Coalgebras

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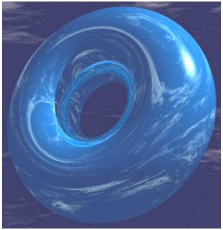

set with algebraic operations	set with transitions and observations
algebra for a functor	coalgebra for a functor
initial algebra	final coalgebra
least fixed point	greatest fixed point
congruence relation	bisimulation equivalence rel'n
equational logic	modal logic
recursion: map out of an initial algebra	corecursion: map into a final coalgebra
Foundation Axiom	Anti-Foundation Axiom
iterative conception of set	coiterative conception of set
useful in syntax	useful in semantics
bottom-up	top-down

We have seen examples of circularly-defined sets such as

the set of streams

the set of infinite trees

One of the main goals of the course is to present a theory of how these "solution spaces" work.
The theory is based on the concept of a coalgebra for a functor and on similar notions from category theory.

Today's lecture includes an introduction to the main concepts which we'll need.

But it is not a systematic presentation of the subject.

$$
\begin{array}{lcl}
x & \approx & \langle 0, y \rangle \\
y & \approx & \langle 1, z \rangle \\
z & \approx & \langle 2, x \rangle
\end{array}
$$

Let us construe such a system as a function
from its set of variables.

So let $X = \{x, y, z\}$.

We regard the system as a function $e : X \to N \times X$.
(e stands for "equation".)

$$
\begin{array}{lcl}
e(x) & = & \langle 0, y \rangle \\
e(y) & = & \langle 1, z \rangle \\
e(z) & = & \langle 2, x \rangle
\end{array}
$$

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\begin{aligned}
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e(y) &= \langle 1, z\rangle \\
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\end{aligned}
$$

Let's write N^∞ for the set of streams on N.

The solution to our system e is a function $e^\dagger : X \to N^\infty$.
Explicitly,

$$
\begin{aligned}
e^\dagger(x) &= (0,1,2,0,1,2,\ldots) \\
e^\dagger(y) &= (1,2,0,1,2,0,\ldots) \\
e^\dagger(z) &= (2,0,1,2,0,1\ldots)
\end{aligned}
$$

Now what we want to do is to talk
in an abstract way what the relation between
e and $e^\dagger$.

And what we say should hold for all systems.

$$
\begin{aligned}
e(x) &= \langle 0, y \rangle & \qquad e^{\dagger}(x) &= (0, 1, 2, 0, 1, 2, \ldots) \\
e(y) &= \langle 1, z \rangle & \qquad e^{\dagger}(y) &= (1, 2, 0, 1, 2, 0, \ldots) \\
e(z) &= \langle 2, x \rangle & \qquad e^{\dagger}(z) &= (2, 0, 1, 2, 0, 1 \ldots)
\end{aligned}
$$

Here is what we want to say:

$$
\begin{array}{ccc}
X & \xrightarrow{\quad e \quad} & N \times X \\
{\scriptstyle e^{\dagger}} \big\downarrow & & \big\downarrow {\scriptstyle id_N \times e^{\dagger}} \\
N^{\infty} & \xrightarrow[\langle hd, tail \rangle]{} & N \times N^{\infty}
\end{array}
$$

More on our diagram

I'm in the middle of explaining the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad e \quad} & N \times X \\ {\scriptstyle e^\dagger}\big\downarrow & & \big\downarrow{\scriptstyle id_N \times e^\dagger} \\ N^\infty & \xrightarrow[\langle hd,tail \rangle]{\qquad} & N \times N^\infty \end{array}$$

The hard part is the function $id_N \times e^\dagger$.

For this, I will need some general notation on products.

If $f : C \to A$ and $g : C \to B$, then we get a new function

$$\langle f, g \rangle : C \to A \times B.$$

It is defined by

$$\langle f, g \rangle(c) \quad = \quad (f(c), g(c))$$

Product functions

If $f : C \to A$ and $g : C \to B$, then we get a new function

$$\langle f, g \rangle : C \to A \times B.$$

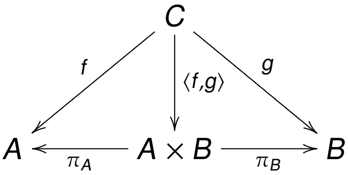
It is defined by

$$\langle f, g \rangle(c) \quad = \quad (f(c), g(c))$$

The product set $A \times B$ itself comes with projections

$$A \xleftarrow{\pi_A} A \times B \xrightarrow{\pi_B} B$$

And then the diagram below commutes:



If $f : C \to A$ and $g : D \to B$, then we get a new function

$$f \times g : C \times D \to A \times B.$$

It is defined by

$$(f \times g)(c, d) \quad = \quad (f(c), g(d))$$

Note the difference between the notations $\langle f, g \rangle$ and $f \times g$.

They must be related, but how?

