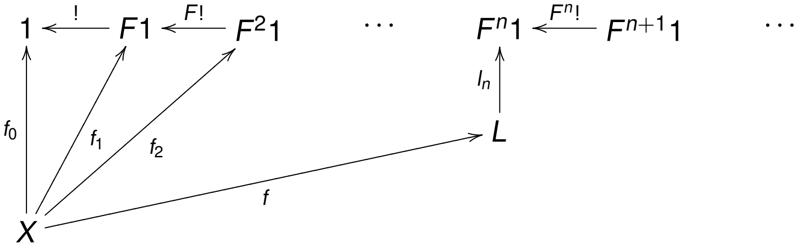
$$



$f_2(x) = 00$, $f_2(y) = 01$, $f_2(z) = 10$.

Suppose we are given $\xi : X \to 2 \times X$, say $X = \{x, y, z\}$

$$\begin{aligned} \xi(x) &= \langle 0, y \rangle \\ \xi(y) &= \langle 0, z \rangle \\ \xi(z) &= \langle 1, x \rangle \end{aligned}$$



$f(x) = 001001001\cdots$, $f(y) = 01001001\cdots$, $f(z) = 100100\cdots$.
Note that f is the coalgebra morphism from (X, ξ) to (L, m).

Finitary Iteration

Let $\mathcal{A}$ be a category with final object 1
and with the property that every ω^{op}-chain in $\mathcal{A}$ has a limit.

Let $F : \mathcal{A} \to \mathcal{A}$ preserve ω^{op}-limits,
and consider the final ω^{op}-chain of F:

$$1 \xleftarrow{\;!\;} F1 \xleftarrow{\;F!\;} F^2 1 \qquad \cdots \qquad F^n 1 \xleftarrow{\;F^n !\;} F^{n+1} 1 \qquad \cdots$$

Let νF be its limit, and let $m : F \to F(\nu F)$ be the factorizing morphism.

Then $(\nu F, m)$ is a final F-coalgebra.

We don't really need *all* limits, only the one shown.
And this is the only limit we need F to preserve.

Finitary iteration gives final coalgebras for all functors on Set built from

- the identity functor
- constant functors
- the discrete measure functor $\mathcal{D}(X)$.

and using

- $+$, $\times$, F^A for fixed sets A
- composition

But it doesn't work for $\mathcal{P}_{fin}$ or its relatives $\mathcal{P}_\kappa$.

Finitary iteration gives final coalgebras for all functors on compact Hausdorff spaces built from

- the identity functor
- constant functors
- the Vietoris functor V.
  V(X) is the hyperspace of X,
  the set of compact subsets of X, with a certain topology.
  For $f : X \to Y$ and $A \in \mathsf{V}X$,

$$(\mathsf{V}f)A = f[A].$$

- the Borel measure functor $\mathcal{B}$. For $f : X \to Y$ and $A \in \mathsf{V}X$,

$$((\mathcal{B}f)\mu)A \;=\; \mu(f^{-1}(A))$$

and using

- $+$, $\times$
- composition

Finitary iteration gives final coalgebras for all functors on MS
1-bounded metric spaces and non-expanding maps,
built from

- the identity functor
- constant functors
- $\varepsilon\mathcal{P}_k$, the scaled version of the closed set functor $\mathcal{P}_k$, using the Hausdorff distance

  $$d(s,t) = \max\{\sup_{x\in s}\inf_{y\in t} d(x,y), \sup_{x\in s}\inf_{y\in t} d(x,y)\}.$$

  The distance from $\emptyset$ to any other closed set is 1.
  $\varepsilon < 1$ scales distances.
- using $+$, $\times$, and composition.

van Breugel: $\mathcal{P}_k$ without scaling has no final coalgebra on MS.

Finitary iteration gives final coalgebras for all functors on CMS
1-bounded complete metric spaces and non-expanding maps,
which are locally weakly contracting:
for $f : X \to X$,

$$d(Ff, id_{FX}) < \varepsilon d(f, id_X) \quad \text{for some } \varepsilon < 1$$

America, Rutten: Assume $F\emptyset \neq \emptyset$.
The inverse of an initial algebra of F is a final coalgebra of F.

