ORDERED ALGEBRAS AND LOGIC

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Abstract. Ordered algebras such as Boolean algebras, Heyting algebras, lattice-ordered groups, and MV-algebras have long played a decisive role in logic, although perhaps only in recent years has the significance of the relationship between the two fields begun to be fully recognized and exploited. The first aim of this survey article is to briefly trace the distinct historical roots of ordered algebras and logic, culminating with the theory of algebraizable logics, based on the pioneering work of Lindenbaum and Tarski and Blok and Pigozzi, that demonstrates the complementary nature of the two fields. The second aim is to explain and illustrate the usefulness of this theory, both from an ordered algebra and logic perspective, in the context of the relationship between residuated lattices and substructural logics. In particular, completions on the ordered algebra side, and Gentzen systems on the logic side, are used to address properties such as decidability, interpolation and amalgamation, and completeness.

1. Introduction

Ordered algebras such as Boolean algebras, Heyting algebras, lattice-ordered groups, and MV-algebras have long played a decisive role in logic, both as the models of theories of first (or higher) order logic, and as algebraic semantics for the plethora of non-classical logics emerging in the twentieth century from linguistics, philosophy, mathematics, and computer science. Perhaps only in recent years, however, has the full significance of the relationship between ordered algebras and logic begun to be recognized and exploited. The first pioneering and revelatory step was taken already by Tarski and Lindenbaum back in the 1930's, who showed that despite their different origins and motivations, Boolean algebras and propositional classical logic may be viewed in a certain sense as two sides of the same coin.

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A crucial element in the development of this relationship was Tarski's definition of an abstract concept of logical consequence. He was guided by the intuition that a consequence relation should specify when a single formula (the conclusion) follows from a set of formulas (the premises), satisfying only three natural constraints: (i) every formula follows from itself (*reflexivity*); (ii) whatever follows from a set of premises also follows from any larger set of premises (*monotonicity*); and (iii) whatever follows from consequences of a set of premises also follows from the set itself (*cut*).

Prototypical sources of consequence relations are axiomatic logical calculi, via the standard definition of formal proof: a formula α is a consequence of a set of formulas Γ (relative to the calculus C) when α is provable in C using assumptions in Γ. However, in Tarski's work there is also the implicit idea that behind each *class of algebras* lies hidden (at least) one consequence relation. In fact, every class $\mathcal{K}$ of algebras has an associated *equational consequence relation* $\vdash_{Eq(\mathcal{K})}$ that holds between a set of equations E and a single equation ε (in the appropriate language) if and only if ε is satisfied in every algebra $\mathbf{A} \in \mathcal{K}$ which satisfies every member of E. In itself, this is not a consequence relation in Tarski's sense – it is defined on equations rather than on formulas – but is readily seen to enjoy similar properties of reflexivity, monotonicity, and cut. A few stipulations, moreover, ensure that it can be converted into a Tarskian consequence relation. First, an equation $\alpha \approx \beta$ can be viewed as a pair (α, β) of formulas. Second, a map τ from formulas to sets of equations should be specified that defines for every $\mathbf{A} \in \mathcal{K}$, a "truth set" T in the following sense: the algebra $\mathbf{A} \in \mathcal{K}$ satisfies $\tau(\alpha)$ just in case every valuation of α on $\mathbf{A}$ maps it to a member of T (intuitively: the meaning of α in $\mathbf{A}$ belongs to the set of "true values" and hence is true). This immediately yields a consequence relation in Tarski's sense, defined by

$$\Gamma \vdash^{\tau}_{\mathcal{K}} \alpha \quad \text{iff} \quad \{\tau(\beta) \mid \beta \in \Gamma\} \vdash_{Eq(\mathcal{K})} \tau(\alpha).$$

Things, of course, get interesting when this consequence relation happens to coincide with the deducibility relation of a well understood logical calculus; this is the situation precisely for the variety $\mathcal{BA}$ of Boolean algebras and classical propositional logic. What Tarski (developing an idea by Lindenbaum) showed, is that the consequence relation $\vdash_{\text{HCL}}$ extracted from the axiomatic calculus HCL for classical propositional logic is exactly $\vdash^{\tau}_{\mathcal{BA}}$ for $\tau(\alpha) = \{\alpha \approx 1\}$. In other words, Boolean algebras form an *algebraic semantics* for classical propositional logic.

Although Tarski's approach was successfully extended to non-classical logics in succeeding years, becoming a standard tool for their investigation, it took until the 1980's and the work of Blok and Pigozzi for a formal account of this general correspondence between logics and classes of algebras to appear. In particular, Blok and Pigozzi pointed out that the algebraic semantics of a given logic need not be unique; moreover, the link provided by this relation is weak in several respects. They introduced a stronger notion of *equivalent algebraic semantics* for a logic, calling a logic *algebraizable* if it has a class $\mathcal{K}$ of algebras (unique, if $\mathcal{K}$ is a quasivariety) as equivalent algebraic semantics. This account was subsequently extended in many directions, and indeed now forms the basis for the active area of research known as *abstract algebraic logic*.

Narrowing our scope for the purposes of this article, we will focus here on the relationship between *substructural logics* and *residuated lattices*. On the one hand, substructural logics encompass many important non-classical logics such as the full Lambek calculus, linear logic, relevance logics, and fuzzy logics. On the other hand, residuated lattices, as well as providing algebraic semantics for these logics, also feature in areas interesting from the order-algebraic perspective such as the theory of lattice-ordered groups, vector lattices (or Riesz spaces), and abstract ideal theory. Intriguingly, in tackling problems from these fields, methods from both algebra and logic seem to be essential. In particular, algebraic methods have been used to address completeness problems for Gentzen systems, while these systems have themselves been used to establish decidability and amalgamation properties for classes of algebras.

The first aim of this survey article is, through Sections 2 and 3, to briefly trace the distinct historical roots of ordered algebras and logic, culminating with the theory of algebraizable logics, based on the pioneering work of Lindenbaum and Tarski and Blok and Pigozzi, that demonstrates the complementary nature of the two fields. The second aim is to explain and illustrate the usefulness of this theory for ordered algebra and logic in the context of residuated lattices and substructural logics, described in Sections 4 and 5, respectively. In particular, we will explain in Section 6, how completions on the ordered algebra side, and Gentzen systems on the logic side, are used to address properties such as decidability, interpolation and amalgamation, and completeness.

We will assume that the reader is familiar with the basic facts, definitions, and terminology from universal algebra and lattice theory. In particular, the notions *partially ordered set* or *poset* for short, *lattice*, and *algebra* are central to this paper, as are the concepts of *congruence relation*, *homomorphism*, and *variety*. For an introduction to universal algebra, the reader may wish to consult [23], [63], or [97], while any of [7], [37], or [62] would serve as a suitable lattice theory reference. A comprehensive treatment of residuated lattices and propositional substructural logics is presented in [51].

2. The Logic of Mathematics

No survey on the relationship between algebra and logic can eschew the more general issue of an assessment of the role of logic in mathematics, and, for that matter, of the role of mathematics in logic. The very expression "mathematical logic" has repeatedly been recognized as ambiguous, at least to some extent, between an investigation into the logical foundations of mathematical theories and an analysis of logical reasoning carried out with the aid of mathematical tools. Roughly expressed, "mathematical logic" is ambiguous between "the mathematics of logic" and the "logic of mathematics."[1] In the first part of this paper, we will trace the antecedents of these two approaches by means of a short and necessarily perfunctory historical survey.