More on product functions

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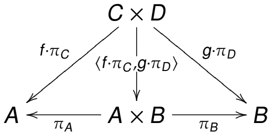
The $f \times g$ notation is a special case of pairing:



So that

$$f \times g \quad = \quad \langle f \cdot \pi_C, g \cdot \pi_D \rangle.$$

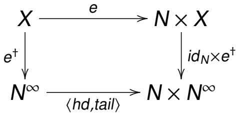
Let's get back to the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad e \quad} & N \times X \\ {\scriptstyle e^\dagger}\downarrow & & \downarrow{\scriptstyle id_N \times e^\dagger} \\ N^\infty & \xrightarrow[\langle hd, tail \rangle]{} & N \times N^\infty \end{array}$$

Now we know about the function $id_N \times e^\dagger$.
id_N is the identity function on N.
The definitions that we have seen tell us
that for $i \in N$ and $w \in X$,

$$(id_N \times e^\dagger)(i, w) \quad = \quad (i, e^\dagger(w)).$$

Let's get back to the diagram



Recall that $X = \{x, y, z\}$ and that

$$\begin{array}{rclcrcl} e(x) & = & \langle 0, y \rangle & \quad & e^\dagger(x) & = & (0, 1, 2, 0, 1, 2, \ldots) \\ e(y) & = & \langle 1, z \rangle & & e^\dagger(y) & = & (1, 2, 0, 1, 2, 0, \ldots) \\ e(z) & = & \langle 2, x \rangle & & e^\dagger(z) & = & (2, 0, 1, 2, 0, 1 \ldots) \end{array}$$

We'll check that the diagram really does commute,

Let's start with y, for example, as a "random" element of X.

Across the top, we get $\langle 1, z \rangle$.
Then going down, we get $\langle 1, (2, 0, 1, 2, 0, 1 \ldots) \rangle$.
But starting again with y and going down, we get
$(1, 2, 0, 1, 2, 0, \ldots)$.
And the head of this stream is 1; the tail is $(2, 0, 1, 2, \ldots)$.

So it really does commute!

Reading the diagram

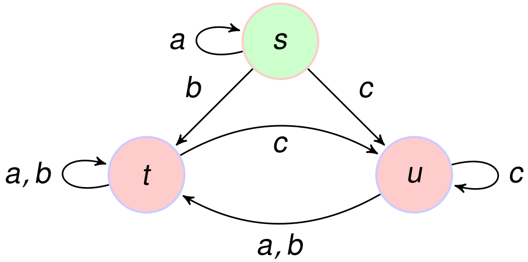
In fact, we can verbalize what it means to say that our diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\quad e \quad} & N \times X \\
{\scriptstyle e^{\dagger}} \downarrow & & \downarrow {\scriptstyle id_N \times e^{\dagger}} \\
N^{\infty} & \xrightarrow[\langle hd, tail \rangle]{\qquad} & N \times N^{\infty}
\end{array}
$$

commutes.

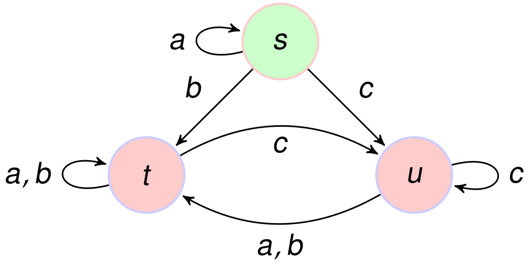
For all $w \in X$, if $e(w) = \langle i, v \rangle$, then
$e^{\dagger}(w)$ is a stream whose head is i, and whose tail is
$e^{\dagger}(v)$.

Here is a deterministic automaton with no start state:



The set of states is $S = \{s, t, u\}$.
We have one accepting state (in green).
The input alphabet is $A = \{a, b, c\}$.

We have a transition function $t : S \times A \to S$,
and also an output function $o : S \to 2$.
(Here $2 = \{0, 1\}$, and $o(s) = 1$ iff s is accepting.)



For all states s, the empty word ε is accepted at s if $acc(s) = 1$.

If w is a word and a an alphabet symbol, then

aw is accepted at s iff w is accepted at $t(s, a)$

So far a deterministic automaton on $\{a, b, c\}$ is

$$(S, s, acc),$$

where S is a set,

$$s : S \times A \to S,$$

and

$$acc : S \to 2.$$

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$$(S, s, acc),$$

where S is a set,

$$s : S \times A \to S,$$

and

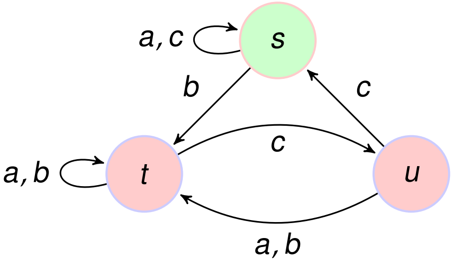
$$acc : S \to 2.$$

We can curry s to get $\widehat{s} : S \to S^A$.

We also use pairing

$$\langle \widehat{s} \times acc \rangle : S \to 2 \times S^A.$$

To match our earlier usage, we write e for $\langle \widehat{s} \times acc \rangle$.



We have re-packaged the picture into a function

$$e : S \to 2 \times S^A$$

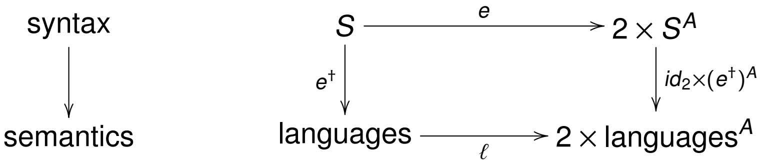
It is

$$\begin{aligned} e(s) &= (1, \{(a, s), (b, t), (c, s)\}) \\ e(t) &= (0, \{(a, t), (b, t), (c, u)\}) \\ e(u) &= (0, \{(a, t), (b, t), (c, s)\}) \end{aligned}$$

A^* is the set of finite words on A, including the empty word ϵ.

$\mathcal{L} = \mathcal{P}(A^*)$ is the set of languages X on A.

We want to think of language acceptance in the same way as we have seen for streams and sets.



But this needs an explanation!