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Example: final coalgebra of $FX = 1 + \frac{1}{2}(X \times X)$
is finite and infinite binary trees with the usual metric and evident structure.

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America, Rutten: Assume $F\emptyset \neq \emptyset$.
The inverse of an initial algebra of F is a final coalgebra of F.

den Hartog and de Vink 2002: Scaled versions of the functor giving compactly supported Borel measures are locally contracting.
(at least in the case of ultrametric spaces).

Note that this category does not have limits in general.
Adámek and Reiterman 1994:
A version of this holds for categories enriched over CMS, too.

On KMS, the compact metric spaces and non-expanding maps, again with functors which are locally weakly contracting: for $f : X \to X$,

$$d(Ff, id_{FX}) < \varepsilon d(f, id_X) \quad \text{for some } \varepsilon < 1$$

Alessi, Baldan, Bellé 1995: Assume $F\emptyset \neq \emptyset$.
The inverse of an initial algebra of F is a final coalgebra of F.
and F has a unique fixed point.

A $\mathrm{CPO}_\perp$ is a complete partial order with $\perp$.

$\mathcal{A}$ is $\mathrm{CPO}_\perp$-enriched if its homsets $\mathcal{A}(X, Y)$
carry the structure of a CPO with $\perp$
and composition is strict (preserves the least element) and
continuous (preserves ω-joins) in both variables.

$F : \mathcal{A} \to \mathcal{A}$ is locally continuous if $F \bigsqcup f_n = \bigsqcup F f_n$
for all ω-chains $f_n \in \mathcal{A}(X, Y)$.

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Theorem (Adamek, based on Smyth and Plotkin 1982)

Every locally continuous $F : \mathcal{A} \to \mathcal{A}$ has a canonical fixed point:
there is an initial algebra and it is the inverse of a final coalgebra.

This result is at the core of Dana Scott's construction of

$$D \cong [D \to D]$$

giving a model of the lambda calculus.

SB = standard Borel spaces,
measurable spaces which use the Borel subsets of a Polish space

$\Delta : \mathrm{SB} \to \mathrm{SB}$ takes M to the set of its probability measures with σ-algebra generated by

$$\{B^p(E) \mid p \in [0,1], E \in \Sigma\},$$

where

$$B^p(E) \quad = \quad \{\mu \in \Delta(M) \mid \mu(E) \geq p\}.$$

SB = standard Borel spaces,
measurable spaces which use the Borel subsets of a Polish space

Kolmogorov Consistency Theorem

Let

$$X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} X_2 \qquad \cdots \qquad X_n \xleftarrow{f_n} X_{n+1} \qquad \cdots$$

be an ω^{op}-chain in SB, and assume in addition that each f_n is surjective. Let $X = \lim X_n$, and let $\pi_n : X \to X_n$ be the projection. Let $\mu_n \in \Delta X_n$ be Borel measures such that $\Delta f_n(\mu_{n+1}) = \mu_n$ for all n. Then there is a unique $\mu \in \Delta X$ so that for all n, $\Delta\pi_n(\mu) = \mu_n$.

Thus Δ : SB $\to$ SB has a final coalgebra, as does a functor like

$$FX = \Delta(X \times [0, 1])$$

What about $\Delta : \mathsf{Meas} \to \mathsf{Meas}$?

Viglizzo 2005

The functor $\Delta : \mathsf{Meas} \to \mathsf{Meas}$ does not preserve limits of ω^{op}-chains.

What about Δ : Meas $\to$ Meas?

Viglizzo 2005

The functor Δ : Meas $\to$ Meas does not preserve limits of ω^{op}-chains.

LM and Viglizzo 2006

Every functor F : Meas $\to$ Meas built from

$$\Delta : \text{Meas} \to \text{Meas}$$

and the usual stuff has a final coalgebra.