2.1. Aristotle and the Stoics.

The study of logic as an independent discipline began with Aristotle (384-322 B.C.E.). In his *Organon* (*Λογικά* or *Ὄργανον*), as is well known, he analyzed the structure of arguments having syllogistic form, providing a canon of valid reasoning that would be virtually undisputed for centuries to come. Despite appearances to the contrary – after all, Aristotelian logic is what the founders of modern mathematical logic wanted to go beyond – it does not seem inappropriate to include Aristotle's contributions among the attempts to carry out a logical analysis of mathematical reasoning. Let us recall, in fact, that he sharply distinguished between the formal development of his general theory of syllogism and its application to scientific discourse on the one hand, carried out in the *Prior Analytics* (*Ἀναλυτικά Πρότερα*) and in the *Posterior Analytics* (*Ἀναλυτικά Ὕστερα*) respectively, and the analysis of the logical structure of everyday reasoning on the other, reserved instead for the *Topics* and the *Sophistical Refutations* (*Τοπικοί* and *Σοφιστικοί Ἔλεγχοι*).

Moreover, it should be observed that, although this may sound blasphemous to a modern ear – syllogistic logic being just a decidable barely expressive fragment of first order logic – Aristotle considered his theory of syllogism completely adequate for the formalization of

[1]This observation is not as trite as it might seem. Kreisel, for one, does not accept the former meaning: "'Mathematical logic' [...] refers to mathematical methods used in logic [...]; in analogy to 'mathematical physics' which means the mathematics of physics (and not the physics of mathematics)" ([87]).

all mathematical inference. Actually, Aristotle even tried to provide a *proof* of this claim in two interesting passages from his *Prior Analytics* (I, 25.41b36-42a32 and I, 23.40b18-41a21).

More precisely, what Aristotle attempts is the following. The definition of syllogism initially given in the *Prior Analytics* is extremely general in nature:

> A syllogism is discourse in which, certain things being stated, something other than what is stated follows of necessity from their being so. I mean by the last phrase that they produce the consequence, and by this, that no further term is required from without in order to make the consequence necessary (I, 1.24b19).

This definition would seem to subsume (although it is of course not limited to) all kinds of mathematical reasoning. Later in the book, however, Aristotle provides a different, more technical, definition of syllogism where precise constraints are introduced both as to the form of the propositions involved and regarding the structure of the inferences themselves. In the above-mentioned passages, Aristotle essentially argues that every syllogism in the first sense is also a syllogism in the second sense. His purported proof is quite obscure and full of holes, relying on a number of controversial assumptions; moreover, Aristotle is forced to constrain somehow his vague initial concept of syllogism in order to compare it with the other one, so much so that it is not at all clear whether all mathematical inferences can still be encompassed by the definition ([33], [89], [123]). Nonetheless, his inconclusive argument is clear evidence that Aristotle considered his theory of syllogism as describing, among other things, *the* logical structure of mathematical reasoning.

This conviction was explicitly challenged already in the Antiquity. The Stoics, who are as a rule collectively mentioned as the second most important contributors to the development of ancient logic, as well as being credited with the first noteworthy analysis of propositional connexion, drew a sharp distinction between syllogisms (whether Aristotle's categorical syllogisms or their own new brands of propositional syllogisms, hypothetical and disjunctive) and non-formally valid enthymemes, called "unmethodically valid arguments" (*λόγοι ἀμεθόδως περαίνοντες*). Examples of such arguments, according to the Stoics, abound in concrete mathematical proofs; therefore Aristotle's syllogistics does not provide a complete basis for mathematical reasoning ([8]).

2.2. **Leibniz and Wolff.** Gottfried Wilhelm Leibniz (1646-1716) is sometimes cited as a forerunner of modern mathematical logic, and even as the thinker who established the discipline long before Boole or Frege. Be this as it may, the importance of Leibniz in the development of logic and his appreciation of logical reasoning as a basis for science are acknowledged and unquestioned. Instead, the contribution of another XVII c. German philosopher, Christian Wolff (1679-1754), is less well-known. Under the influence of Leibniz, he disavowed his own early devaluative views on the role of logic in mathematics and came to embrace a revised version of Aristotle's thesis: the theory of categorical syllogism, as supplemented by the Scholastics, is sufficient to formalize the inferences contained in *all* mathematical proofs. In his *Philosophia Rationalis Sive Logica* (1728), he boldly tried to prove his claim following a two-step strategy:

(1) First, he tried to show that some representative examples of mathematical proofs[2] were reducible to a finite sequence of sentences, each of which was either a definition, or an axiom, or a previously established theorem, or could be obtained from preceding sentences through the application of a (categorical, hypothetical, or disjunctive) syllogism or of a one-premise propositional inference.
(2) Second, he completed his reduction process with an argument to the effect that hypothetical and disjunctive syllogisms and one-premise inferences can be reformulated as categorical syllogisms.

True to form, Wolff's attempt was far from successful. If we read his analysis carefully, we see that he interspersed his reconstructions of geometrical proofs with "intuitive judgments" (*iudicia intuitiva*), introduced by such phrases as "It is intuitively evident that..." or "Looking at the figure, we intuitively know that...", and drawn from the observation of the figure or from the constructions he had previously carried out. Of course, these "intuitive judgments" do not count as definitions or axioms; hence their presence invalidates the first step of Wolff's strategy from the start (by the way, his second step was just

[2]Interestingly enough, all the examples of proofs analyzed by Wolff belong to elementary Euclidean geometry, although he was familiar (as witnessed by his teaching syllabi and notes collected in his *Ratio Praelectionum Wolffianarum*) with the latest developments of calculus and algebra. Most probably, only Euclidean geometry met the standards of rigor he deemed necessary for his investigation.

as gappy). Influenced by the traditional Euclidean model of geometrical proof, which provided for a sharp distinction between the logico-deductive part – the *apódeixis* (ἀπόδειξις), or proof proper – and the constructive, synthetic part embodied by the constructions carried out on the figure – the *ékthesis* (ἔκθεσις) and the *kataskeué (κατασκευή):* ([70]) – Wolff was only concerned with a logical reconstruction of the former, while he regarded the latter as somehow foreign to the body of the proof. In sum: categorical syllogistics may or may not have been sufficient for a formalization of the narrow strictly deductive fragment of a standard Euclidean proof, but of course it was not enough for a formalization of the proof as a whole[3].

2.3. **Bolzano.** Bernard Bolzano (1781-1848) brought reflection on the logical foundations of mathematics to unprecedented levels of awareness and depth. A mathematician by trade, he sought from the very beginning of his career (e.g., in his *Beyträge zu einer begründeteren Darstellung der Mathematik*, 1810) to establish on a firmer ground the foundations of the mathematical disciplines. In accordance with the time-honored tradition of "doctrine of method" leading from the French XVII c. theorists (Pascal, Descartes, Arnauld) to his more recent antecedents Lambert, Crusius, and Kant, Bolzano believed that logic is instrumental for mathematics in that it serves as a preliminary methodological framework for stating the rules that properly found each mathematical discipline.