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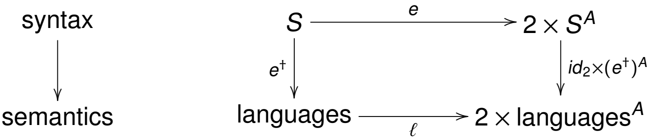

But this needs an explanation!
Let's write $\mathcal{L}$ for the set of all languages on A.
(This is just $\mathcal{P}(A^*)$.)

We make $\mathcal{L}$ into an automaton (!) in our cooked sense by

$$\ell : \mathcal{L} \to 2 \times \mathcal{L}^A.$$

where

$$\ell(X) = (1 \text{ iff } \epsilon \in X, a \mapsto \{w : aw \in X\})$$

Taking $f : X \to Y$ to $f^A : X^A \to Y^A$

To explain the map $(e^\dagger)^A$,
here is a general definition.

If $f : X \to Y$, then $f^A : X^A \to Y^A$ is given by

$$g : A \to X \quad \mapsto \quad f \cdot g.$$

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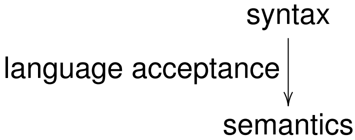
Now we understand



$$\begin{CD} S @>{e}>> 2 \times S^A \\ @V{e^\dagger}VV @VV{id_2 \times (e^\dagger)^A}V \\ \mathcal{L} @>>{\ell}> 2 \times \mathcal{L}^A \end{CD}$$

syntax
language acceptance ↓
semantics

$$\begin{array}{ccc} S & \xrightarrow{e} & 2 \times S^A \\ {\scriptstyle e^\dagger}\downarrow & & \downarrow{\scriptstyle id_2 \times (e^\dagger)^A} \\ \mathcal{L} & \xrightarrow[\ell]{} & 2 \times \mathcal{L}^A \end{array}$$

For all states s, language accepted at s has two features:

- it contains the empty word iff s is an accepting state; that is, if $\pi_2(e(s)) = 1$.
- for all words w and all a, it contains aw iff w is in the language accepted at $\pi_{S^A}(e(s))(a)$.

The reason for all these diagrams is that they enable us see the same kind of pattern coming up again and again.
We want an overall language to talk about it.

We have seen:

streams	$e : X \to A \times X$
languages	$e : S \to 2 \times S^A$

Let's think of $A \times X$ and $\mathcal{P}X$ as cooked versions of X.

So the kind of systems that we have seen are

functions from a raw object to a cooked version of it

Very soon, we'll start calling this a coalgebra.

Let C be a category.

An initial object is an object c such that
for all d, there is unique morphism $f : c \to d$.

An final object is an object c such that
for all d, there is unique morphism $f : d \to c$.

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for all d, there is unique morphism $f : d \to c$.

In Set, $\emptyset$ is initial, and every singleton $\{x\}$ is final.

Note that there is more than one final object, but they are all isomorphic.

Let C be a category.

An initial object is an object c such that
for all d, there is unique morphism $f : c \to d$.

An final object is an object c such that
for all d, there is unique morphism $f : d \to c$.

In Pos, the empty poset is initial, and the one-point poset $\{x\}$ is final.

The same basically works for MS.

Let C be a category.

An initial object is an object c such that
for all d, there is unique morphism $f : c \to d$.

An final object is an object c such that
for all d, there is unique morphism $f : d \to c$.

In a poset P, an initial object would be a minimal element,
and a final object would be a maximal element.

(These may or may not exist.)

Let C be a category.

An initial object is an object c such that
for all d, there is unique morphism $f : c \to d$.

An final object is an object c such that
for all d, there is unique morphism $f : d \to c$.

In BiP, the initial object is any object
based on a two-element set: $(\{\top, \bot\}, \top, \bot)$.

We often write 0 for an initial object and 1 for a final one.

So in the category of pointed sets, $0 = 1$.

Let $F : C \to C$ be an endofunctor.
A coalgebra for F is a pair (A, a), where $a : A \to FA$ in C.
We have already seen many examples!

A morphism from (A, a) to (B, b) is $h : A \to B$ such that

$$\begin{array}{ccc} A & \xrightarrow{a} & FA \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Fh} \\ B & \xrightarrow[b]{} & FB \end{array}$$

commutes.

So now we have a category of coalgebras for an endofunctor.

Let $(A, a : FA \to A)$ and $(B, b : FB \to B)$ be algebras.
A morphism in the algebra category of F is $f : A \to B$ in the same underlying category so that

$$\begin{array}{ccc} FA & \xrightarrow{a} & A \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle f} \\ FB & \xrightarrow[b]{} & B \end{array}$$

commutes.

Let $(A, a : A \to FA)$ and $(B, b : B \to FB)$ be coalgebras.
A morphism in the coalgebra category of F is $f : A \to B$ in the same underlying category so that

$$\begin{array}{ccc} A & \xrightarrow{a} & FA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Ff} \\ B & \xrightarrow[b]{} & FB \end{array}$$

commutes.

Initial algebras and final coalgebras

An initial algebra is an initial object of the algebra category.
A final coalgebra is a final object of the coalgebra category.

initial algebra

$$\begin{array}{ccc} FA & \xrightarrow{a} & A \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle f} \\ FB & \xrightarrow[b]{} & B \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{a} & FA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Ff} \\ B & \xrightarrow[b]{} & FB \end{array}$$

final coalgebra

(One could also consider final algebras and initial coalgebras, but they turn out to be much less interesting.)

Recall that an endofunctor on a poset $(P, \leq)$
is a monotone function $f : P \to P$.
An algebra for f is some p such that $f(p) \leq p$.

A coalgebra for f is some p such that $p \leq f(p)$.