The proof used a version of probabilistic modal logic, using the set of all theories of all points in all spaces, and also using the π-λ Theorem of measure theory.

But the method does not always work!

Consider $\mathcal{P}_{fin}$ on Set, and the terminal sequence:

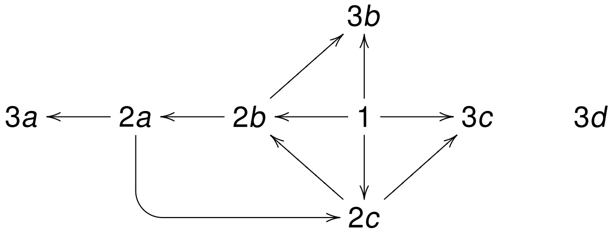
$$1 \xleftarrow{!} \mathcal{P}_{fin}1 \xleftarrow{\mathcal{P}_{fin}!} \mathcal{P}^2_{fin}1 \qquad \cdots \qquad \mathcal{P}^n_{fin}1 \xleftarrow{\mathcal{P}^n_{fin}!} \mathcal{P}^{n+1}_{fin}1 \qquad \cdots$$

It happens that $m : FL \to L$ is not surjective.
[Worrell 2005]
So m cannot be part of a final coalgebra structure.

Lambek's Lemma

The structure morphisms of initial algebras and final coalgebras are always isomorphisms.

These are finitely branching graphs, suitably re-packaged:



is a picture of the coalgebra (G, e), with $G = \{1, 2a, 2b, \ldots, 3c\}$

$$\begin{array}{lclclcl} e(1) & = & \{2b, 3b, 2c\} & \quad & e(3a) & = & \emptyset \\ e(2a) & = & \{2c, 3a\} & & e(3b) & = & \emptyset \\ e(2b) & = & \{2a, 3b\} & & e(3c) & = & \emptyset \\ e(2c) & = & \{2b, 3c\} & & e(3d) & = & \emptyset \end{array}$$

The coalgebra morphisms in this case are not
the usual graph morphisms
(edge preserving maps).

They are rather the "*p*-morphisms" of modal logic,
done without atomic sentences:

$\varphi : G \to H$ would be a morphism if for all $g \in G$,

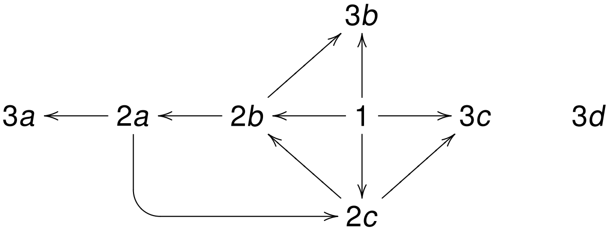
$$\{\varphi(g') : g \to g' \text{ in } G\} = \{h : \varphi(g) \to h \text{ in } H\}.$$

In words, φ preserves sets of children.

Let $(G, \to)$ be a graph.
A relation R on G is a bisimulation iff the following holds:
whenever xRy,

(ZIG) If $x \to x'$, then there is some $y \to y'$ such that $x'Ry'$.

(ZAG) If $y \to y'$, then there is some $x \to x'$ such that $x'Ry'$.



The largest bisimulation on our graph G is the relation that relates 1 to itself,
all 2-points to all 2-points,
and all 3-points to all 3-points.

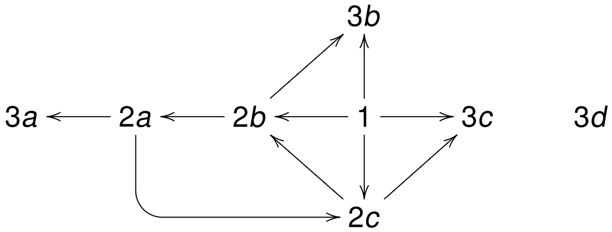
Note that this is an equivalence relation: reflexive, symmetric, and transitive.

The modal sentences are the smallest collection containing a constant true and closed under the boolean $\neg$, $\wedge$, and $\vee$ and a unary modal operator $\Box$.