The second volume of his monumental work *Wissenschaftslehre* (1837) contains a detailed and not yet fully appreciated development of a powerful logical system, including an analysis of logical consequence, viewed by some as an anticipation of Tarski. In particular, he considers the two relations of *derivability* (*Ableitbarkeit*) and *consecution* (*Abfolge*) among propositions. The former relation is, somewhat surprisingly, a multiple-conclusion one: the propositions $q_1, ..., q_m$ are said to be derivable from the propositions $p_1, ..., p_n$ with respect to the component concepts $a_1, ..., a_r$ iff the following two conditions are satisfied:

(1) There exists a sequence of concepts $b_1, ..., b_r$ such that, denoting by $p_i(a_1/b_1, ..., a_r/b_r)$ the result of uniformly substituting in p_i every occurrence of a_j by an occurrence of b_j, all the propositions

$$p_1(a_1/b_1, ..., a_r/b_r), ..., p_n(a_1/b_1, ..., a_r/b_r)$$

[3]According to some Kantian interpreters, Kant's claim to the effect that mathematics is synthetic a priori does not contradict Wolff's thesis that mathematics is based on syllogistic logic: Kant is referring to mathematical proof as a whole, Wolff to the *apódeixis* ([75], [49]).

are true (*compatibility clause*).

(2) Every such sequence of concepts $b_1, ..., b_r$ also makes all the conclusions true, i.e., the propositions

$$q_1(a_1/b_1, ..., a_r/b_r), ..., q_m(a_1/b_1, ..., a_r/b_r)$$

are true.

This notion is meant to formalize several features Bolzano requires of "good" mathematical proofs: the compatibility clause reflects Bolzano's desire, inherited from Arnauld and Kant, to assign direct proofs a higher status than ex absurdo proofs, while the fact that the derivability relation is relativized to a sequence of components – possibly a *proper* subsequence of the sequence of all non-logical concepts contained in the propositions at issue – is an attempt to describe an *enthymematic* consequence relation broad enough to encompass forms of reasoning currently employed in standard mathematical practice (cf. the Stoics' "unmethodically conclusive arguments" mentioned above).

The relation of consecution is even more interesting, since it goes some way towards investigating a stronger *causal* concept of consequence. To illustrate the difference between the two notions, consider the following propositions:

(1) In Rome the temperature is higher in August than in January.
(2) In Rome the mercury columns of thermometers are higher in August than in January.

(1) and (2) are derivable from each other (w.r.t. the component concepts "Rome", "August", "January"), but it is only (1) that causally implies (2), not vice versa. According to Bolzano, only the best mathematical proofs are made up by causal inferences, while in other cases there is at most a "transfer of evidence" from the premises down to the conclusions.

Consecution is not formally defined by Bolzano, but only characterized by means of a list of properties. Among such properties there is a remarkable one: if p and q are mathematical propositions and p causally implies q, p cannot be "more complex" than q. In one of the most extraordinary passages of his *Wissenschaftslehre* (§§ 216-221), Bolzano investigates in detail the formal structure of proof trees explicitly taken as mirroring the structure of actual mathematical proofs, and viewed as bottom-up proof processes of "going from a consequence up to its reasons": nodes in the trees correspond to mathematical propositions, and arcs to instances of the consecution relation. Taking advantage of the above-mentioned postulate of non-decreasing complexity, he tries to show that such proof trees must be analytic and therefore finite.

His proof turns out to be conclusive only under additional assumptions; however, it remains one of the earliest and most limpid examples of formal analysis of the logical structure of mathematical proofs.

2.4. **From Frege to Hilbert.** Gottlob Frege (1848-1925) articulated a view of the relation between logic and mathematics that was at once clear-cut and utterly controversial: mathematics *is* logic. Frege, indeed, defended a strong reductionist thesis nowadays known as *logicism*: mathematical concepts can be defined in terms of purely logical concepts, and mathematical principles can be derived from the laws of logic alone. As part of the job needed to prove this claim, he had of course to define numbers in logical terms. Frege did not start from scratch: he built upon the work of Dedekind, Cantor, and Weierstrass, who had managed around 1870 to reduce real numbers to rational numbers (Cantor, e.g., defined real numbers as certain equivalence classes of Cauchy sequences of rationals), and given similar reductions of rational numbers to integers and of integers to natural numbers. What remained then to complete the process was a reduction of natural numbers to logic.

To do this, Frege needed an adequate logical framework. Indeed, one of his greatest accomplishments was the introduction of a logical system that closely resembles an axiom system for second order quantifier logic in the modern sense, appearing first in his early work *Begriffsschrift* (1879) and then, in a more mature formulation, in his two-volume *Grundgesetze der Arithmetik* (1893-1903). Within this framework, he was able to provide a powerful argument for the logicist claim, proceeding via a (supposedly) logical definition of the concept of natural number.

It is well-known that Frege's attempt was not crowned with success. His logic relied on an unrestricted comprehension principle asserting the existence of a set (or a concept, as Frege would have put it) for every open formula with one free variable. This existential claim, apart from casting a shadow on the purely logical nature of his system, was shown to be inconsistent by Russell. The ultimate failure of Frege's program, however, should not obscure its merits and partial achievements. Frege, in fact, succeeded in deriving the Dedekind-Peano axioms for arithmetic in a consistent subsystem of his logic ([71], [19]). Moreover, as a by-product of his foundational work, he laid the groundwork for modern mathematical logic as an investigation into the logical foundations of mathematical theories.

In *Principia Mathematica* (1910-1913), coauthored by Bertrand Russell (1872-1970) and Alfred N. Whitehead (1861-1947), Frege's seminal

A1. $\alpha \to (\beta \to \alpha)$
A2. $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
A3. $\alpha \wedge \beta \to \alpha$
A4. $\alpha \wedge \beta \to \beta$
A5. $(\alpha \to \beta) \to ((\alpha \to \gamma) \to (\alpha \to \beta \wedge \gamma))$
A6. $\alpha \to \alpha \vee \beta$
A7. $\beta \to \alpha \vee \beta$
A8. $(\alpha \to \gamma) \to ((\beta \to \gamma) \to (\alpha \vee \beta \to \gamma))$
A9. $(\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$
A10. $\alpha \to (\neg\alpha \to \beta)$
A11. $\neg\neg\alpha \to \alpha$
A12. $(\alpha \to \alpha) \to 1$
A13. $1 \to (\alpha \to \alpha)$
A14. $0 \to \neg 1$
A15. $\neg 1 \to 0$

R1. $\dfrac{\alpha \quad \alpha \to \beta}{\beta}$ (Modus ponens)

Table 1. Hilbert-style calculus HCL for classical propositional logic.

work on the codification of logical principles in a formal calculus was improved, both notationally and conceptually. It is Russell and Whitehead's, not Frege's, formalization of logic that constituted the backdrop against which most logical research in the first thirty years of the last century – including Gödel's completeness and incompleteness theorems – was carried out. Building on this monumental piece of work, by the end of the 1920's David Hilbert (1862-1943) had essentially formulated the modern concepts of logical language, axiomatic calculus, and formal proof. Today, as a matter of fact, axiomatic logical calculi are antonomastically named *Hilbert-style calculi* in his honor. Later, he also codified in his two-volume *Grundlagen der Mathematik* [74], written with Paul Bernays, what is now considered to be the standard presentation of first order classical logic. Its propositional fragment HCL is reproduced, with a few inessential variants, in Table 1.

