Algebras

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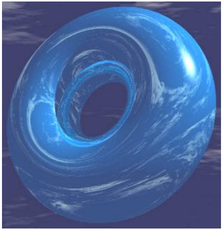
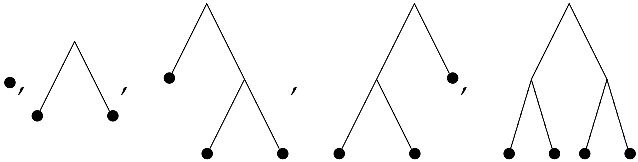


Let T be the set which starts out as



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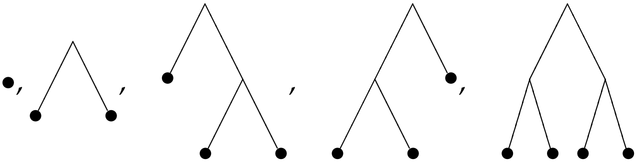

Then $T = 1 + (T \times T)$,
where 1 is an arbitrary singleton, and $+$ is disjoint union.

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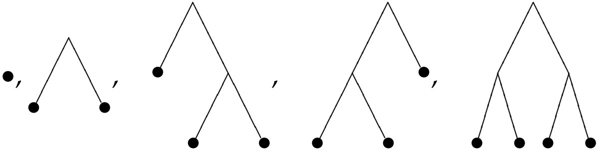
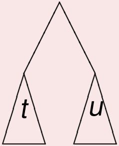


Recursion Principle for Finite Trees

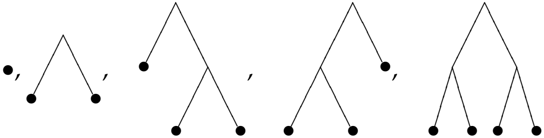
For all sets X, all $x \in X$, all $f : X \times X \to X$,
there is a unique $\varphi : T \to X$
so that φ is

one-point tree $\mapsto$ x



$\mapsto$ $f(\varphi(t), \varphi(u))$

Let T be the set which starts out as



Recall $T = 1 + (T \times T)$.

Recursion Principle for Finite Trees

For all sets X, all $f : 1 + (X \times X) \to X$,
there is a unique $\varphi : T \to X$ so that

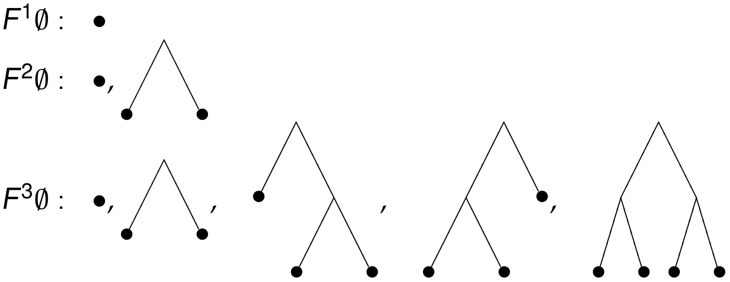
$$\begin{array}{ccc} 1 + (T \times T) & \xrightarrow{id} & T \\ \downarrow{\scriptstyle 1+(\varphi\times\varphi)} & & \downarrow{\scriptstyle \varphi} \\ 1 + (X \times X) & \xrightarrow[f]{} & X \end{array}$$

commutes, where $(\varphi \times \varphi)(t, u) = (\varphi(t), \varphi(u))$.

If we only knew about the set operation

$$F(X) = X \mapsto 1 + (X \times X),$$

we could get our hands on T by taking $\bigcup_i F^i(\emptyset)$



That is, T is the least fixed point of F on sets.

Let *HF* be the set which starts out as

$$\emptyset \cup \{\emptyset\} \cup \{\emptyset, \{\emptyset\}\} \cup \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\{\emptyset\}\}\} \cup \cdots$$

Note that every finite subset of *HF* is an element of *HF*.
(Usual notation is V_ω.)

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(Usual notation is V_ω.)
Then $HF = \mathcal{P}_{fin}(HF)$.