That is, the modal sentences are the initial algebra of a functor related to the signature $H_{\Sigma_{\text{modal}}}$, where Σ_{modal} contains true, $\neg$, $\wedge$, $\Box$.

Given a $\mathcal{P}$-coalgebra (X, e), we define $x \models \varphi$, by recursion on $\mathcal{L}$ as follows:

$x \models \text{true}$		always
$x \models \text{false}$		never
$x \models \neg\varphi$	iff	it is not the case that $x \models \varphi$
$x \models \varphi \wedge \psi$	iff	$x \models \varphi$ and $x \models \psi$
$x \models \Box\varphi$	iff	for all $y \in e(x)$, $y \models \varphi$



$3a \models \text{true} \wedge \Box\text{false}$
$2b \models (\Diamond\Box\text{false}) \wedge \Diamond\Diamond\text{true}.$

The theory of a point is the set of modal sentences it satisfies.
Bisimilar points have the same theory (but not conversely)
But in finitely branching graphs, points with the same theory are bisimilar.
(the Hennessey-Milner property).

A functor $F : \mathsf{Set} \to \mathsf{Set}$ is finitary if
any of the following hold:

1. There is a (finitary) signature Σ and a natural transformation $\eta : H_\Sigma \to F$ with η_X surjective for non-empty X.
2. For all X, and all $x \in FX$, there is a finite set S and some $f : S \to X$ and some $y \in FS$ such that $x = Ff(y)$.
3. Etc. (lots of others)

Every functor built from $\mathcal{P}_{fin}$, $\mathcal{D}$, and the signature functors using composition is finitary.

For any set S of modal sentences, let us write ∇S for

$$\Box \bigvee_{\varphi \in S} \varphi \;\wedge\; \bigwedge_{\varphi \in S} \Diamond \varphi$$

So a point x satisfies ∇S if

- every φ in S is satisfied by some child of x.
- every child of x satisfies some sentence in S.

Now write

$$\begin{array}{lcllcl} 1 & = & \{\text{true}\} & & & \\ F_1 & = & \{\nabla(a) : a \subseteq 1\} & = & \{\nabla\emptyset, \nabla\{\text{true}\}\} \\ F_2 & = & \{\nabla(a) : a \subseteq F_1\} & & \\ F_{n+1} & = & \{\nabla(a) : a \subseteq F_n\} & & \end{array}$$

$$F_{n+1} \approx \mathcal{P} F_n$$

FINE 1974

For all n,
every point in every graph satisfies a unique $\varphi \in F_n$.

If $\varphi, \psi \in F_n$, then either $\models \varphi \leftrightarrow \psi$, or $\models \varphi \rightarrow \neg\psi$.

For all $\varphi \in F_{n+1}$, there is a unique $\varphi' \in F_n$ such that
$\models \varphi \rightarrow \varphi'$.

Every ordinary modal sentence of modal height n
is equivalent to
some disjunction of elements of F_n.

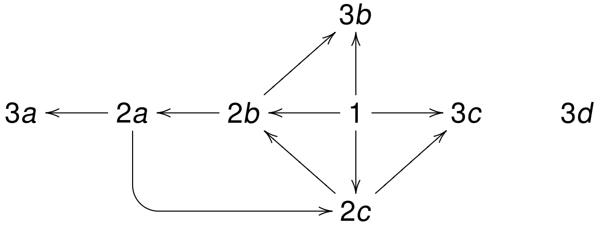
When dealing with $F = \mathcal{P}_{fin}$, we plan to replace

