Some small categories

- Let $(P, \leq)$ be a *partially ordered set* (i.e., $\leq$ is reflexive, transitive, and antisymmetric). Then P is a category, where the *objects* are the elements of P, and there exists a *unique* morphism $f : x \to y$ iff $x \leq y$.

- Let $(M, \bullet, e)$ be a *monoid* (i.e., $\bullet$ is an associative operation with unit e). Then M is a category, where there is a *unique* object $*$, and one morphism $x : * \to *$ for each element $x \in M$, with composition $x \circ y = x \bullet y$ and identity $\mathrm{id} = e$.

- Let X be a *set*. Then X is a category, called the *discrete* category, where the objects are the elements of X, and the only morphisms are identities $\mathrm{id}_x : x \to x$.

Cartesian product of two categories

If $\mathbf{C}, \mathbf{D}$ are categories, then $\mathbf{C} \times \mathbf{D}$ is a category defined as:

- Objects: (A, B) where $A \in |\mathbf{C}|$ and $B \in |\mathbf{D}|$;
- Morphisms: $(f, g) : (A, B) \to (A', B')$ where $f : A \to A'$ and $g : B \to B'$;
- Composition and identities: componentwise.

Duality

If $\mathbf{C}$ is a category, then its *dual category* $\mathbf{C}^{\mathrm{op}}$ is defined by

- Objects: $\mathbf{C}^{\mathrm{op}}$ has the same objects as $\mathbf{C}$;
- Morphisms: $\mathbf{C}^{\mathrm{op}}(A, B) = \mathbf{C}(B, A)$;
- Identities: same as those of $\mathbf{C}$;
- Composition: in reverse order, i.e.: $g \circ_{\mathbf{C}^{\mathrm{op}}} f = f \circ_{\mathbf{C}} g$.

For every definition or theorem about categories, there is a *dual* definition or theorem, obtained by replacing the category by its dual.

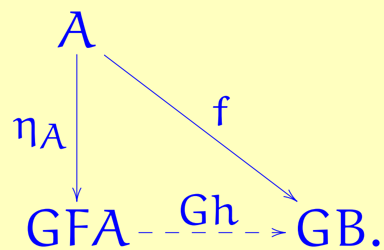
Adjunctions

Suppose $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ are two functors, and further assume that there is a *natural isomorphism of hom-sets*

$$\mathbf{D}(FA, B) \cong \mathbf{C}(A, GB).$$

Then F is called a *left adjoint* of G, and G is called a *right adjoint* of F. We write $F \dashv G$.

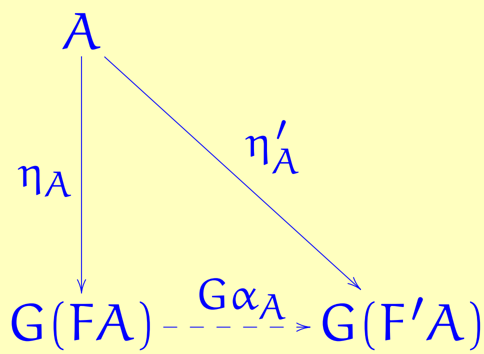
Equivalently (and more concretely), this means that there is $\eta_A : A \to G(FA)$, and for every $f : A \to GB$, there exists a unique $h : FA \to B$ satisfying



Adjoints arise everywhere in mathematics. For example: if G is a *forgetful functor*, and F is its left adjoint, then F is a *free functor*.

Uniqueness of adjoints

Theorem. Suppose that $G : \mathbf{D} \to \mathbf{C}$ is a functor, and that $F, F' : \mathbf{C} \to \mathbf{D}$ are two left adjoints of G. Then there exists a natural isomorphism $\alpha : F \to F'$ such that $\eta' = G\alpha \bullet \eta$.



Adjoints between posets

Remark. Let P, Q be partially ordered (or preordered) sets, and let $f : P \to Q$ and $g : Q \to P$ be monotone functions. Then f is left adjoint to g if and only if for all $x \in P$, $y \in Q$:

$$f(x) \leq y \quad \iff \quad x \leq g(y).$$

(Equivalently, f is "residuated", or f and g form a "Galois connection").