An initial algebra for f is some p such that

- $f(p) \leq p$.
- If $f(q) \leq q$, then $p \leq q$.

A final algebra for f is some p such that

- $p \leq f(p)$.
- If $q \leq f(q)$, then $q \leq p$.

These correspond to
least fixed points and greatest fixed points,
respectively.

$$\begin{array}{c} \text{syntax} \\ \downarrow \\ \text{semantics} \end{array} \qquad\qquad \begin{array}{ccc} S & \xrightarrow{\quad e \quad} & 2 \times S^A \\ {\scriptstyle e^\dagger}\downarrow & & \downarrow{\scriptstyle id_2 \times (e^\dagger)^A} \\ \text{languages} & \xrightarrow[\quad \ell \quad]{} & 2 \times \text{languages}^A \end{array}$$

$e^\dagger$ takes a states s to the language of all words accepted if we start at s.

It is important to check that the coalgebra morphisms are exactly the usual morphisms of automata.

In general, final coalgebras are like the “semantic observation spaces” for the type of coalgebra.

Stream systems as coalgebras, their solutions as coalgebra morphisms

Let $FX = N \times X$.
Coalgebras of F are stream systems;
that is, maps of the form $e : X \to FX$.

Even more, the solution $e^\dagger : X \to N^\infty$ would be a coalgebra morphism:

$$
\begin{array}{ccc}
X & \xrightarrow{\;e\;} & FX \\
{\scriptstyle e^\dagger}\downarrow & & \downarrow{\scriptstyle Fe^\dagger} \\
N^\infty & \xrightarrow[\;id\;]{} & FN^\infty
\end{array}
$$

The point is that for $x \in X$,

$$
\begin{array}{rcl}
Fe^\dagger(e(x)) & = & Fe^\dagger\langle \mathrm{fst}(e(x)), \mathrm{snd}(e(x))\rangle \\
 & = & \langle \mathrm{fst}(e(x)), e^\dagger(\mathrm{snd}(e(x)))\rangle
\end{array}
$$

We have seen this formulation before.

Here A is a fixed set.

The initial algebra is the empty set.

The final coalgebra is the set of streams on A, with a structure

$$\langle head, tail \rangle : A^\infty \to A \times A^\infty.$$

$F(X) = A \times X$ on Set

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The final coalgebra is the set of streams on A, with a structure

$$\langle head, tail \rangle : A^\infty \to A \times A^\infty.$$

If we change to $F(X) = (A \times X) + 1$, then what would we get?

Let's use finality to define two functions.

First, the constant embedding $c : A \to A^\infty$.
Second, for a fixed $f : A \to A$, the function $map_f : A^\infty \to A^\infty$.

To start, what equations do we want these to satisfy?
Remember that streams are pairs, and that we have structure

$$\langle head, tail \rangle : A^\infty \to A \times A^\infty.$$

Let's also write the inverse with a colon : in infix notation.
So if $a \in A$ and $s \in A^\infty$, then $a : s \in A^\infty$, and

$$head(a : s) = a \qquad\qquad tail(a : s) = s.$$

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So if $a \in A$ and $s \in A^\infty$, then $a : s \in A^\infty$, and

$$head(a : s) = a \qquad tail(a : s) = s.$$

$c(a) = a : c(a).$

$map_f(s) = f(head(s)) : map_f(tail(s)).$

We start with two coalgebras of $A \times X$:

$$A \xrightarrow{\Delta} A \times A \qquad\qquad A^\infty \xrightarrow{\langle h,t \rangle} A \times A^\infty \xrightarrow{f \times id} A \times A^\infty$$

These immediately drive corecursions, by finality:

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \times A \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle id \times c} \\ A^\infty & \xrightarrow[\langle h,t \rangle]{} & A \times A^\infty \end{array} \qquad \begin{array}{ccccc} A^\infty & \xrightarrow{\langle h,t \rangle} & A \times A^\infty & \xrightarrow{f \times id} & A \times A^\infty \\ {\scriptstyle map_f}\downarrow & & & & \downarrow{\scriptstyle id \times map_f} \\ A^\infty & & \xrightarrow[\langle h,t \rangle]{} & & A \times A^\infty \end{array}$$

Now we should be able to use the definitions and finality,
and general facts about functions on sets,
and nothing much else,
including nothing about the connections of streams and functions,
to prove general facts.

What is the connection of c and map_f?

Now we should be able to use the definitions and finality,
and general facts about functions on sets,
and nothing much else,
including nothing about the connections of streams and functions,
to prove general facts.

What is the connection of c and map_f?

$c \cdot f = map_f \cdot c$.

We want to put down one F-coalgebra (A, g) and then show that

$$c \cdot f \quad \text{and} \quad map_f \cdot c$$

are coalgebras morphisms from our coalgebra to the final one; thus they are equal.

$$A \xrightarrow{\Delta} A \times A \xrightarrow{f \times id} A \times A$$

USING THE DEFINITIONS c AND map_f

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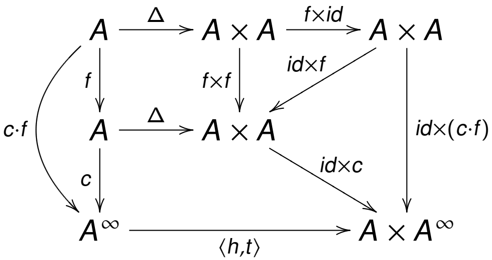
$$A \xrightarrow{\Delta} A \times A \xrightarrow{f \times id} A \times A$$