A common feature of the logicians we have just mentioned was their insistence on *syntactic* aspects of logic. Actually, it should be said that, prior to Frege, the distinction between syntax and semantics did not make any sense at all, in logic or any other mathematical theory: there was no difference, say, between a provable and a valid sentence in syllogistic logic, or between a consistent axiom system for geometry and a system admitting models. The revolution of the axiomatic method changed this state of affairs once and for all. Following Hilbert, a set of axioms for a mathematical theory was no longer a body of self-evident truths about an intended domain of objects, and proofs were no longer viewed as means to transfer evidence from axioms to theorems. Rather, axioms were thought of as arbitrarily designated sentences which implicitly define their own objects, and proofs as means to ensure that the relations enunciated by theorems hold in every possible domain of entities in which the axioms also hold. In logic, the availability of a rigorous concept of logical calculus made it possible to reduce all the claims concerning truth or validity, which otherwise would have to be checked in an "outer" domain of objects, to provability claims that it was possible to verify *within* the calculus itself.

3. The Mathematics of Logic

Uncovering the logical structure of mathematical proofs and mathematical theories is not the same as trying to formalize reasoning – on *any* subject matter – by mathematical means. Although the mainstream tradition in early XX c. mathematical logic, leading from Frege and Russell to Hilbert and Gödel, can be categorized under the former heading, outstanding contributions to the shaping of contemporary logic were also made by a second important stream, generally referred to as "algebra of logic", which eventually converged with the first stream into a unique research domain. George Boole (1815-1864) was a pioneer of this approach, while Stanley Jevons (1835-1882), Charles Sanders Peirce (1839-1914) and Ernst Schröder (1841-1902) followed in his footsteps. Two leading figures in the foundational research of the pre-World War Two period, Leopold Löwenheim (1878-1957) and Thoralf Skolem (1887-1963) can also be seen as belonging, at least to some extent, to this tradition.

3.1. **The early tradition in the algebra of logic.** According to many interpreters, Boole's *Mathematical Analysis of Logic*, published in 1847, and the expanded version of the treatise appearing in 1854 under the title *An Investigation of the Laws of Thought,* mark the official birth of modern mathematical logic. In these milestone volumes, Boole admittedly sought no less than to "investigate the fundamental laws of those operations of the mind by which reasoning is performed, to give expression to them in the symbolical language of a calculus, and upon this foundation to establish the science of Logic and construct its method." Despite this somewhat bombastic statement of purpose, Boole did not depart from tradition as radically as it might seem, because one of his main concerns – as for his contemporary Augustus De Morgan (1806-1871) – was providing an algebraic treatment of Aristotelian syllogistic logic, which occupies about one third of his *Mathematical Analysis of Logic*. Boole started with the assumption that ordinary logic is concerned with assertions that can be considered assertions about *classes* of objects. He then translated the latter into equations in the language of classes. His approach contained numerous errors, partly due to his insistence that the algebra of logic should behave like ordinary algebra, but offered significant new perspectives.

Although the name "Boolean algebra" might suggest that the inventor of this concept was Boole, there is by now widespread agreement among the scholarly community that he was not. Not that it is always easy to clearly understand what Boole had in mind when working on his calculus of classes: as Hailperin puts it ([68], p. 61),

> If we look carefully at what Boole actually did [...], we find him carrying out operations, procedures, and processes of an algebraic character, often inadequately justified by present-day standards and, at times, making no sense to a modern mathematician. [...] Boole considered this acceptable so long as the end result could be given a meaning.

The reason why Boole's calculus cannot be considered as a first incarnation of a Boolean algebra of sets is the fact that the two operations of *combination* of two classes x and y (written xy in Boole's notation) and of *aggregation* of x and y (written $x + y$) do correspond to intersection and union, respectively, but the latter only makes sense when x and y are disjoint classes. The algebras to be found in Boole's work bear therefore some resemblance to partial algebras in the modern sense, except for the fact that often Boole happily disregarded his disjointness condition throughout his calculations, seemingly finding such a procedure unobjectionable provided the final result did not violate the condition itself.

Boole preceded an array of researchers who tried to develop further his idea of turning logical reasoning into an algebraic calculus. The already mentioned Stanley Jevons, Charles Sanders Peirce, and Ernst Schröder took their cue from Boole's investigations, yet all of them suggested their own improvements and modifications to the work of their predecessor.

Jevons, in particular, was dissatisfied with Boole's choice of primitive set-theoretical operations. He did not like the fact that aggregation was a partial operation, undefined on pairs of classes with nonempty intersection. In his *Pure Logic* (1864), he suggested a variant of Boole's calculus which he proudly advertised as based only on "processes of self-evident meaning and force." He viewed + as a total operation making sense for any pair of classes, and essentially corresponding to set-theoretic union. He also showed that all the expressions he used remained interpretable throughout the intermediate steps of his calculations, thereby overcoming one of the main drawbacks of Boole's work ([67]).

The main contribution to logic of the versatile Peirce – a philosopher, mathematician, and authority on several pure and applied sciences – is usually attributed to the foundation of the algebra of relations, for which Schröder also made significant developments. However, in the 12,000 pages of his published work – rising to an astounding 90,000 if we take into account his unpublished manuscripts – much more can be found. He investigated the laws of propositional logic, discovering that all the usual propositional connectives were definable in term of the single connective NAND; he introduced quantifiers, although, unlike Frege, he did not go so far as to suggest an axiomatic calculus for quantified logic. He conceived of complex and fascinating graphs by which he could represent logical syntax in two or even three dimensions. For all these achievements, however, his impact on logic would not be even remotely comparable to that exerted by Frege ([61]).

Let us finally mention that, contrary to the dominant paradigm of Hilbertian formalism described in Subsection 2.4, adherents to the algebra of logic tradition inherited from Boole a markedly anti-formalistic stance on logic and mathematics: mathematical language was seen as a system of *interpreted* symbols, and semantical notions like validity or satisfiability were accorded priority over their syntactical counterparts. This view had some unfortunate consequences. Skolem, for example, came very close to proving the completeness theorem for first order logic, but refrained from giving it an explicit formulation because he viewed consistency of a system as equivalent *by definition* to satisfiability [24].

3.2. **Lindenbaum and Tarski.** Roughly at the same time as Hilbert and Bernays wrote what is now considered as the standard presentation of first order classical logic, Emil Leon Post (1897-1954), Jan Łukasiewicz (1878-1956), and Clarence Irving Lewis (1883-1964), among others, introduced the first Hilbert-style calculi for some propositional (many-valued or modal) non-classical logics. If we also count Heyting's calculus for intuitionistic logic, we can see how already in the

1930's classical logic had quite a number of competitors, each one of which tried to capture a different concept of *logical consequence*. But what should count, abstractly speaking, as a concept of logical consequence? In answering this question, the Polish logicians Adolf Lindenbaum (1904-1941) and Alfred Tarski (1901-1983) initiated a confluence of the Fregean and algebra of logic traditions into one unique stream. Lindenbaum and Tarski showed how it is possible to associate in a canonical way, at least at the propositional level, logical calculi (and their attendant consequence relations) with classes of algebras.

To give some idea of their accomplishments, we will subordinate historical accuracy to the needs of a more systematic treatment. As a first step, we will define the general concept of consequence relation along the lines of Tarski's 1936 paper ([130]). A (propositional) *language* over a countably infinite set X, whose members are referred to as *variables*, is a nonempty set $\mathcal{L}$ (disjoint from X), whose members are called *connectives*, such that a nonnegative integer n is assigned to each member c of $\mathcal{L}$. This integer is called the *arity* of c. The set Fm of $\mathcal{L}$*-formulas* over X is defined as follows:

- *Inductive beginning*: Every member p of X is a formula.
- *Inductive step*: If c is a connective of arity n and $\alpha_1, ..., \alpha_n$ are formulas, then so is $c(\alpha_1, ..., \alpha_n)$.