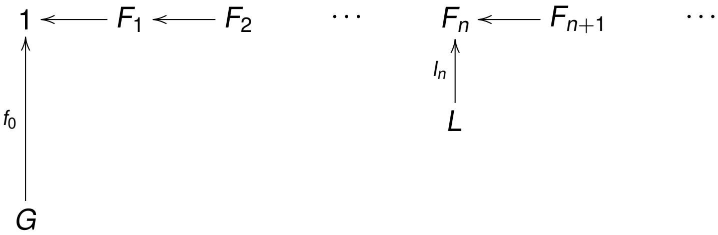
$$1 \xleftarrow{!} F1 \xleftarrow{F!} F^2 1 \qquad \cdots \qquad F^n 1 \xleftarrow{F^n !} F^{n+1} 1 \qquad \cdots$$

by

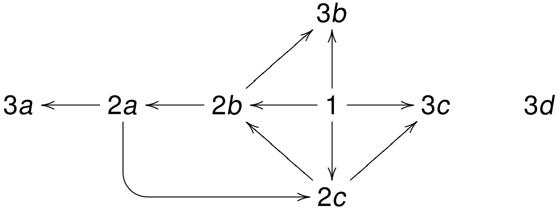
$$1 \xleftarrow{!} F_1 \longleftarrow F_2 \qquad \cdots \qquad F_n \longleftarrow F_{n+1} \qquad \cdots$$

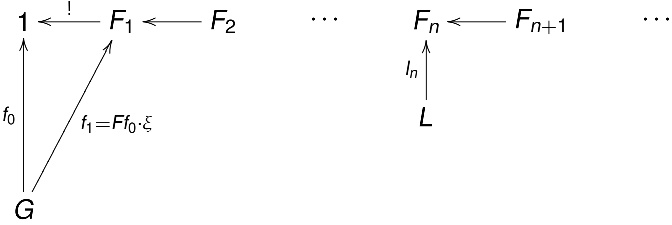
The maps take a sentence in F_{n+1} to the sentence in F_n which it implies.



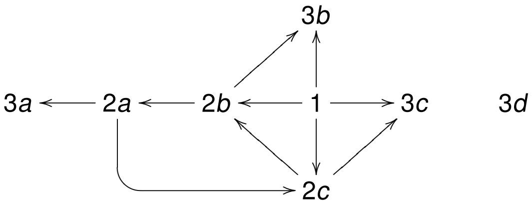


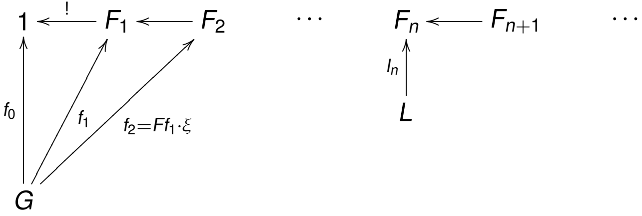
$1 = \{true\}$, so $f_1(g) =$ true for all $g \in G$.





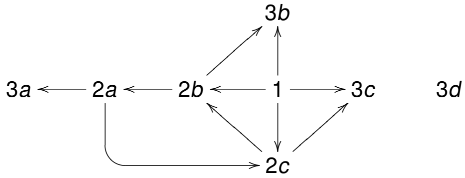
$f_1(3a) = \nabla\emptyset = f_1(3b) = f_1(3c) = f_1(3d)$.
$f_1(1) = \nabla\{\text{true}\} = f_1(2a) = f_1(2b) = f_1(2c)$.

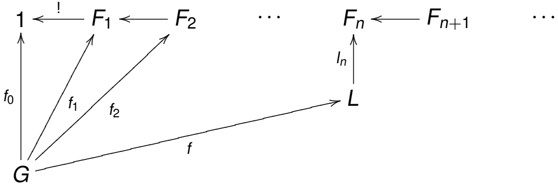




$f_2(1) = \nabla\{f_1(2b), f_1(2c), f_1(3b)\} = \nabla\{\nabla\{\text{true}\}\}$.

In general, $f_2(g)$ is the most informative modal sentence of height 2 that g satisfies.





$f(1) \approx$ the theory of the point 1 in the graph.

Let's check that the set of points in L is the set of worlds in the canonical model of the modal logic K.

And f is the theory map.