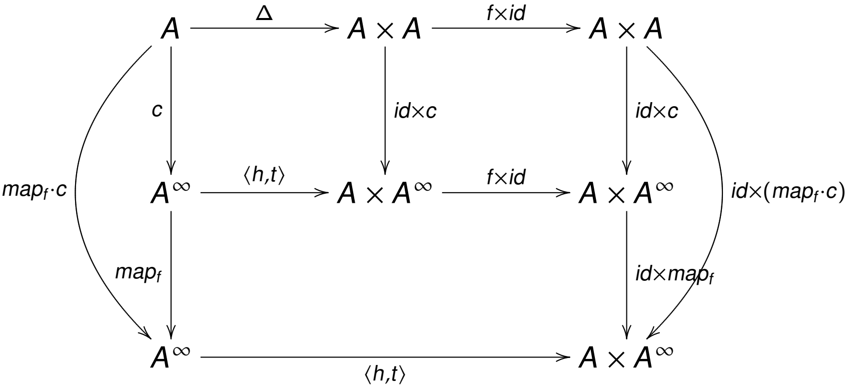
So we need to prove that both diagrams below commute:

$$\begin{array}{ccccc}
A & \xrightarrow{\Delta} & A \times A & \xrightarrow{f \times id} & A \times A \\
{\scriptstyle c \cdot f}\big\downarrow & & & & \big\downarrow{\scriptstyle id \times (c \cdot f)} \\
A^{\infty} & & \xrightarrow[\langle h, t \rangle]{} & & A \times A^{\infty}
\end{array}
\qquad
\begin{array}{ccccc}
A & \xrightarrow{\Delta} & A \times A & \xrightarrow{f \times id} & A \times A \\
{\scriptstyle map_f \cdot c}\big\downarrow & & & & \big\downarrow{\scriptstyle id \times (map_f \cdot c)} \\
A^{\infty} & & \xrightarrow[\langle h, t \rangle]{} & & A \times A^{\infty}
\end{array}$$

Expand both diagrams, and fill them in with stuff

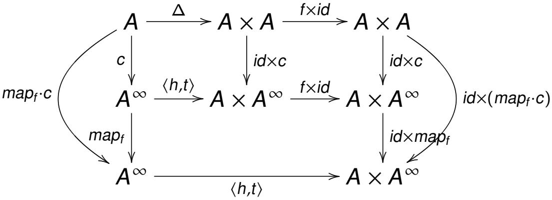


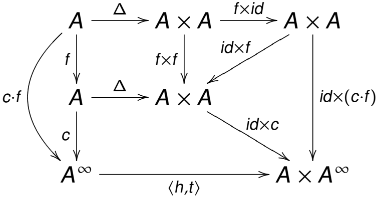
Why do all parts of the diagram commute?


A
Δ
A × A
f×id
A × A
c
id×c
id×c
map_f·c
A^∞
⟨h,t⟩
A × A^∞
f×id
A × A^∞
id×(map_f·c)
map_f
id×map_f
A^∞
A × A^∞
⟨h,t⟩

Expand both diagrams, and fill them in with stuff

So we have two coalgebra maps from the same coalgebra in to the final one.





Thus

$$map_f \cdot c \quad = \quad ((f \times id) \cdot \Delta)^{\dagger} \quad = \quad c \cdot f.$$

Let $FX = A \times X \times X$.

The final coalgebra is the set T of infinite binary trees with all points labeled by an element of A.

We have a structure $t : T \to A \times T \times T$.

The trees are ordered, with a left child and a right child:

$$t \quad = \quad head(t) : left(t) : right(t)$$

Those children are both themselves trees.
Let $swap : T \times T \to T \times T$ be

$$swap(\langle t, u \rangle) = \langle u, t \rangle.$$

Now consider

$$T \xrightarrow{t} A \times T \times T \xrightarrow{id \times swap} A \times T \times T$$

$$T \xrightarrow{t} A \times T \times T \xrightarrow{id \times swap} A \times T \times T$$

It's a coalgebra for F.

So what is its map into the final coalgebra??

$$\begin{array}{ccccc}
T & \xrightarrow{t} & A \times T \times T & \xrightarrow{id \times swap} & A \times T \times T \\
{\scriptstyle mirror}\downarrow & & & & \downarrow{\scriptstyle id_A \times mirror \times mirror} \\
T & & \xrightarrow[t]{\hspace{6em}} & & A \times T \times T
\end{array}$$

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$$\begin{array}{ccc} T & \xrightarrow{t} A \times T \times T \xrightarrow{id \times swap} & A \times T \times T \\ {\scriptstyle mirror}\downarrow & & \downarrow{\scriptstyle id_A \times mirror \times mirror} \\ T & \xrightarrow[t]{\qquad\qquad\qquad\qquad} & A \times T \times T \end{array}$$

$$mirror(t) = head(t) : mirror(right(t)) : mirror(third(t))$$

How to define the left and right branch of a tree?

Let $F(X) = A \times X \times X$, with final coalgebra $(T, t = \langle head, left, right \rangle)$.

Let $G(X) = A \times X$, with final coalgebra $(A^\infty, \langle head, tail \rangle)$.

How can we define the left branch function $lb : T \to A^\infty$?

How to define the left and right branch of a tree?

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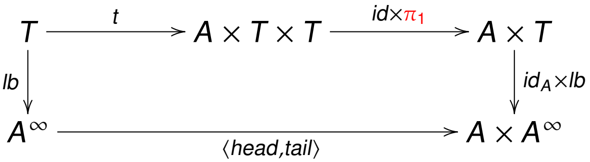


We write down the coalgebra on the top, and then *lb* comes automatically
by finality of the streams as a G-coalgebra.