We will confine ourselves to considering cases where $\mathcal{L}$ is finite; we also observe that most connectives used in logic have arities 0 through 2. In this last case – i.e., for *binary* connectives – the customary infix notation will be employed.

On the algebraic side, the same concept of language can be adopted to specify the symbols denoting the primitive operations of an algebra (or class of algebras). In this case, we sometimes use the phrase "similarity type" or simply "type" in place of "language". If $\mathcal{L} = \{c_1, ..., c_n\}$ is a language over X, then by the inductive definition of formula

$$\mathbf{Fm} = \langle Fm, c_1, ..., c_n \rangle$$

is an algebra of type $\mathcal{L}$, called the *formula algebra* of $\mathcal{L}$. In the following, given a formula $\alpha(p_1, ..., p_n)$ containing at most the indicated variables, an algebra $\mathbf{A}$ of type $\mathcal{L}$ and elements $a_1, ..., a_n \in A$, we will denote by $\alpha^{\mathbf{A}}(a_1, ..., a_n)$ (or $\alpha^{\mathbf{A}}(\overrightarrow{a})$ when the length of the string is either clear from the context or inessential) the result of the application to α of the unique homomorphism $h : \mathbf{Fm} \to \mathbf{A}$ such that $h(p_i) = a_i$ for all $i \leq n$. This notation will be sometimes extended to sets of formulas in the obvious way. An *equation* of type $\mathcal{L}$ is a pair (α, β) of $\mathcal{L}$-formulas, written as $\alpha \approx \beta$. Endomorphisms on $\mathbf{Fm}$ are called *substitutions*, and whenever there is some substitution σ such that $\alpha = \sigma(\beta)$, we say that α is a *substitution instance* of β. The dependence of $\mathbf{Fm}$ on $\mathcal{L}$ will be tacitly acknowledged in what follows.

We now have all we need to define consequence relations. A *consequence relation* over the formula algebra $\mathbf{Fm}$ is a relation $\vdash \subseteq \wp(Fm) \times Fm$ with the following properties:

(1) $\alpha \vdash \alpha$ *(reflexivity)*;
(2) If $\Gamma \vdash \alpha$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \alpha$ *(monotonicity)*;
(3) If $\Gamma \vdash \alpha$ and $\Delta \vdash \gamma$ for every $\gamma \in \Gamma$, then $\Delta \vdash \alpha$ *(cut)*.

A (propositional) *logic* is a pair $\mathrm{L} = (\mathbf{Fm}, \vdash_{\mathrm{L}})$, where $\mathbf{Fm}$ is the formula algebra of some given type $\mathcal{L}$ and $\vdash_{\mathrm{L}}$ is a *substitution-invariant* consequence relation over $\mathbf{Fm}$; in other words, if $\Gamma \vdash_{\mathrm{L}} \alpha$ and σ is a substitution on $\mathbf{Fm}$, then $\sigma(\Gamma) \vdash_{\mathrm{L}} \sigma(\alpha)$ (where $\sigma(\Gamma) = \{\sigma(\gamma) | \gamma \in \Gamma\}$). Informally speaking, we have a logic whenever we can specify a logical language and a concept of consequence among formulas of that language according to which: (i) every formula follows from itself; (ii) whatever follows from a set of premises also follows from any larger set of premises; (iii) whatever follows from consequences of a set of premises also follows from the set itself; (iv) whether a conclusion follows or not from a set of premises only depends on the *logical form* of the premises and the conclusion themselves. A formula α is a *theorem* of the logic $\mathrm{L} = (\mathbf{Fm}, \vdash_{\mathrm{L}})$ if $\emptyset \vdash_{\mathrm{L}} \alpha$.

Our next step will consist of rigorously defining Hilbert-style calculi. An *inference rule* over $\mathbf{Fm}$ is a pair $\mathsf{R} = (\Gamma, \alpha)$, where Γ is a finite (possibly empty) subset of Fm and $\alpha \in Fm$. If Γ is empty, the rule is an *axiom*; otherwise, it is a *proper rule*. A *Hilbert-style calculus* (over $\mathbf{Fm}$) is a set of inference rules over $\mathbf{Fm}$ that contains at least one axiom and at least one proper rule. For axioms, we will henceforth omit outer brackets and the empty set symbol; also, proper rules $(\{\alpha_1, ..., \alpha_n\}, \alpha)$ will be written in the fractional form

$$\frac{\alpha_1 \quad \dots \quad \alpha_n}{\alpha}$$

If $\Delta \cup \{\beta\} \subseteq Fm$ and HL is a Hilbert-style calculus over $\mathbf{Fm}$, then a *derivation* of β from Δ in HL is a finite sequence $\beta_1, ..., \beta_n$ of formulas in Fm such that $\beta_n = \beta$ and for each β_i $(i \leq n)$:

(1) either β_i is a member of Δ; or
(2) β_i is a substitution instance of an axiom of HL; or else
(3) there are a substitution σ and an inference rule $(\Gamma, \alpha) \in \mathrm{HL}$ such that $\beta_i = \sigma(\alpha)$ and, for every $\gamma \in \Gamma$, $\sigma(\gamma) \in \{\beta_1, ..., \beta_{i-1}\}$.

From any Hilbert-style calculus HL over $\mathbf{Fm}$ we can extract a logic $(\mathbf{Fm}, \vdash_{\mathrm{HL}})$ by specifying that $\Gamma \vdash_{\mathrm{HL}} \alpha$ whenever there is a derivation of

α from Γ in HL. Such logics are called *deductive systems* and have the property that whenever $\Gamma \vdash_{\text{HL}} \alpha$, there is always some *finite* $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{\text{HL}} \alpha$ (this much is clear from the very definition of derivation: after we prune Γ of all that is not necessary to derive α, we are left with a finite set). This compactness property, on the other hand, need not be available in general. Logics having the property are called *finitary*; hence, we may rephrase the above by saying that deductive systems are finitary logics.

An example of the preceding rather abstract discussion is the Hilbert-style calculus HCL for classical propositional logic illustrated in Table 1 of Subsection 2.4. Its language $\mathcal{L}$ contains the connectives $\neg, \wedge, \vee, \to, 0$, and 1. Classical propositional logic can be now identified with the deductive system $\text{CL} = (\mathbf{Fm}, \vdash_{\text{HCL}})$.

Developing an idea by Lindenbaum, Tarski showed in 1935 in what sense Boolean algebras can be considered the algebraic counterpart of CL ([129]). Actually, Tarski pointed out a rather weak kind of correspondence between Boolean algebras and classical logic: he showed that the former are an *algebraic semantics* for the latter, a notion that we now proceed to explain in full generality.