Let T be a maximal consistent set in K.
By completeness, there is a model $(W, \rightarrow)$ and a point $w \in W$ such that

$$\varphi \in T \quad \text{iff} \quad w \models \varphi \text{ in } W.$$

Let x_T be the sequence so that

$$x_T(n) = \text{the normal form of height } n \text{ satisfied by } w \text{ in } W.$$

Then $x_T \in \mathcal{P}^{\omega}_{fin}1$.
In the other direction, let $x \in L$, and consider

$$\{\varphi : \text{for some } n, \vdash I_n(x) \rightarrow \varphi\}.$$

This will be maximal consistent, by the properties of the normal forms.

The limit L is too big to be the final coalgebra

The limit L comes with $m : L \to FL$.

In the case $F = \mathcal{P}_{fin}$, m will not be surjective.
And so by Lambek's Lemma, $(L, m : L \to FL)$ will not be a final coalgebra.

In fact, the theory maps $f : G \to L$ will not in general be coalgebra morphisms.

It is the set of all theories of all points in all coalgebras.
This means: the theories of all points in finitely branching models.

It is the set of all theories of all points in all coalgebras.
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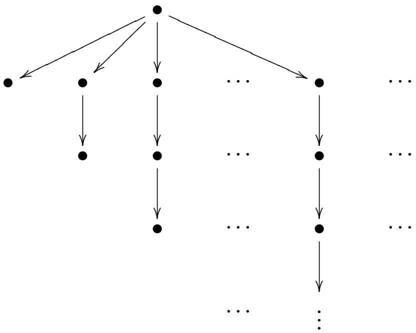
So it would exclude the theory of the top point in



The structure map in the final coalgebra is familiar from modal logic:
take a theory T to the set of theories U such that
if $\Box\varphi \in T$, then $\varphi \in U$.

For any functor F, we can consider the final sequence

$$1 \xleftarrow{!} F_1 \longleftarrow F_2 1 \qquad \cdots \qquad F^n 1 \longleftarrow F_{n+1} 1 \qquad \cdots$$

And in Set, we can always take the limit cone,
and we always get $m : FL \to L$.

WORRELL'S THEOREM 2005

If F : Set → Set is finitary, then m is one-to-one.
So if we march forward with

$$L \xleftarrow{m} FL \xleftarrow{Fm} F^2 L \qquad \cdots \qquad F^n L \xleftarrow{F^n m} F^{n+1} L \qquad \cdots$$

we get a decreasing sequence of sets,
the intersection is the limit, and F does preserve it.
Indeed, this smaller limit is a final coalgebra.

That is , the final coalgebra of F is $F^{\omega+\omega} 1$.

Another idea for $\mathcal{P}_{fin}$ is to take the disjoint union of all coalgebras
and then take the quotient by some equivalence relation.

Before taking the quotient, we have pairs (G, g) such that $g \in G$, and

$$(G, g) \to (H, h) \quad \text{iff} \quad G = H \text{ and } g \to h \text{ in } G.$$

The equivalence notion is maximal bisimulation:

$$(G, g) \equiv (H, h)$$

if there is a bisimulation between G and H which relates g to h.

It has many faces:

- ⋆ the set of theories in *K*
- ⋆ the Cauchy completion of *HF*
- ⋆ the carrier of the final coalgebra of V on compact Hausdorff spaces.

These were shown by Abramsky 2005.

The final coalgebra is smaller, and also has many faces:

- ⋆ the theories of points in finitely-branching graphs.
- ⋆ $\mathcal{P}^{\omega+\omega}_{fin}1$, by Worell's Theorem.

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All aspects of this development generalize.

Although final coalgebras are very interesting,
$F^{\omega}1$ is often also an interesting object!