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Note that every finite subset of *HF* is an element of *HF*.
(Usual notation is V_ω.)

RECURSION PRINCIPLE FOR HEREDITARILY FINITE SETS

For all sets X, all $f : \mathcal{P}_{fin}(X) \to X$,
there is a unique $\varphi : HF \to X$ so that

$$\varphi(\{a_1, \ldots, a_n\}) = f(\{\varphi(a_1), \ldots, \varphi(a_n)\}).$$

Let *HF* be the set which starts out as

$$\emptyset \cup \{\emptyset\} \cup \{\emptyset, \{\emptyset\}\} \cup \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\{\emptyset\}\}\} \cup \cdots$$

Note that every finite subset of *HF* is an element of *HF*.
(Usual notation is V_ω.)

RECURSION PRINCIPLE FOR HEREDITARILY FINITE SETS

For all sets X, all $f : \mathcal{P}_{fin}(X) \to X$,
there is a unique $\varphi : HF \to X$ so that

$$\begin{array}{ccc} \mathcal{P}_{fin}(HF) & \xrightarrow{id} & HF \\ {\scriptstyle \mathcal{P}_{fin}\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathcal{P}_{fin}(X) & \xrightarrow[f]{} & X \end{array}$$

commutes, where $\mathcal{P}_{fin}\varphi(A) = \{\varphi(a) : a \in A\}$.

If we only knew about the set operation

$$\mathcal{P}_{fin}(X),$$

we could get our hands on *HF* by taking $\bigcup_i \mathcal{P}^i_{fin}(\emptyset)$

$$\mathcal{P}^1_{fin}\emptyset : \quad \{\emptyset\}$$

$$\mathcal{P}^2_{fin}\emptyset : \quad \{\emptyset, \{\emptyset\}\}$$

$$\mathcal{P}^3_{fin}\emptyset : \quad \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\{\emptyset\}\}\}$$

That is, *HF* is the least fixed point of $\mathcal{P}_{fin}$ on sets.

Let N be the set which starts out as

$$0 = \emptyset, 1 + 0, 1 + (1 + 0), 1 + (1 + (1 + 0)), \ldots,$$

$1 + X$ is the disjoint union of a singleton set and X.

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$1 + X$ is the disjoint union of a singleton set and X.
Then

$$N = 1 + N.$$

Or depending on how you "implement" disjoint unions,

$$N \cong 1 + N.$$

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RECURSION PRINCIPLE FOR N

For all sets X, all $x \in X$, all $f : X \to X$,
there is a unique $\varphi : N \to X$ so that

$$\begin{array}{lcl} \varphi(0) & = & x \\ \varphi(n+1) & = & f(\varphi(x)) \end{array}$$

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RECURSION PRINCIPLE FOR N

For all sets X, all $x \in X$, all $f : X \to X$,
there is a unique $\varphi : N \to X$ so that

$$\begin{array}{ccc} 1 + N & \xrightarrow{t} & N \\ {\scriptstyle 1+\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ 1 + X & \xrightarrow[f]{} & X \end{array}$$

commutes, where, for all $n \in N$, $(1 + \varphi)(n) = \varphi(n)$ inside $1 + X$.

If we only knew about the set operation

$$F(X) = 1 + X$$

we could get our hands on N by taking $\bigcup_i F^i(\emptyset)$

$$F^1\emptyset : \quad 1 + \emptyset$$

$$F^2\emptyset : \quad 1 + F^1\emptyset$$

$$F^3\emptyset : \quad 1 + F^2\emptyset$$

That is, N is the least fixed point of $X + 1$ on sets.

Recursion on N is tantamount to Initiality

Recursion on N: For all sets A, all $a \in A$, and all $f : A \to A$, there is a unique $\varphi : N \to A$ so that
$\varphi(0) = a$, and $\varphi(n+1) = f(\varphi(n))$ for all n.

Initiality of N: For all (A, a), there is a unique homomorphism $\varphi : (N, t) \to (A, a)$:

$$\begin{array}{ccc} 1 + N & \xrightarrow{\cong} & N \\ {\scriptstyle 1+\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ 1 + A & \xrightarrow[a]{} & A \end{array}$$

These are equivalent in set theory without Infinity.