Let $\text{L} = (\mathbf{Fm}, \vdash_{\text{L}})$ be a logic in the language $\mathcal{L}$, and let $\tau = \{\gamma_i(p) \approx \delta_i(p)\}_{i \in I}$ be a set of equations in a single variable of $\mathcal{L}$. To avoid overloading notation, it will be convenient to think of τ as a function which maps formulas in Fm to sets of equations of the same type. Therefore we let $\tau(\alpha)$ stand for the set

$$\{\gamma_i(p/\alpha) \approx \delta_i(p/\alpha)\}_{i \in I}.$$

Now let $\mathcal{K}$ be a class of algebras also of the same type. We say that $\mathcal{K}$ is an algebraic semantics for L if, for some such τ, the following condition holds for all $\Gamma \cup \{\alpha\} \in Fm$:

$$\begin{array}{lll} \Gamma \vdash_{\text{L}} \alpha & \text{iff} & \text{for every } \mathbf{A} \in \mathcal{K} \text{ and every } \overrightarrow{a} \in A^n, \\ & & \text{if } \tau(\gamma)^{\mathbf{A}}(\overrightarrow{a}) \text{ for all } \gamma \in \Gamma, \text{ then } \tau(\alpha)^{\mathbf{A}}(\overrightarrow{a}), \end{array}$$

a condition which will sometimes be rewritten as

$$\Gamma \vdash_{\text{L}} \alpha \quad \text{iff} \quad \{\tau(\gamma) | \gamma \in \Gamma\} \vdash_{Eq(\mathcal{K})} \tau(\alpha).$$

In particular, if $\mathcal{L}$ contains a nullary connective 1, it is sometimes possible to choose τ to be the singleton $\{p \approx 1\}$. Informally speaking, what we do in such cases is the following: given an algebra $\mathbf{A} \in \mathcal{K}$, we interpret its elements as "meanings of propositions" or "truth values" and, in particular, the element $1^{\mathbf{A}}$ as "true"; homomorphisms $h : \mathbf{Fm} \to \mathbf{A}$, where $\mathbf{A} \in \mathcal{K}$, are interpreted as "assignments of meanings" to elements of Fm. Moreover, we want α to follow from the set of premises Γ just in case, whenever we assign the meaning "true" to all the premises in some algebra in $\mathcal{K}$, the conclusion is also assigned the meaning "true".

Let us now apply this definition to our classical logic example. We just remind the reader that a Boolean algebra is usually defined as an algebra $\mathbf{A} = \langle A, \wedge, \vee, \neg, 1, 0\rangle$ such that $\langle A, \wedge, \vee, 1, 0\rangle$ is a bounded distributive lattice and, for every $a \in A$, $a \wedge \neg a = 0$ and $a \vee \neg a = 1$. To apply the definition of algebraic semantics, Boolean algebras must be algebras of the same type as the formula algebra of CL, whence it is expedient to include in the type the derived operation symbol $\to$ (defined via $p \to q = \neg p \vee q$). Once this is done, setting $\tau = \{p \approx 1\}$ it is possible to show that:

Theorem 3.1. *The class $\mathcal{BA}$ of Boolean algebras is an algebraic semantics for* CL.

Proof. (Sketch). The left-to-right implication can be established by induction on the length of a derivation of α from Γ in HCL: we show that axioms A1 to A15 are always evaluated at $1^{\mathbf{A}}$ for every $\mathbf{A} \in \mathcal{BA}$ and that the proper inference rule R1 preserves this property.

The converse implication is trickier. We show the contrapositive: we suppose that $\Gamma \nvdash_{\mathrm{CL}} \alpha$ and prove that there exist a Boolean algebra $\mathbf{A}$ and a sequence of elements $\overrightarrow{a}$ such that $\gamma(\overrightarrow{a}) = 1^{\mathbf{A}}$ for all $\gamma \in \Gamma$, yet $\alpha(\overrightarrow{a}) \neq 1^{\mathbf{A}}$.

Let T be the smallest set of $\mathcal{L}$-formulas that includes Γ and is closed under the consequence relation of CL (that is, for every $\beta \in Fm$, $T \vdash_{\mathrm{HCL}} \beta$ implies that $\beta \in T$). Define a binary relation Θ_T on Fm by stipulating that

$$(\beta, \gamma) \in \Theta_T \quad \text{iff} \quad \beta \to \gamma, \gamma \to \beta \in T.$$

The whole business of finishing our completeness proof amounts to establishing the following two assertions:

(1) Θ_T is a congruence on $\mathbf{Fm}$, and the coset $\left[1^{\mathbf{Fm}}\right]_{\Theta_T}$ is just T;
(2) the quotient $\mathbf{Fm}/\Theta_T$ is a Boolean algebra.

Proofs of (1) and (2) make heavy use of syntactic lemmas established for HCL. Once this laborious task has been carried out, in order to construct our falsifying model, it suffices to take $\mathbf{A} = \mathbf{Fm}/\Theta_T$ (we are justified in so doing by (2) above) and evaluate each variable p in $\Gamma \cup \{\alpha\}$ as its own congruence class modulo Θ_T: then $\gamma^{\mathbf{A}}(\overrightarrow{[p]_{\Theta_T}}) = 1^{\mathbf{A}}$ for all $\gamma \in \Gamma$ (since $\Gamma \subseteq T$) yet $\alpha^{\mathbf{A}}(\overrightarrow{[p]_{\Theta_T}}) \neq 1^{\mathbf{A}}$ (since $\alpha \notin T$). □

3.3. **Blok and Pigozzi: algebraizable logics.** Algebraic logic rapidly developed after World War Two, once again to the credit of Polish logicians. Although Tarski had permanently settled in the States before that time, establishing in Berkeley what would later become the leading research group in algebraic logic worldwide, his compatriots Jerzy Łoś, Roman Suszko, Helena Rasiowa, and Roman Sikorski kept the flag of Polish algebraic logic flying, developing in detail throughout the 1950's and 1960's the theory of logical matrices initiated twenty years earlier by Łukasiewicz and Tarski himself. A major breakthrough in the discipline came about in 1989, when Wim Blok and Don Pigozzi (one of Tarski's students) published their monograph on algebraizable logics [12], considered a milestone in the area of abstract algebraic logic. This subsection summarizes some of the main results of their text.

The concept of algebraic semantics described above is too weak in at least two respects. First, in the relationship between a given logic and a candidate algebraic semantics there is room for much promiscuity. In fact:

- There are logics with no algebraic semantics. For example, let HI be the Hilbert-style calculus whose sole axiom is $\alpha \to \alpha$ and whose sole inference rule is modus ponens. Then the deductive system $(\mathbf{Fm}, \vdash_{\mathrm{HI}})$ has no algebraic semantics [13].
- The same logic can have more than one algebraic semantics. As we have recalled, every Boolean algebra is a complemented distributive lattice. A Boolean algebra $\mathbf{B}$ is, in particular, a bounded distributive lattice such that for every $a, b \in B$, $a \to b$ (defined in this particular case as $\neg a \vee b$) is (w.r.t. the induced order of the underlying lattice) the top element in the set

$$\{x \in B \mid a \wedge x \leq b\}.$$

A *Heyting algebra* can be defined as an algebra $\mathbf{H} = \langle B, \wedge, \vee, \to, 1, 0 \rangle$ satisfying exactly the above conditions. Moreover, the class of Heyting algebras can be equationally defined and so forms a variety $\mathcal{HA}$. Any Boolean algebra therefore forms a Heyting algebra by letting $\neg x = x \to 0$; more precisely, an equational basis for Boolean algebras relative to Heyting algebras is given by the single equation $x \approx (x \to 0) \to 0$, expressing the fact that negation is an *involution*. Now, by Glivenko's Theorem ([59]), CL admits not only $\mathcal{BA}$ as an algebraic semantics, but also $\mathcal{HA}$, by choosing $\tau = \{\neg\neg p \approx 1\}$ ([13]).
- There can be different logics with the same algebraic semantics. Since Heyting algebras are an algebraic semantics for intuitionistic logic IL, the previous example shows that both CL and IL have $\mathcal{HA}$ as an algebraic semantics.