$lb \cdot mirror = rb$

The right branch function *rb* is the coalgebra map for

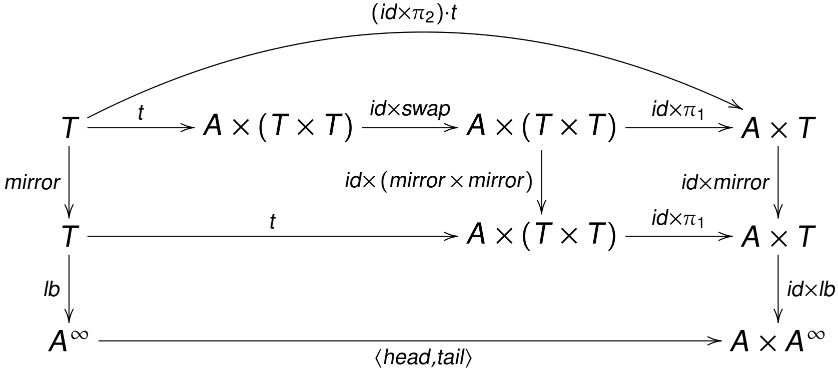
$$T \xrightarrow{\;t\;} A \times T \times T \xrightarrow{id \times \pi_2} A \times T$$

and this is the same as

$$T \xrightarrow{\;t\;} A \times T \times T \xrightarrow{id \times swap} A \times T \times T \xrightarrow{id \times \pi_1} A \times T$$

So we need to show that the diagram below commutes:

$$\begin{array}{ccccccc}
T & \xrightarrow{\;t\;} & A \times T \times T & \xrightarrow{id \times swap} & A \times T \times T & \xrightarrow{id \times \pi_1} & A \times T \\
\Big\downarrow {\scriptstyle lb \cdot mirror} & & & & & & \Big\downarrow {\scriptstyle id \times (lb \cdot mirror)} \\
A^{\infty} & & & \xrightarrow[\;t\;]{\hspace{6em}} & & & A \times A^{\infty}
\end{array}$$


(id×π₂)·t
T
t
A × (T × T)
id×swap
A × (T × T)
id×π₁
A × T
mirror
id× (mirror × mirror)
id×mirror
T
t
A × (T × T)
id×π₁
A × T
lb
id×lb
A^∞
⟨head,tail⟩
A × A^∞

$\mathbb{R}$ here is the set of real numbers.
For $FX = \mathbb{R} \times X$ on Set,

- Initial algebra is the empty set
- Another representation of the final coalgebra:
  Let RA be the set of functions which are real analytic at 0: $f^{(n)}(0)$ exists for all n, and f agrees with its Taylor series in a neighborhood of 0.

  The coalgebra structure $\varphi : RA \to \mathbb{R} \times RA$ is given by

$$f \quad \mapsto \quad (f(0), f').$$

Consider a coalgebra $(A, a : A \to \mathbb{R} \times A)$, where $A = \{\alpha, \beta, \gamma, \delta\}$, and

$$\begin{array}{rclcrcl} a(\alpha) & = & (0, \beta) & \qquad & a(\gamma) & = & (0, \delta) \\ a(\beta) & = & (1, \gamma) & \qquad & a(\delta) & = & (-1, \alpha) \end{array}$$

Query: What is the map $h = a^{\dagger}$ below?

$$\begin{array}{ccc} A & \xrightarrow{\;a\;} & \mathbb{R} \times A \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Fh} \\ RA & \xrightarrow[\;\varphi\;]{} & \mathbb{R} \times RA \end{array}$$

Consider a coalgebra $(A, a : A \to \mathbb{R} \times A)$, where $A = \{\alpha, \beta, \gamma, \delta\}$, and

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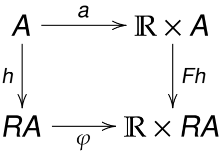
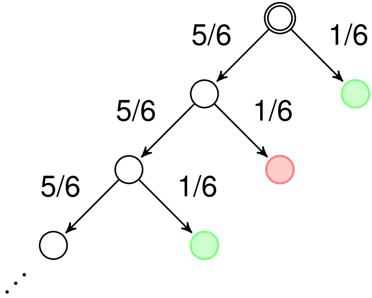


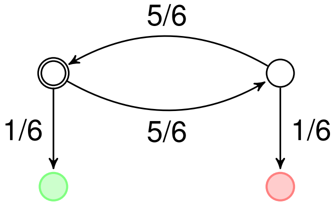
It is

$$\alpha \mapsto \sin\ x, \quad \beta \mapsto \cos\ x, \quad \gamma \mapsto -\sin\ x, \quad \gamma \mapsto -\cos x$$

h is defined by corecursion.

What functor are these two coalgebras of?





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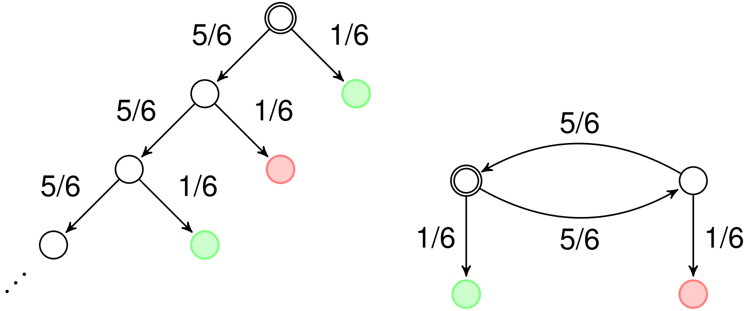
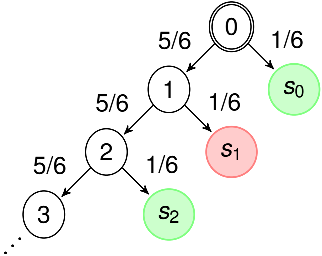


$FX = \{\text{fail}, \text{success}\} + \Delta X.$

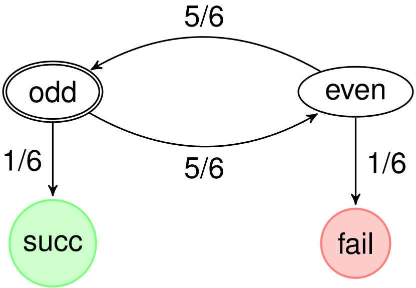
We'll call the coalgebras of this F
Markov chains with observations (MCO's).