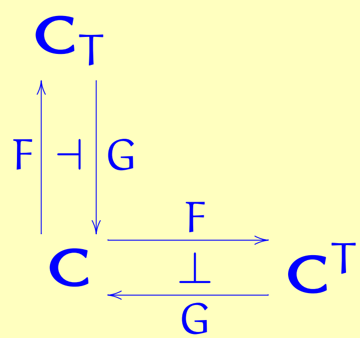
Adjunctions and monads

Every adjunction $F \dashv G$, where $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$, defines a monad on $\mathbf{C}$, via

$$T = G \circ F.$$

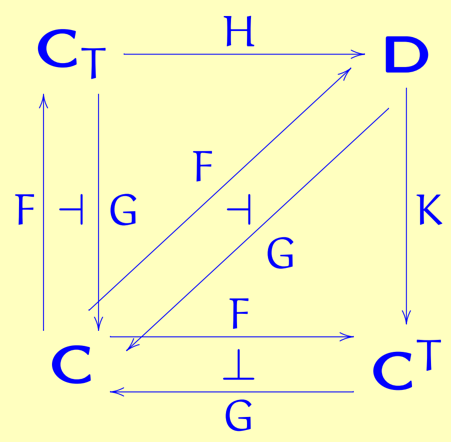
(Note: lots of details omitted).

Conversely, every monad arises in this way: actually in *two* different ways that are canonical: If T is any monad on a category $\mathbf{C}$, then both the *Kleisli construction* and the *Eilenberg-Moore construction* give rise to adjuctions, each satisfying $T = G \circ F$:



Adjunctions and monads, continued

Both the *Kleisli construction* and the *Eilenberg-Moore construction* give rise to adjuctions, each satisfying $T = G \circ F$. Moreover, they are *universal*: given any third adjunction $F \dashv G$ between $\mathbf{C}$ and some category $\mathbf{D}$, there exist *unique* functors $H : \mathbf{C}_T \to \mathbf{D}$ and $K : \mathbf{D} \to \mathbf{C}^T$ such that



Constructions within categories (blackboard)

- Monomorphism (dual: epimorphism)
- Isomorphism
- Terminal object (dual: initial object)
- Finite products (dual: coproducts)
- Equalizers (dual: coequalizers)
- Limits (dual: colimits)

Monomorphisms and epimorphisms

Let $f : A \to B$ be a morphism in a category. Then f is called a *monomorphism* (or *monic*) if:

for all objects X, and all morphisms $g, h : X \to A$,

$$f \circ g = f \circ h \quad \Rightarrow \quad g = h.$$

$$X \overset{g}{\underset{h}{\rightrightarrows}} A \xrightarrow{f} B.$$

The dual concept is called an *epimorphism* (or *epic*).

Isomorphisms

A morphism $f : A \to B$ in a category is called an *isomorphism* if it is invertible, i.e., there exists some $g : B \to A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$.

A natural transformation $\alpha : F \to G$ is called a *natural isomorphism* if $\alpha_A : FA \to GA$ is an isomorphism for all A.

A category in which *all* morphisms are invertible is called a *groupoid*. (Or in case there is only one object, it is called a *group*).

Example: in **Set**, monomorphism = injective, epimorphism = surjective, isomorphism = bijective.

In **Top**, monomorphism = injective, epimorphism = dense, isomorphism = homeomorphism.

Terminal object

An object A in a category is called *terminal* if:

for all objects X, there exists a *unique* morphism $g : X \to A$.

Note: a terminal object, if it exists, is unique up to isomorphism.

The dual concept is called an *initial* object.