Second, the relation between a logic L and its algebraic semantics $\mathcal{K}$ is asymmetric. In fact, the property of belonging to the set of "true values" of an algebra $\mathbf{A} \in \mathcal{K}$ must be definable by means of the set of equations τ, whence the class $\mathcal{K}$ has the expressive resources to indicate when a given *formula* is valid in L. On the other hand, the logic L need not have the expressive resources to indicate when a given *equation* holds in $\mathcal{K}$. For example, there is no way in CL to express, by means of a condition involving a set of formulas, when it is the case that an equation $\alpha \approx \beta$ holds in its algebraic semantics $\mathcal{HA}$. This means that there is a sense of "faithful representation" according to which the relation $\vdash_{Eq(\mathcal{HA})}$ faithfully represents the consequence relation of CL, but not conversely. This makes a sharp contrast to the situation we have with the other algebraic semantics of CL we examined, namely $\mathcal{BA}$: we gather from the proof of Theorem 3.1 that an equation $\alpha \approx \beta$ holds in $\mathcal{BA}$ just in case $\vdash_{\text{CL}} \alpha \rightarrow \beta$ and $\vdash_{\text{CL}} \beta \rightarrow \alpha$. The notion of *algebraizability* ([12]) aims at making precise this stronger relation between a logic and a class of algebras which holds between CL and $\mathcal{BA}$, but not between CL and its "unofficial" semantics $\mathcal{HA}$.

Before stating the formal definition of this concept, let us establish some further notational conventions: given an equation $\alpha \approx \beta$ and a set of formulas in two variables $\rho = \{\alpha_j(p, q)\}_{j \in J}$, we will abbreviate by $\rho(\alpha, \beta)$ the set $\{\alpha_j(p/\alpha, q/\beta)\}_{j \in J}$, and ρ will be regarded as a function mapping equations to sets of formulas. If Γ, Δ are sets of formulas, by $\Gamma \vdash_{\text{L}} \Delta$ we will mean $\Gamma \vdash_{\text{L}} \alpha$ for every $\alpha \in \Delta$; similarly, if E, E' are sets of equations, by $E \vdash_{Eq(\mathcal{K})} E'$ we will mean $E \vdash_{Eq(\mathcal{K})} \varepsilon$ for every $\varepsilon \in E'$.

A logic $\text{L} = (\mathbf{Fm}, \vdash_{\text{L}})$ is said to be *algebraizable* with *equivalent algebraic semantics* $\mathcal{K}$ (where $\mathcal{K}$ is a class of algebras of the same type as $\mathbf{Fm}$) iff there exist a map τ from formulas to sets of equations, and a map ρ from equations to sets of formulas such that the following conditions hold for any $\alpha, \beta \in Fm$:

AL1: $\Gamma \vdash_{\text{L}} \alpha$ iff $\tau(\Gamma) \vdash_{Eq(\mathcal{K})} \tau(\alpha)$;
AL2: $E \vdash_{Eq(\mathcal{K})} \alpha \approx \beta$ iff $\rho(E) \vdash_{\text{L}} \rho(\alpha, \beta)$;
AL3: $\alpha \dashv\vdash_{\text{L}} \rho(\tau(\alpha))$;
AL4: $\alpha \approx \beta \dashv\vdash_{Eq(\mathcal{K})} \tau(\rho(\alpha, \beta))$.

The sets $\tau(p)$ and $\rho(p, q)$ are respectively called a *system of defining equations* and a *system of equivalence formulas* for L and $\mathcal{K}$. A logic L is algebraizable (tout court) iff, for some $\mathcal{K}$, it is algebraizable with equivalent algebraic semantics $\mathcal{K}$.

Put differently, a logic L is algebraizable with equivalent algebraic semantics $\mathcal{K}$ when:

- the relation $\vdash_{\mathrm{L}}$ is faithfully interpretable via the map τ into the relation $\vdash_{Eq(\mathcal{K})}$ (AL1);
- the relation $\vdash_{Eq(\mathcal{K})}$ is faithfully interpretable via the map ρ into the relation $\vdash_{\mathrm{L}}$ (AL2);
- the two maps ρ and τ are mutually inverse, meaning that if we apply them in succession we end up with a formula, respectively an equation, which is equivalent to the one we started with according to $\vdash_{\mathrm{L}}$, respectively $\vdash_{Eq(\mathcal{K})}$ (AL3 and AL4).

This definition can be drastically simplified, in that one can show that a logic L is algebraizable with equivalent algebraic semantics $\mathcal{K}$ iff it satisfies either AL1 and AL4, or else AL2 and AL3.

As an example, we can strengthen Theorem 3.1 by showing that

Theorem 3.2. CL *is algebraizable with equivalent algebraic semantics* $\mathcal{BA}$.

Proof. Let

$$\begin{aligned} \tau(p) &= \{p \approx 1\}; \\ \rho(p,q) &= \{p \to q, q \to p\}. \end{aligned}$$

By the previous observation, we need only check that τ and ρ satisfy conditions AL1 and AL4 above. However, AL1 is just Theorem 3.1. As for AL4,

$$\begin{aligned} \alpha \approx \beta \dashv\vdash_{Eq(\mathcal{BA})} \tau(\rho(\alpha,\beta)) \quad &\text{iff} \quad \alpha \approx \beta \dashv\vdash_{Eq(\mathcal{BA})} \tau(\alpha \to \beta, \beta \to \alpha) \\ &\text{iff} \quad \alpha \approx \beta \dashv\vdash_{Eq(\mathcal{BA})} \{\alpha \to \beta \approx 1, \beta \to \alpha \approx 1\}. \end{aligned}$$

However, given any $\mathbf{A} \in \mathcal{BA}$ and any $\overrightarrow{a} \in A^n$, $\alpha^{\mathbf{A}}(\overrightarrow{a}) = \beta^{\mathbf{A}}(\overrightarrow{a})$ just in case $\alpha \to \beta^{\mathbf{A}}(\overrightarrow{a}) = 1^{\mathbf{A}}$ and $\beta \to \alpha^{\mathbf{A}}(\overrightarrow{a}) = 1^{\mathbf{A}}$, which proves our conclusion. □

Every equivalent algebraic semantics for L is, in particular, an algebraic semantics for L in virtue of AL1. However, as we have seen, the converse need not hold. The concept of equivalent algebraic semantics is therefore a genuine strengthening of Tarski's definition.

If L is algebraizable with equivalent algebraic semantics $\mathcal{K}$, then $\mathcal{K}$ might not be the unique equivalent algebraic semantics for L. However, in case L is finitary, any two equivalent algebraic semantics for L generate the same quasivariety. Clearly, this quasivariety is in turn an equivalent algebraic semantics for the same logic, whence we are justified in talking about *the* equivalent quasivariety semantics for L. On the other hand, it is not uncommon to find different algebraizable logics with the same equivalent algebraic semantics (see, e.g., [111]);

however, if L and L$'$ are algebraizable with equivalent quasivariety semantics $\mathcal{K}$ and with the *same set* of defining equations $\tau(p)$, then L and L$'$ must coincide.

