Composing functors

Horizontal composition (functors):

$$\mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{G} \mathbf{E}$$

If F, G are functors, then so is $G \circ F$. Defined on objects as $(G \circ F)(A) = G(F(A))$ and on morphisms as $(G \circ F)(f) = G(F(f))$.

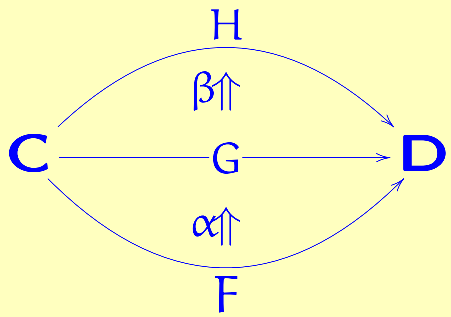
Identity (functors):

$$\mathbf{C} \xrightarrow{1_{\mathbf{C}}} \mathbf{C}$$

The identity functor $1_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ is defined as $1_{\mathbf{C}}(A) = A$ on objects and $1_{\mathbf{C}}(f) = f$ on morphisms.

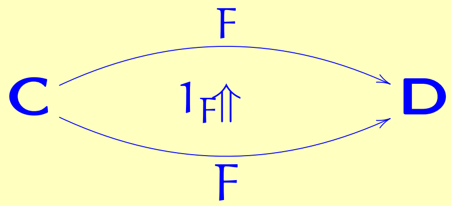
Composing natural transformations

Vertical composition (natural transformations):



If $\alpha : F \to G$ and $\beta : G \to H$ are natural transformations, then so is $\beta \bullet \alpha : F \to H$. If it defined by $(\beta \bullet \alpha)_A = \beta_A \circ \alpha_A : FA \to HA$.

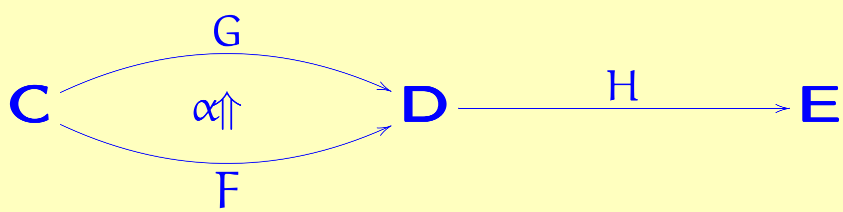
Identity (natural transformations):



The identity natural transf. $1_F : F \to F$ is defined as $(1_F)_A = 1_{FA}$. By abuse of notation, we sometimes denote 1_F by 1, or even F.

Composing natural transformations, continued

Whiskering (right):



If $F, G : \mathbf{C} \to \mathbf{D}$ and $H : \mathbf{D} \to \mathbf{E}$ are functors, and if $\alpha : F \to G$ is a natural transformation, the *right whiskering*

$$H \circ \alpha : H \circ F \to H \circ G$$

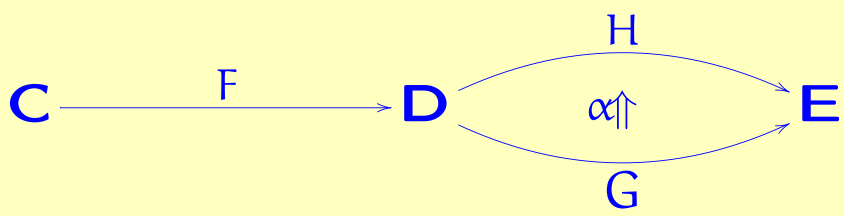
is defined as $(H \circ \alpha)_A : H(FA) \to H(GA)$ by $(H \circ \alpha)_A = H(\alpha_A)$. This is indeed a natural transformation, i.e.,

$$\begin{array}{ccc} H(FA) & \xrightarrow{H(\alpha_A)} & H(GA) \\ {\scriptstyle H(Ff)}\downarrow & & \downarrow{\scriptstyle H(Gf)} \\ H(FB) & \xrightarrow{H(\alpha_B)} & H(GB). \end{array}$$

In this case, it follows from the naturality of α and the functoriality of H.