The set S of states is $\{0, 1, 2, \ldots, s_0, s_1, s_2, \ldots\}$.
$m(0)$ is given by $m(0)(1) = 5/6$, $m(0)(s_0) = 1/6$.
$m(1)$ is given by $m(1)(2) = 5/6$, $m(1)(s_1) = 1/6$.
$m(s_0) = \mathit{success}$
$\vdots$

We'll call this MCO S.

The fact that 0 is a start state is not reflected, but adding it gives a pointed Markov chain with observations.



The set T of states is $\{odd, even, succ, fail\}$.
We'll call this structure T.

We have two Markov chain with observations, and we want to say why the smaller one is a quotient of the larger one.

---

quotient: a surjective image preserving relevant structure

We need a notion of a mapping between Markov chains with observations

$$\varphi : S \to T.$$

Given $S = (S, m)$, and $T = (T, n)$, a map between them is a coalgebra morphsim.
This is a function $\varphi : S \to T$ so that

$$\begin{array}{ccc} S & \xrightarrow{m} & \Delta S + Obs \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \Delta\varphi + id_{Obs}} \\ T & \xrightarrow[n]{} & \Delta T + Obs \end{array}$$

Maps between MCOs

Given $S = (S, m)$, and $T = (T, n)$, a map between them is a coalgebra morphsim.
This is a function $\varphi : S \to T$ so that

$$\begin{array}{ccc} S & \xrightarrow{m} & \Delta S + Obs \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \Delta\varphi + id_{Obs}} \\ T & \xrightarrow[n]{} & \Delta T + Obs \end{array}$$

One can check that in our examples, the following is a map between MCOs:

$$\begin{array}{lcl} 2k & \mapsto & even \\ 2k+1 & \mapsto & odd \\ s_{2k} & \mapsto & succ \\ s_{2k+1} & \mapsto & fail \end{array}$$

Given a Markov chain without observations, say
$(S, m : S \to \Delta S)$,
and some measure $\mu \in \Delta S$,
we might want to talk about the next step measure $next(\mu)$.

The way we get this is via

$$\Delta S \xrightarrow{\Delta m} \Delta\Delta S \xrightarrow{mix_S} \Delta S$$

Here $mix_S : \Delta\Delta S \to \Delta S$ takes a
discrete measure on discrete measures on S
to another discrete measure on S by mixing.

Given a Markov chain without observations, say
$(S, m : S \to \Delta S)$,
and some measure $\mu \in \Delta S$,
we might want to talk about the next step measure $next(\mu)$.

The function mix_S is then a composition:

$$\Delta S \xrightarrow{\Delta m} \Delta\Delta S \xrightarrow{mix_S} \Delta S, \qquad next = mix_S \circ \Delta m$$

$$\mu \qquad \mapsto \qquad next(\mu)$$

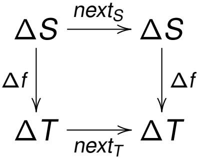
Since you have been learning about natural transformations

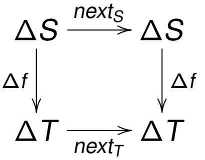
Let's check that if $f : M \to N$, then $\Delta_f \cdot next_S = next_T \cdot \Delta_f$:

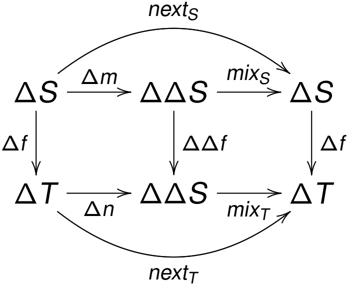
$$
\begin{array}{ccc}
\Delta S & \xrightarrow{next_S} & \Delta S \\
{\scriptstyle \Delta f}\downarrow & & \downarrow{\scriptstyle \Delta f} \\
\Delta T & \xrightarrow[next_T]{} & \Delta T
\end{array}
$$

Since you have been learning about natural transformations

Let's check that if $f : M \to N$, then $\Delta_f \cdot next_S = next_T \cdot \Delta_f$:

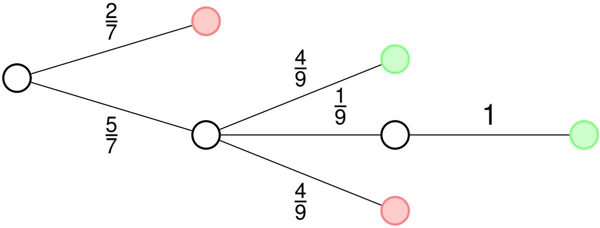






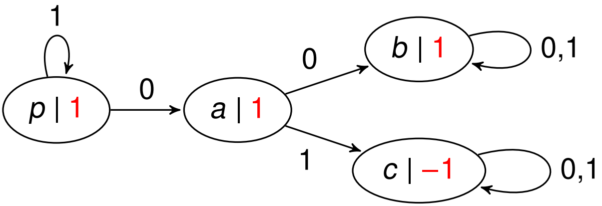
For our functor $FX = \{\text{fail}, \text{success}\} + \Delta X$,