At this point, we want to generalize the three examples which we have seen.

The basic ingredients are

- Initial objects in categories.
- Functors, especially functors $F : \mathrm{Set} \to \mathrm{Set}$.
- Algebras of functors.
- Initial algebras.

I will introduce these in an expansive way,
much more generally than we actually need for the
three examples which we have seen.

After this, we turn to other examples, and then to coalgebraic variations on what we have seen.

A category C consists of

1. objects $c, d, \dots$
   The collection of objects might be a proper class.
2. For each two objects c and d, a collection of morphisms $f, g, \dots$.
   with emphdomain c and codomain d.
   We write $f : c \to d$ to say that f is such a morphism.
3. identity morphisms id_a for all objects.
4. a composition operation:
   if $f : a \to b$ and $g : b \to c$, then $g \cdot f : a \to c$.

subject to the requirements that

- Composition is associative.
- If $f : a \to b$, then $id_b \cdot f = f = f \cdot id_a$.

The objects of Set are sets (all of them).

A morphism from X to Y is a function from X to Y.

The identity morphism id_a for a set a is the identity function on a.

The composition operation of morphisms
is the one we know from sets.

Second example: the category Pos of posets

The objects of Pos are posets $(P, \leq)$.
(That is, $\leq$ is reflexive, transitive, and anti-symmetric.)

A morphism from P to Q is a monotone function f from X to Y.
(If $p \leq p'$ in P, then $f(p) \leq f(p')$ in Q.

The identity morphism id_a for a poset P is the identity function on the underlying set P.

The composition operation of morphisms
is again the one we know from sets.

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The identity morphism id_a for a poset P is the identity function on the underlying set P.

The composition operation of morphisms
is again the one we know from sets.

The anti-symmetry plays no role,
and we might as well generalize to preorders

Third example: every poset is itself a category

Let $(P, \leq)$ be a poset.

We consider P to be a poset by taking its elements as the objects.

The morphisms $f : p \to q$ are just the pairs (p, q) with $p \leq q$.

Unlike sets, between any two objects there is either 0 or 1 morphisms.

The morphism id_p is (p, p).

$(q, r) \cdot (p, q) = (p, r)$.

MS is the category of metric spaces (X, d),
with $d : X \times X \to [0, 1]$ satisfying the metric properties:

- $d(x, x) = 0$
- If $d(x, y) = 0$, then $x = y$.
- $d(x, y) = d(y, x)$.
- $d(x, z) \leq d(x, y) + d(y, z)$.

A morphisms from (X, d) to (Y, d') is a *non-expanding function* $f : X \to Y$.
This means that

$$d'(f(x), f(y)) \quad \leq \quad d(x, y)$$

The category CMS is the same, but we use *complete* metric spaces.

The category BiP of bi-pointed sets

Objects are $(X, \top, \bot)$, where X is a set and $\top$ and $\bot$ are elements of X.
We require $\bot \neq \top$.

A morphism $f : (X, \top, \bot) \to (Y, \top, \bot)$ is a function $f : X \to Y$ such that $f(\top) = \top$ and $f(\bot) = \bot$.

The rest of the structure is as in Set, or any other concrete category.

Let C be a category.

An initial object is an object c such that
for all d, there is unique morphism $f : c \to d$.

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In Set, $\emptyset$ is initial.

Recall that the empty function is a function from $\emptyset$ to any set.
Also, there is no function from any non-empty set to $\emptyset$.

Let C be a category.

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for all d, there is unique morphism $f : c \to d$.

In Pos, the empty poset is initial.

The same basically works for MS.

Let C be a category.

An initial object is an object c such that
for all d, there is unique morphism $f : c \to d$.

In a poset P, an initial object would be a minimal element.

(This may or may not exist.)

Let C be a category.

An initial object is an object c such that
for all d, there is unique morphism $f : c \to d$.

In BiP, the initial object is any object
based on a two-element set: $(\{\top, \bot\}, \top, \bot)$.

We often write 0 for an initial object.