Composing natural transformations, continued

Whiskering (left):



If $F : \mathbf{C} \to \mathbf{D}$ and $G, H : \mathbf{D} \to \mathbf{E}$ are functors, and if $\alpha : G \to H$ is a natural transformation, the *left whiskering*

$$\alpha \circ F : G \circ F \to H \circ F$$

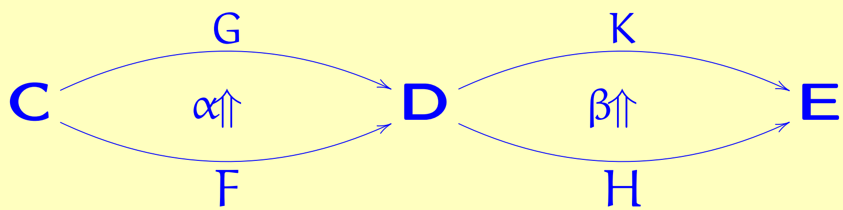
is defined as $(\alpha \circ F)_A : G(FA) \to H(FA)$ by $(\alpha \circ F)_A = \alpha_{FA}$. This is indeed a natural transformation, i.e.,

$$\begin{array}{ccc} G(FA) & \xrightarrow{\alpha_{FA}} & H(FA) \\ {\scriptstyle G(Ff)}\downarrow & & \downarrow{\scriptstyle H(Ff)} \\ G(FB) & \xrightarrow{\alpha_{FB}} & H(FB). \end{array}$$

In this case, it follows from the naturality of α.

Composing natural transformations, continued

Horizontal composition (natural transformations):



If $F, G : \mathbf{C} \to \mathbf{D}$ and $H, K : \mathbf{D} \to \mathbf{E}$ are functors, and if $\alpha : F \to G$ and $\beta : H \to K$ are natural transformations, the *horizontal composition*

$$\beta \circ \alpha : H \circ F \to K \circ G$$

can be defined in two different ways:

- Right whiskering followed by left whiskering:
  $\beta \circ \alpha = (\beta \circ G) \bullet (H \circ \alpha)$

- Left whiskering followed by right whiskering:
  $\beta \circ \alpha = (K \circ \alpha) \bullet (\beta \circ F)$.

Composing natural transformations, continued

- Right whiskering followed by left whiskering:
  $\beta \circ \alpha = (\beta \circ G) \bullet (H \circ \alpha)$

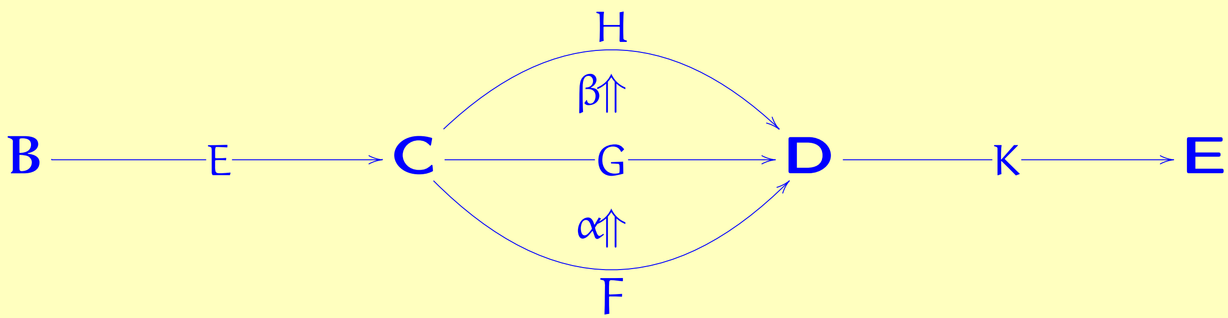
- Left whiskering followed by right whiskering:
  $\beta \circ \alpha = (K \circ \alpha) \bullet (\beta \circ F)$.

The two definitions coincide, because
$[(\beta \circ G) \bullet (H \circ \alpha)]_A = \beta_{GA} \circ H(\alpha_A)$, and
$[(K \circ \alpha) \bullet (\beta \circ F)]_A = K(\alpha_A) \circ \beta_{FA}$, and

$$\begin{CD} H(FA) @>{H(\alpha_A)}>> H(GA) \\ @V{\beta_{FA}}VV @VV{\beta_{GA}}V \\ K(FA) @>>{K(\alpha_A)}> K(GA). \end{CD}$$

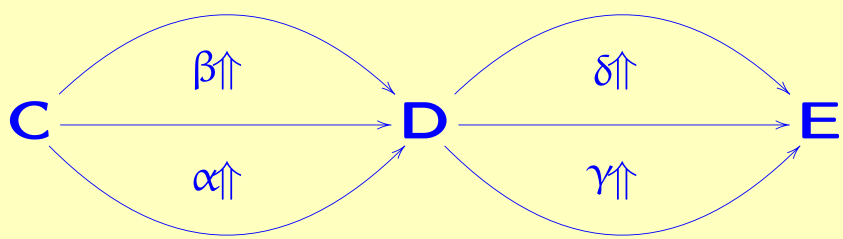
by naturality of β.