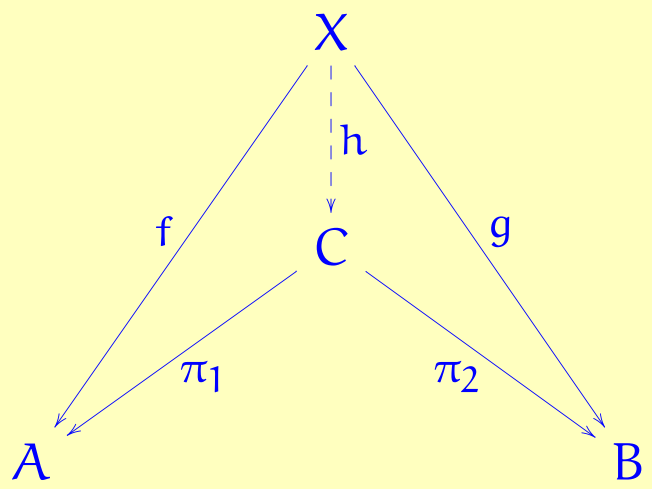
Example: in **Set**, $1 = \{*\}$ is terminal and $0 = \emptyset$ is initial.

In **Vec**, **Grp**, and **Set**$_\perp$, 1 is initial and terminal.

Categorical product

Let A, B be objects in a category. A *categorical product* of A and B is a triple (C, π_1, π_2), where C is an object, $\pi_1 : C \to A$ and $\pi_2 : C \to B$ are morphisms, and such that the following property holds:

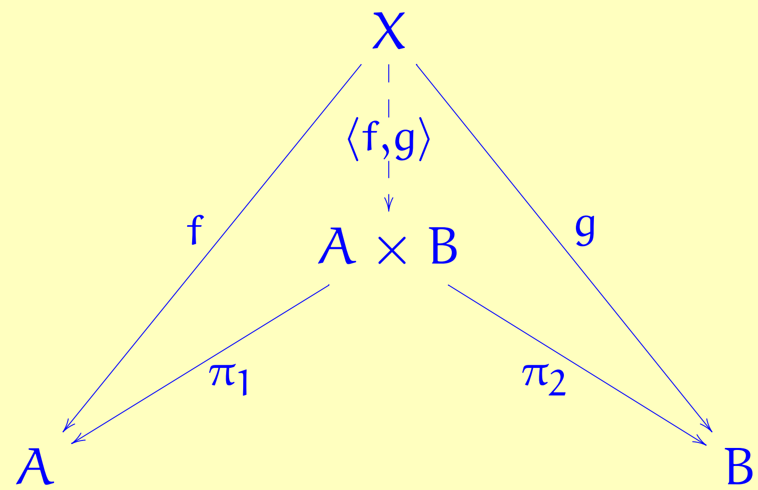
For all objects X and all morphisms $f : X \to A$ and $g : X \to B$, there exists a *unique* morphism $h : X \to C$ such that $f = \pi_1 \circ h$ and $g = \pi_2 \circ h$.



Categorical product, continued

Note: a categorical product, if it exists, is unique up to isomorphism.

Notation: we often write $C = A \times B$, $h = \langle f, g \rangle$.



Example: In **Set**, **Grp**, **Top**, **Vec**, **Pos**, categorical product is cartesian product (with the pointwise structure).

In a poset, categorical product is *meet*, i.e., *greatest lower bound*.

Products, continued

Proposition. In any category where they exist, categorical products satisfy

$$(A \times B) \times C \cong A \times (B \times C), \qquad A \times B \cong B \times A$$

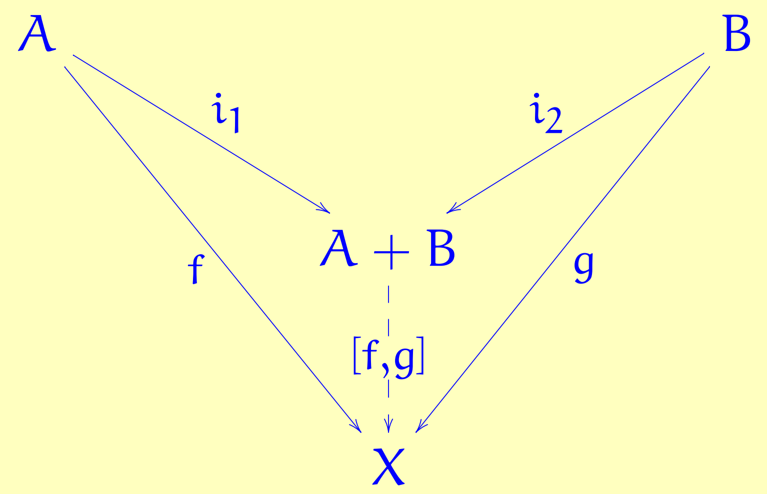
If 1 is a terminal object, then we also have