$\mathsf{Set}_{\neq\emptyset}$: the non-empty sets, as a full subcategory of Set.
(This means that we use all morphisms in the big category.)

Set_p: the pointed sets.

Objects: (X, x), where X is a set and $x \in X$.
Morphisms $f : (X, x) \to (Y, y)$ is a function $f : X \to Y$ such that $f(x) = y$.

Do these have initial objects?

Let C and D be categories.
A functor from C to D consists of

- An object mapping $a \mapsto Fa$, taking objects of C to objects of D.
- A morphism mapping $f \mapsto Ff$, taking morphisms of C to morphisms of D.

such that

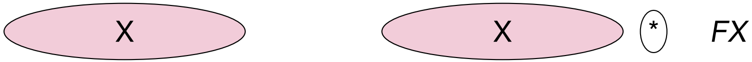
- If $f : a \to b$, then $Ff : Fa \to Fb$.
- $Fid_a = id_{Fa}$.
- $F(f \cdot g) = Ff \cdot Fg$.

A functor from C to itself is an endofunctor.
(In our terminology, an endofunctor is a “recipe for cooking”.

$FX = 1 + X$ as a functor Set $\to$ Set

1 here is *any* one-element set, and to emphasize the arbitrariness
one often writes it as $\{*\}$.

$+$ is disjoint union, the categorical coproduct.



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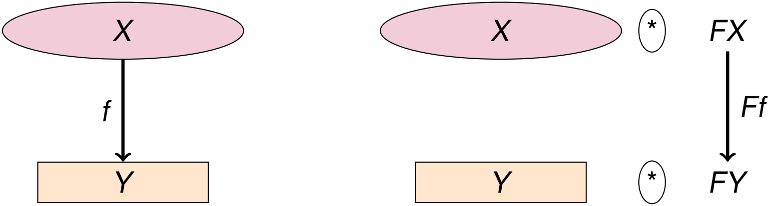

Ff is defined to preserve the new points.

The power set endofunctor $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$

For any set X, $\mathcal{P}X$ is the set of subsets of X.
$\mathcal{P}$ extends to an endofunctor, taking

$$f : X \to Y$$

to

$$\mathcal{P}f : \mathcal{P}X \to \mathcal{P}Y$$

given by direct images: for $a \subseteq X$, $\mathcal{P}f(a) = f[a] = \{f(x) : x \in a\}$.

We similarly have functors such as the finite power set functor $\mathcal{P}_{fin}$.

Examples of $F : \mathsf{Set} \to \mathsf{Set}$ which we will use later

- For any set A, $FX = A \times X$.
  If $f : X \to Y$, then $Ff : A \times X \to A \times Y$ is

$$Ff(a, x) \quad = \quad (a, fx).$$

- $F(X) = X^A$, where A is a fixed set.
  (Here X^A is the set of functions from A to X.)

  If $f : X \to Y$, then $Ff : X^A \to Y^A$ is given by

$$Ff \quad = \quad g \mapsto f \cdot g.$$

  That is,

$$Ff(g) \quad = \quad f \cdot g.$$

A discrete measure on a set A is a function $m : A \to [0,1]$ such that

1. m has *finite support*: $\{a \in A \mid m(a) > 0\}$ is finite.
2. $\sum_{a \in A} m(a) = 1$.

$\mathcal{D}(A)$ is the set of discrete measures on A.
We make $\mathcal{D}$ into a functor by setting,
for $f : A \to B$, $\mathcal{D}f(m)(b) = m(f^{-1}(b))$;
this is $\sum\{m(a) : f(a) = b\}$.

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Incidentally, this Δ is a monad:
$\eta_X : X \to \Delta XC$ is $\eta_X(x) = \delta_x$ (the Dirac δ):

$$\eta_X(x)(y) = 1 \text{ iff } y = x, \text{ otherwise } 0$$

$\mu_X : \Delta\Delta X \to \Delta X$ is "mixing":

$$\mu_X(m)(x) = \sum_{q \in \Delta(X)} m(q)(x)$$

- $\text{upclosed} : \text{Pos} \to \text{Pos}$
  taking a poset P to the set of upward closed subsets, under $\subseteq$.