- Initial algebra is finite probabilistic trees ending in observations, such as



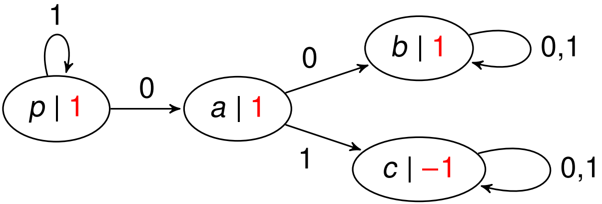
- Final coalgebra is harder to come by, but it does exist (One way to get it is to use coalgebraic generalizations of modal logic.)

What functor is this a coalgebras of?



The set of states is the carrier of the coalgebra.
On the first day, we saw how to map
states to streams on $\{1, -1\}$,
using binary expansions.

What functor is this a coalgebras of?



The set of states is the carrier of the coalgebra.
On the first day, we saw how to map
states to streams on $\{1, -1\}$,
using binary expansions.

$F(X) = \{1, -1\} \times X \times X$.

But, we are now used to thinking of the final
coalgebra of this F as the trees, not the streams!

A different final coalgebra of $FX = A \times X \times X$

$$A \times A^\infty \times A^\infty \to A^\infty$$

given by taking the inverse of the one-to-one function

$$a, s, t \quad \mapsto \quad a : zip(s, t)$$

For each coalgebra $(A, e : X \to A \times X \times X)$,
the map $e^\dagger : A \to A^\infty$
takes a state $a \in A$
to the stream whose nth term is obtained
by writing n in binary, reading the digits into the coalgebra
starting with the least significant one,
and then taking the output at the end.

(Kupke and Rutten 2011)

- Kripke model
- conditional frame
- Markov chain with observations
- belief space

- A Kripke model is a set W of worlds, together with an accessibility relation $R \subseteq W \times W$ and a valuation telling which atomic sentences are true at which worlds.

- A conditional frame is a set W together with a selection function $f : W \times \mathcal{P}(W) \to \mathcal{P}(W)$.

- A Markov chain with observations is a set S such that each $s \in S$ either
  1. comes with an observation set from $\{$*success, fail*$\}$
  2. or there are outgoing arrows $s \to t$ whose labels sum to 1.

- A belief space (for two players, over a space S) is a pair

$$(T_1, T_2, m_1, m_2)$$

  where $m_1 : T_1 \to \Delta(S \times T_2)$, and $m_2 : T_2 \to \Delta(S \times T_1)$. (This is from Heifetz and Samet 1998; other definitions exist.)

To put things under one roof, we use some standard "repackaging":

- trade in a relation $R \subseteq A \times B$ for $f_R : A \to \mathcal{P}B$.
- trade in the valuation on a Kripke model for a function from worlds to sets of atomic sentences.
- trade in $f : A \times B \to C$ for $f^* : A \to C^B$.
- trade two functions $f : A \to B$ and $g : A \to C$ for $\langle f, g \rangle : A \to B \times C$.
- trade in a choice between
  a member of A and a member of B
  for a member of $A + B$ (disjoint union).

- A Kripke model is $(W, f : W \to \mathcal{P}W \times \mathcal{P}(\mathrm{AtSen}))$
- A conditional frame is a

$$(W, f : W \to \mathcal{P}(W)^{\mathcal{P}(W)})$$

- A Markov chain with observations is

$$(S, m : S \to \Delta S + Obs),$$

  where $Obs = \{success, fail\}$,
  and Δ is described below.
- A belief space (for two players, over a space S) is

$$(T, m : T \to FT)$$

  where T is a pair (T_1, T_2) of measurable spaces,
  and $FT = (\Delta(S \times T_2), \Delta(S \times T_1))$.

Morphisms of Kripke models are the coaglebraic morphisms:

$$\begin{array}{ccc} W & \longrightarrow & \mathcal{P}W \times \mathcal{P}(\mathrm{AtSen}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{P}f \times id} \\ W' & \longrightarrow & \mathcal{P}W' \times \mathcal{P}(\mathrm{AtSen}) \end{array}$$

These are usually called *p*-morphisms.

Caution

Morphisms of Kripke models are the coaglebraic morphisms:

$$\begin{array}{ccc} W & \longrightarrow & \mathcal{P}W \times \mathcal{P}(\text{AtSen}) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{P}f \times id} \\ W' & \longrightarrow & \mathcal{P}W' \times \mathcal{P}(\text{AtSen}) \end{array}$$

These are usually called p-morphisms.

Indeed, based on what we have seen for automata, we
might expect
the final coalgebra to be the canonical model C of modal logic

$$\begin{array}{ccc} W & \longrightarrow & \mathcal{P}W \times \mathcal{P}(\text{AtSen}) \\ {\scriptstyle th}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{P}th \times id} \\ C & \longrightarrow & \mathcal{P}C \times \mathcal{P}(\text{AtSen}) \end{array}$$

with the “theory” map $th : W \to C$ defined by

$$th(w) = \{\varphi : w \models \varphi \text{ in the model } W\}$$

Unfortunately this is false!
th isn’t a coalgebra morphism,
and in the first place, the functor doesn’t have a final coalgebra.

Morphisms of Kripke models are the coaglebraic morphisms:

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These are usually called *p*-morphisms.

Next time I'll try to tell you quite a bit about the general constructions of final coalgebras.