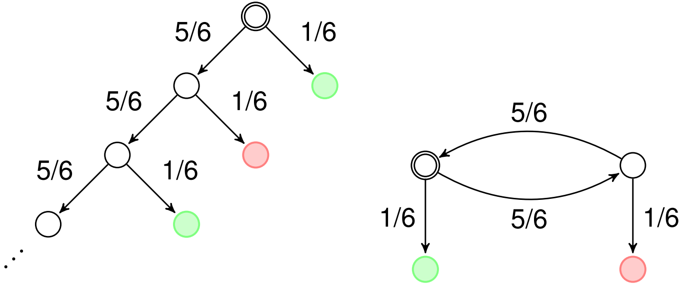
Let BiP be the category of bi-pointed sets.
These are $(X, \top, \bot)$ with $\top, \bot \in X$ and $\top \neq \bot$.

F : BiP $\rightarrow$ BiP takes the disjoint union of two copies of X,

$$(\{0\} \times X) \cup (\{1\} \times X),$$

then identifies $(0, \top)$ with $(1, \bot)$.

Come to my lecture tomorrow
to hear about the final coalgebra of this functor.



The second is the quotient of the first
by the maximum coalgebraic bisimulation.

That is, we can generalize:

$$\frac{\text{bisimulation}}{\text{graphs}} = \frac{???}{\text{coalgebras of a functor } F}$$

We can generalize:

$$\frac{\text{modal logic}}{\text{graphs}} = \frac{\text{???}}{\text{coalgebras of a functor } F}$$

This subject of coalgebraic modal logic is one of the most active areas of coalgebra.

For more on it, one should see papers of

Alexander Kurz
Dirk Pattinson
Lutz Schröder

A good portion of the papers get presented at an annual conference,
Coalgebraic Methods in Computer Sciecne.

Consider the category $\mathcal{A}$ of classes.

(A class is like a set, but it could be "too big" to be a set.
For example, the class V of all sets is a set.

Classes can be taken to be formulas in the language of set theory,
allowing sets as parameters.)

$\mathcal{P} : \mathcal{A} \to \mathcal{A}$ gives the class of sub*sets* of a given class.

Note that $\mathcal{P}V = V$.

Work in ZF – *Foundation*

The Foundation Axiom (FA) is equivalent to the assertion that

$$(V, id : \mathcal{P}V \to V)$$

is an initial algebra of $\mathcal{P}$.

The Anti-Foundation Axiom (AFA) is equivalent to the assertion that

$$(V, id : V \to \mathcal{P}V)$$

is a final coalgebra of $\mathcal{P}$.

Take a roulette wheel labeled with points in $[0, 1]$.
Spin it successively, until the total of the spins is ≥ 1.

It might happen in 2 spins, or 3, or 6238.

What is the average number of spins that it would take to get a total of > 1?

Let $E(t)$ = the average number of spins that it would take to get a total of $> t$.

So $E(0) = 1$.

How can we get a formula for $E(t)$?

Fix a number t.

If we spin the wheel once, we get some number, say x.

If $x > t$, we're done on the first spin.

If $x \leq t$, we need to continue.
How many further spins are needed, on average?
For $x \leq t$, we on average will need $E(t - x)$.

We would want to take the probability of getting x, and then multiply it by $1 + E(t - x)$.

But the probability of getting x exactly is 0, and thus we integrate.

$$
\begin{aligned}
E(t) &= \int_t^1 1\, dx \quad + \quad \int_0^t 1 + E(t-x)\, dx \\
&= 1 + \int_0^t E(t-x)\, dx \\
&= 1 + \int_0^t E(u)\, du
\end{aligned}
$$

(We made a substitution $u = t - x$.)

By the Fundamental Theorem of Calculus, $E'(t) = E(t)$.

Combined with $E(0) = 1$, we see that

$$E(t) \quad = \quad e^t,$$

and the answer to the original problem is e.

set with algebraic operations	set with transitions and observations
algebra for a functor	coalgebra for a functor
initial algebra	final coalgebra
least fixed point	greatest fixed point
congruence relation	bisimulation equivalence rel’n
equational logic	modal logic
recursion: map out of an initial algebra	corecursion: map into a final coalgebra
Foundation Axiom	Anti-Foundation Axiom
iterative conception of set	coiterative conception of set
typical of syntactic objects	typical of semantic spaces
bottom-up	top-down

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