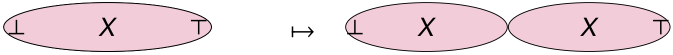
  If $f : (P, \leq) \to (Q, \leq)$, you might like to think about how Ff should work.

- On a particular poset P, a functor $F : P \to P$ is the same thing as a monotone function $F : P \to P$.

  In fact the monotonicity property of an endofunction corresponds to the functoriality property $F(f \cdot g) = Ff \cdot Fg$

Let $(X, \top, \bot)$ be a bipointed set.
We define $X \oplus X$ to be

- two separate copies of X which I'll write as X_1 and X_2.
- The $\bot$ of $X \oplus X$ is the $\bot$ of X_1.
- The $\top$ of $X \oplus X$ is the $\top$ of X_2.
- The $\top$ of X_1 is identified with the $\bot$ of X_2.
  (This is called the midpoint of $X \oplus X$.)



identify $\top$ of left with $\bot$ of right

We get a functor $F : \mathrm{BiP} \to \mathrm{BiP}$ by

$$FX = X \oplus X$$

If $f : X \to Y$ is a BiP morphism, then $Ff : X \oplus X \to Y \oplus Y$
works in the obvious way, preserving the midpoint.

Let d be an object of D.
We get $F : C \to D$, the constant functor d by:
$Fc = d$,
$Ff = id_d$.

The composition of functors is again a functor.

Let $\mathcal{A}$ be a category, and let $F : \mathcal{A} \to \mathcal{A}$ be a functor.

An F-algebra is a pair (A, a), where $a : FA \to A$.

Technically, these are pairs.
a is called the structure of the algebra, and A the carrier.
Let $(A, a : FA \to A)$ and $(B, b : FB \to B)$ be algebras.

An F-algebra morphism is $f : A \to B$ in the same underlying category $\mathcal{A}$ so that

$$\begin{array}{ccc} FA & \xrightarrow{a} & A \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle f} \\ FB & \xrightarrow[b]{} & B \end{array}$$

commutes.

So now we have a category of algebras for an endofunctor.

Let $\mathcal{A}$ be a category, and let $F : \mathcal{A} \to \mathcal{A}$ be a functor.

An F-algebra is a pair (A, a), where $a : FA \to A$.

Technically, these are pairs.
a is called the structure of the algebra, and A the carrier.
An initial F-algebra is one with a unique morphism to any algebra.

$$\begin{array}{ccc} FA & \xrightarrow{\;a\;} & A \\ {\scriptstyle F\varphi}\downarrow & & \downarrow{\scriptstyle \text{there is a unique morphism } \varphi} \\ FB & \xrightarrow[\text{for all } b]{} & B \end{array}$$

An Initial Algebra for $FX = 1 + X$ on Set

For $FX = 1 + X$ on Set,

- Initial algebra is (N, ν), with N the set of natural numbers, and $\nu : 1 + N \to N$ given by

$$\begin{aligned} * &\mapsto 0 \\ n &\mapsto n + 1 \end{aligned}$$

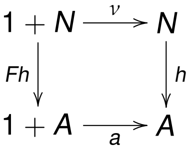
  Initiality "is" recursion, the most important definition principle for functions on numbers.

Initiality at work: $FX = 1 + X$

Consider an algebra $(A, a : 1 + A \to A)$, where $A = \{\alpha, \beta, \gamma, \delta\}$, and

$$\begin{array}{rclcrcl} a(*) & = & \gamma & \quad & a(\gamma) & = & \delta \\ a(\alpha) & = & \beta & \quad & a(\delta) & = & \alpha \\ a(\beta) & = & \gamma & & & & \end{array}$$

Query: What is the map h below?



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Query: What is the map h below?

$$\begin{array}{ccc} 1 + N & \xrightarrow{\nu} & N \\ {\scriptstyle Fh}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ 1 + A & \xrightarrow[a]{} & A \end{array}$$

It is

$$0 \mapsto \gamma \quad 1 \mapsto \delta \quad 2 \mapsto \alpha \quad 3 \mapsto \beta \quad 4 \mapsto \gamma \quad 5 \mapsto \delta \quad \ldots$$

h is defined by recursion.