Some laws about whiskering



$$K \circ (\beta \bullet \alpha) = (K \circ \beta) \bullet (K \circ \alpha)$$
$$K \circ 1_F = 1_{K \circ F}$$
$$1_K \circ \alpha = K \circ \alpha$$
$$1_K \circ 1_F = 1_{K \circ F}$$
$$(\beta \bullet \alpha) \circ E = (\beta \circ E) \bullet (\alpha \circ E)$$
$$1_F \circ E = 1_{F \circ E}$$
$$\alpha \circ 1_E = \alpha \circ E$$

The double interchange law



$$(\delta \circ \beta) \bullet (\gamma \circ \alpha) = (\delta \bullet \gamma) \circ (\beta \bullet \alpha)$$

Example: The list monad

Recall that $F(A) = A^*$ is the *list monad*. Here A^* is the set of finite *lists* (also known as *words*, *strings*) of elements from A.

F is a monad as follows:

- Functor: for $f : A \to B$, define $f^* : A^* \to B^*$ by
$$f^*[a_1, \ldots, a_n] = [f(a_1), \ldots, f(a_n)].$$
- Unit: we define $\eta_A : A \to A^*$ by
$$\eta_A(a) = [a] \quad \text{(singleton).}$$
- Multiplication: we define $\mu_A : A^{**} \to A^*$ by
$$\mu_A([l_1, l_2 \ldots, l_n]) = l_1 \cdot l_2 \cdot \ldots \cdot l_n.$$

Verify the monad laws.

Free algebras

Let Σ be a *signature*, and let E be a set of *equations* (both in the sense of universal algebra).

A signature consists of a set $|\Sigma| = \{f, g, \ldots\}$ of *function symbols*, together with an assignment $\mathsf{ar} : |\Sigma| \to \mathbb{N}$ of an *arity* to each function symbol.

Fix a signature. For example, let h be a function symbol of arity 2, and let g be a function symbol of arity 1.

Let V be a set of *variables*. Then we can form the set of *terms*, e.g.:

$$\begin{array}{l} x, \quad y, \quad g(x), \quad g(y), \quad h(x,x), \quad h(x,y), \\ h(g(x),y), \quad h(g(g(x)),x), \quad g(h(x,g(h(y,x)))), \ldots \end{array}$$

Let $\mathsf{Terms}_\Sigma(V)$ be this set of terms.

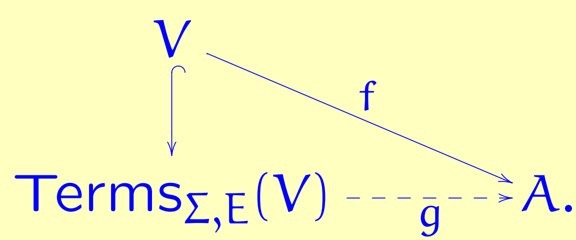
Free algebras, continued

On the set $\mathrm{Terms}_\Sigma(V)$, consider the smallest equivalence relation $\sim_E$ such that:

$$\frac{(t = s) \in E}{t' \sim_E s'} \qquad \frac{t_1 \sim_E s_1, \quad \ldots, \quad t_n \sim_E s_n}{f(t_1, \ldots, t_n) \sim_E f(s_1, \ldots, s_n)}$$

Then $\mathrm{Terms}_\Sigma(V)/\sim_E$ is a (Σ, E)-algebra. We denote it by $\mathrm{Terms}_{\Sigma,E}(V)$.

In fact, it is the *free* (Σ, E)-algebra generated by V. Concretely, this means: for any (Σ, E)-algebra A, and any function $f : V \to A$, there exists a unique homomorphism of (Σ, E)-algebras $g : \mathrm{Terms}_{\Sigma,E}(V) \to A$ such that



The term monad

Fix Σ and E. Consider the functor $T : \mathbf{Set} \to \mathbf{Set}$ given by

$$T(V) = \mathrm{Terms}_{\Sigma,E}(V).$$

This is a monad:

- Functor: for $f : V \to W$, define $T(f) : \mathrm{Terms}_{\Sigma,E}(V) \to \mathrm{Terms}_{\Sigma,E}(W)$ by "renaming" all the variables in a term.

- Unit: $\eta_V : V \to \mathrm{Terms}_{\Sigma,E}(V)$ maps a variable x to the term x.