$$1 \times A \cong A \cong A \times 1.$$

Moreover, the above isomorphisms are natural.

Categorical coproduct

The dual concept of a product is a *coproduct*.

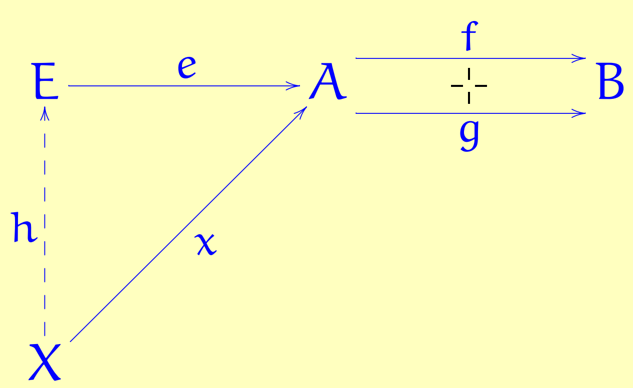


Example: In **Set**, coproduct is disjoint union. In **Vec** and **Ab** coproduct is direct sum. What is the coproduct, if any, in $\mathbf{Set}_\perp$?

Equalizers

Let $f, g : A \to B$ be morphisms in a category. An *equalizer* of f and g is a pair (E, e) where E is an object, $e : E \to A$ is a morphism, and such $f \circ e = g \circ e$, and such that the following property holds:

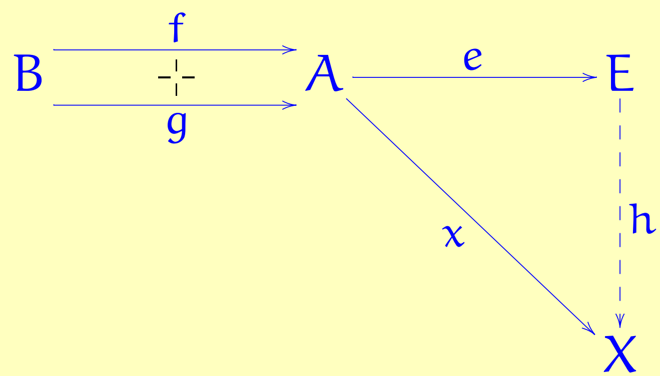
For all objects X and all morphisms $x : X \to A$ with $f \circ x = g \circ x$, there exists a *unique* morphism $h : X \to E$ such that $x = e \circ h$.



Example: in **Set**, an equalizer is the *graph of an equation*, i.e., $E = \{x \in A \mid f(x) = g(x)\}$.

Coequalizers

The dual concept of an equalizer is a coequalizer. In **Set**, this corresponds to the quotient of an *equivalence relation*.



Products and adjoints

Let $\mathbf{C}$ be a category with products. Consider the functors

$$F : \mathbf{C} \to \mathbf{C} \times \mathbf{C}, \quad G : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$$

given by

- $F(A) = (A, A)$ (and similarly on morphisms),
- $G(A, B) = A \times B$.

Then F is a *left adjoint* of G:

$$(\mathbf{C} \times \mathbf{C})((X, X), (A, B)) \cong \mathbf{C}(X, A \times B).$$

Concretely, this means that morphisms $(f, g) : (X, X) \to (A, B)$ in $\mathbf{C} \times \mathbf{C}$ are in natural bijective correspondence with morphisms $h : X \to A \times B$.

Moreover, because adjoints are unique, this property is *equivalent* to the definition of products.