For $FX = \mathcal{P}_{fin}X$, we have an algebra

$$\mathcal{P}_{fin}(HF) \xrightarrow{id} HF$$

and then initiality is the recursion principle for *HF*.

$$\begin{array}{ccc} \mathcal{P}_{fin}(HF) & \xrightarrow{id} & HF \\ {\scriptstyle \mathcal{P}_{fin}\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathcal{P}_{fin}(X) & \xrightarrow[f]{} & X \end{array}$$

The commutativity of the diagram means that for $s \in HF$,

$$\varphi(s) = \{\varphi(x) : x \in s\}.$$

For $FX = 1 + (X \times X)$, initial algebra is the set T of finite binary trees.
Again, we can take the structure to be the identity.

Recall that an endofunctor on a poset $(P, \leq)$
is a monotone function $f : P \to P$.

An algebra for f is some p such that $f(p) \leq p$.

An initial algebra for f is some p such that

- $f(p) \leq p$.
- If $f(q) \leq q$, then $p \leq q$.

These are least fixed points of the functor.

$F(X) = X \times X$ on Set

Initial algebra is the empty set $\emptyset \times \emptyset = \emptyset$
together with the empty function $F\emptyset \to \emptyset$.

Actually, if $F0 = 0$, then $(0, id_0)$ is an initial algebra.

In any category C, objects a and b are isomorphic if there are

$$f : a \rightarrow b$$
$$g : b \rightarrow a$$

such that $g \cdot f = id_a$, and $f \cdot g = id_b$.

Isomorphic objects have the same categorical properties.

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Isomorphic objects have the same categorical properties.

Proposition

If a is an initial object of C, then the only $f : a \to a$ is id_a.

Therefore:
If a and b are initial objects of C, then they are isomorphic.

In any category C, objects a and b are isomorphic if there are

$$f : a \to b$$
$$g : b \to a$$

such that $g \cdot f = id_a$, and $f \cdot g = id_b$.

Isomorphic objects have the same categorical properties.

Proposition

If (a, f) and (b, g) are initial algebras of $F : C \to C$, then these algebras are isomorphic (as algebras), and in particular a and b are isomorphic as objects.

Let's study $FX = 1 + X$ on another category

MS is category of metric spaces (X, d) with distances ≤ 1, and non-expanding maps:

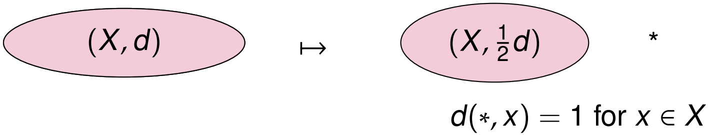
$$d(fx, fy) \leq d(x, y)$$

$FX = 1 + X$ is the disjoint union of X with a one-point space, with distance 1 to the new point.

Initial algebra is the discrete metric space on the natural numbers.

LET'S STUDY $FX = 1 + \frac{1}{2}X$ ON MS

$\frac{1}{2}X$ IS THE SPACE X, BUT WITH DISTANCES SCALED BY $\frac{1}{2}$.



Initial algebra of this F turns out to be

$$((\mathrm{Nat}, d), \varphi),$$

where Nat is the set of natural numbers, and for $n \neq m$,

$$d(n, m) = 2^{-\min(n,m)}.$$

The structure $\varphi : 1 + \frac{1}{2}(\mathrm{Nat}, d) \to (\mathrm{Nat}, d)$ is the same as for N.

$1 + X$ and $1 + \frac{1}{2}X$ on CMS

CMS is the subcategory of *complete* metric spaces.
(Again, distances bounded by 1, and non-expanding maps.)

Initial algebra of $1 + X$ is same as for MS: $(N, d_{discrete})$.

For $1 + \frac{1}{2}X$: it's N^{∞}, again with

$$d(n, m) = 2^{-\min(n,m)},$$

treating ∞ as larger than all $n \in N$.