- Multiplication: $\mu_V : T(T(V)) \to T(V)$ takes a term whose "variables" are other terms. It is defined by "flattening" this structure into a single term.

Check the monad laws!

The list monad as a term monad

In fact, the list monad $A \mapsto A^*$ is the term monad for operations "$\cdot$" (arity 2), e (arity 0), with equations

$$(x \cdot y) \cdot z = x \cdot (y \cdot z), \quad e \cdot x = x, \quad x \cdot e = x.$$

In other words, A^* is the *free monoid* on A. Also:

- $T(A) = A + \bot$ is the term monad over the signature $\Sigma = \{\bot\}$ (arity 0, no equations);
- $T(A) = \mathscr{P}^{\mathrm{fin}}(A)$ is like the list monad, with the additional equations

$$x \cdot x = x, \quad x \cdot y = y \cdot x;$$

- $T(A) = \mathscr{P}^{\mathrm{fin},+}$ is the same, but without the constant e;
- $T(A) = A \times \Sigma^*$ is the term monad over the signature $\{w_c \mid c \in \Sigma\}$, each with arity 1.

An alternative definition of monad [E. Manes]

Let $\mathbf{C}$ be a category, and let $T : |\mathbf{C}| \to |\mathbf{C}|$ be a function on objects (here *not* a priori assumed to be a functor).

Suppose that T is equipped with the following two operations:

$$\frac{}{\eta_A : A \to TA} \qquad \frac{f : A \to TB}{\mathrm{lift}(f) : TA \to TB}$$

Satisfying:

$(a)\ \mathrm{lift}(\eta_A) = 1_{TA} \quad (b)\ (\mathrm{lift} f)\circ\eta_A = f \quad (c)\ \mathrm{lift}((\mathrm{lift} g)\circ f) = (\mathrm{lift} g)\circ(\mathrm{lift} f)$

Note: then T can be made into a functor like this:

$$\frac{\dfrac{f : A \to B}{\eta_B \circ f : A \to TB}}{\mathrm{lift}(\eta_B \circ f) : TA \to TB}$$

Exercise: prove that this is an equivalent definition of monad.

Kleisli category of a monad: $\mathbf{C}_T$

Let (T, η, μ) be a monad on a category $\mathbf{C}$. Its *Kleisli category* $\mathbf{C}_T$ is defined as follows:

- Objects: $\mathbf{C}_T$ has the same objects as $\mathbf{C}$.
- Morphisms: $\mathbf{C}_T(A, B) = \mathbf{C}(A, TB)$.
- Identities and composition:

$$\mathrm{id}_T : A \to TA \qquad\qquad \frac{f : A \to TB \quad g : B \to TC}{g \circ_T f : A \to TC}$$

Then $\mathbf{C}_T$ is a well-defined category. Moreover, there is a canonical functor $F : \mathbf{C} \to \mathbf{C}_T$ mapping A to A and f to $\eta_B \circ f$, and a canonical functor $G : \mathbf{C}_T \to \mathbf{C}$ mapping A to TA and g to $\mathrm{lift}(g)$.

Algebras of a monad: $\mathbf{C}^T$

Let (T, η, μ) be a monad on a category $\mathbf{C}$.

Definition. An *algebra* for T is a pair (A, a), where A is an object of $\mathbf{C}$, and $a : TA \to A$ is a morphism, satisfing

$$\begin{array}{ccc} T^2A & \xrightarrow{Ta} & TA \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle a} \\ TA & \xrightarrow{a} & A, \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & {\scriptstyle 1_A}\searrow & \downarrow{\scriptstyle a} \\ & & A. \end{array}$$

Given two algebras (A, a) and (B, b), a *homomorphism* is given by a map $f : A \to B$ satisfying

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow{f} & B. \end{array}$$

Consider what this means in case of the term monad for (Σ, E).

Eilenberg-Moore category of a monad: $\mathbf{C}^T$

Let (T, η, μ) be a monad on a category $\mathbf{C}$. Its *Eilenberg-Moore category* $\mathbf{C}^T$ is defined as follows:

- Objects: algebras (A, a) for the monad T.
- Morphisms: algebra homomorphisms.
- Identities and composition: as in $\mathbf{C}$.

Then $\mathbf{C}^T$ is a well-defined category. Moreover, there is a canonical functor $F : \mathbf{C} \to \mathbf{C}^T$ mapping A to (TA, μ_A) and f to Tf. There is also a canonical functor $G : \mathbf{C}^T \to \mathbf{C}$ mapping (A, a) to A.