So it's a Cauchy sequence together with its limit.

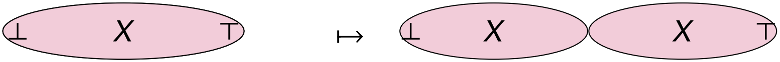
$1 + X$ and $1 + \frac{1}{2}X$ on KMS

KMS is the subcategory of compact metric spaces.
(Again, distances bounded by 1, and non-expanding maps.)

Initial algebra of $1 + X$ doesn't exist.

For $1 + \frac{1}{2}X$: it's the same space as for this functor on CMS.

Recall F works as



identify $\top$ of left with $\bot$ of right

Initial algebra is not so easy to guess, and we'll develop tools to compute it later.

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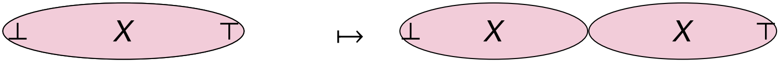


identify $\top$ of left with $\bot$ of right

Initial algebra is not so easy to guess, and we'll develop tools to compute it later.

It may be described as the set $\mathcal{D}$ of
dyadic rational numbers in $[0, 1]$, with $0 = \bot$ and $1 = \top$.
What do you think the structure is?

LEMMA (LAMBEK'S LEMMA)

Let C be a category, let $F : C \to C$ be a functor, and let (a, f) be an initial algebra for F.

Then f is an isomorphism: there is a morphism $g : Fa \to a$ such that $g \cdot f = id_a$ and $f \cdot g = id_{Fa}$.

The same statement holds for final coalgebras of F.

Note first that (Fa, Ff) is an algebra for F. The square below commutes:

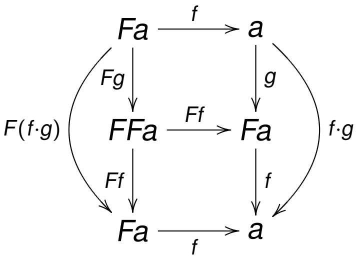
$$\begin{array}{ccc}
FFa & \xrightarrow{Ff} & Fa \\
{\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle f} \\
Fa & \xrightarrow[f]{} & a
\end{array}$$

By initiality, there is a morphism $g : a \to Fa$ so that the square on the top commutes:

$$\begin{array}{ccc}
Fa & \xrightarrow{f} & a \\
{\scriptstyle Fg}\downarrow & & \downarrow{\scriptstyle g} \\
FFa & \xrightarrow{Ff} & Fa \\
{\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle f} \\
Fa & \xrightarrow[f]{} & a
\end{array}$$

The bottom is obvious, the outside of the figure thus commutes.

By initiality, there is a morphism $g : a \to Fa$ so that the square on the top commutes:



By initiality, we see that $f \cdot g = id_a$.
And then from that top square again,

$$g \cdot f = Ff \cdot Fg = F(f \cdot g) = Fid_a = id_{Fa}.$$

This completes the proof.

THERE ARE NO INITIAL ALGEBRAS OF $\mathcal{P}$: Set $\to$ Set

An isomorphism in Set is a bijection.

And there are no maps from any set onto its power set (Cantor's Theorem).

Together with Lambek's Lemma, we see that $\mathcal{P}$ on Set has no initial algebra.
To get around this, one either

1. moves from Set to the category Class of classes.
2. moves from $\mathcal{P}$ to $\mathcal{P}_\kappa$, the functor giving the subsets of a set of size $< \kappa$.

(H_κ, id) is an initial algebra of $\mathcal{P}_\kappa$,
where H_κ is the set of sets of herediary cardinality $< \kappa$.
(So when $\kappa = \aleph_0$, we get *HF* from before.)

Recall that (N, ν) is the initial algebra of $FX = 1 + X$.

A subalgebra of (N, ν) is an F-algebra (M, μ)
such that
there is a one-to-one algebra morphism $i : M \to N$.

Induction on N

If (M, μ) is a subalgebra of (N, ν),
then $(M, \mu) \cong N, \nu)$.