In [12], one finds several elegant equivalent characterizations of algebraizability. Some of them use concepts and tools from the *theory of logical matrices*, which was, as we have recalled, one of the early developments in abstract algebraic logic (see [140]). We just mention a syntactic characterization which, unlike the others, does not presuppose any technical prerequisites:

Theorem 3.3. *A logic* $\mathrm{L} = (\mathbf{Fm}, \vdash_{\mathrm{L}})$ *is algebraizable iff there exist a set* $\rho(p, q)$ *of formulas in two variables and a set of equations* $\tau(p)$ *in a single variable such that, for any* $\alpha, \beta, \gamma \in Fm$*, the following conditions hold:*

(1) $\vdash_{\mathrm{L}} \rho(\alpha, \alpha)$*;*
(2) $\rho(\alpha, \beta) \vdash_{\mathrm{L}} \rho(\beta, \alpha)$*;*
(3) $\rho(\alpha, \beta), \rho(\beta, \gamma) \vdash_{\mathrm{L}} \rho(\alpha, \gamma)$*;*
(4) For every n*-ary connective* c^n *and for every* $\overrightarrow{\alpha}, \overrightarrow{\beta} \in (Fm)^n$*,*
$$\rho(\alpha_1, \beta_1), ..., \rho(\alpha_n, \beta_n) \vdash_{\mathrm{L}} \rho(c^n(\overrightarrow{\alpha}), c^n(\overrightarrow{\beta}));$$
(5) $\alpha \dashv\vdash_{\mathrm{L}} \rho(\tau(\alpha))$*.*

In this case $\rho(p, q)$ *and* $\tau(p)$ *are, respectively, a set of defining equations and a set of equivalence formulas for* L*.*

4. The algebras of logic

The focus of this section is *residuated lattices*, algebraic counterparts of the propositional substructural logics discussed in the next section. The defining properties that describe the class $\mathcal{RL}$ of residuated lattices are few and easy to grasp, and concrete examples are readily constructed that illustrate their key features. However, the theory is also sufficiently robust that the class $\mathcal{RL}$ encompasses a large portion of the ordered algebras arising in logic. Notably, the rich algebraic theory of residuated lattices has produced powerful tools for the comparative study of substructural logics. Moreover, the bridge provided by algebraic logic yields significant benefits to algebra. In fact, one can argue convincingly that an in-depth study of residuated lattices is impossible at this time without the use of logical (in particular, proof-theoretic) techniques.

Our primary aim in this section is to present basic facts from the theory of residuated lattices, including the description of their congruence relations, and provide a brief historical account of the development of the concept of residuated structure in algebra.

4.1. **Preliminaries.** A subset F of a poset $\mathbf{P}$ is said to be an *order-filter* of $\mathbf{P}$ if whenever $y \in P$, $x \in F$, and $x \leq y$, then $y \in F$. Note that the empty set $\emptyset$ is an order-filter. For an element $a \in P$, we write $\uparrow a = \{x \in P \mid a \leq x\}$ for the *principal* order-filter generated by a; more generally, for $A \subseteq P$, $\uparrow A = \{x \in P \mid a \leq x, \text{ for some a} \in \text{A}\}$ denotes the *order-filter generated by* A. *Order-ideals* are defined dually.

We denote the *least* element of a poset $\mathbf{P}$, if it exists, by $\bot_{\mathbf{P}}$. Similarly, $\top_{\mathbf{P}}$ denotes the *greatest* element. Obviously, least elements and greatest elements, when they exist, are unique. Let $X \subseteq P$ be any subset (possibly empty). We use $\bigvee_{\mathbf{P}} X$ and $\bigwedge_{\mathbf{P}} X$, respectively, to denote the *join* (or least upper bound) and *meet* (or greatest lower bound) of X in $\mathbf{P}$ whenever they exist. We use the terms *isotone* and *order-preserving* synonymously to describe a map $\varphi\colon P \to Q$ between posets $\mathbf{P}$ and $\mathbf{Q}$ with the property that for all $x, y \in P$, if $x \leq y$ then $\varphi(x) \leq \varphi(y)$. If for all $x, y \in P$, $x \leq y$ implies $\varphi(x) \geq \varphi(y)$, then φ will be called *anti-isotone* or *order-reversing*. The poset subscripts appearing in some of the notation of this paragraph will henceforth be omitted whenever there is no danger of confusion.

4.2. **Residuated maps and residuated lattices.** Let $\mathbf{P}$ and $\mathbf{Q}$ be posets. A map $\varphi : P \to Q$ is called *residuated* provided there exists a map $\varphi_* : Q \to P$ such that $\varphi(x) \leq y \iff x \leq \varphi_*(y)$, for all $x \in P$ and $y \in Q$. We refer to φ_* as the *residual* of φ.

We have the following simple but useful result (see, e.g., [56]):

Lemma 4.1.

(1) *If $\varphi_* : Q \to P$ is residuated with residual φ_*, then φ preserves all existing joins in $\mathbf{P}$ and φ_* preserves any existing meets in $\mathbf{Q}$.*

(2) *Conversely, if $\mathbf{P}$ is a complete lattice and $\varphi : \mathbf{P} \to \mathbf{Q}$ preserves all joins, then it is residuated.*

A binary operation $\cdot$ on a partially ordered set $\mathbf{P} = \langle P, \leq\rangle$ is said to be *residuated* if there exist binary operations $\backslash$ and $/$ on P such that for all $x, y, z \in P$,

$$x \cdot y \leq z \;\text{ iff }\; x \leq z/y \;\text{ iff }\; y \leq x\backslash z.$$

Note that $\cdot$ is residuated if and only if, for all $a \in P$, the maps $x \mapsto ax$ ($x \in P$) and $x \mapsto xa$ ($x \in P$) are residuated in the sense of the preceding definition. Their residuals are the maps $y \mapsto a\backslash y$ ($y \in P$) and $y \mapsto y/a$ ($y \in P$), respectively. The operations $\backslash$ and $/$ are referred to as the *right residual* and the *left residual* of $\cdot$, respectively. Observe also that $\cdot$ is residuated if and only if it is order-preserving in each argument and, for all $x, y \in P$, the sets $\{z \mid x \cdot z \leq y\}$ and $\{z \mid z \cdot x \leq y\}$ both contain greatest elements, $x \backslash y$ and y/x, respectively. In particular, note that $\cdot$ and $\leq$ uniquely determine $\backslash$ and $/$.

It is suggestive to think of the residuals as generalized division operations. The expression y/x is read as "y over x" while $x \backslash y$ is read as "x under y." In either case, y is considered the *numerator* and x the *denominator*. We tend to favor $/$ in our calculations, but any statement about residuated structures has a "mirror image" obtained by replacing $x \cdot y$ by $y \cdot x$ and interchanging x/y with $y \backslash x$. It follows directly from the preceding definition that a statement is equivalent to its mirror image, and we often state results in only one form. As usual, we write xy for $x \cdot y$ and adopt the convention that, in the absence of parenthesis, $\cdot$ is performed first, followed by $\backslash$ and $/$, and finally $\vee$ and $\wedge$. We also define $x^1 = x$ and $x^{n+1} = x^n \cdot x$.

As a consequence of Lemma 4.1, multiplication preserves all existing joins in each argument, and $\backslash$ and $/$ preserve all existing meets in the "numerator." Moreover, it is easy to check that they convert all existing joins in the "denominator" to meets. More specifically, we have the following result:

Proposition 4.2. *Let $\cdot$ be a residuated map on a poset $\mathbf{P}$ with residuals $\backslash$ and $/$.*

(1) The operation $\cdot$ preserves all existing joins in each argument; i.e., if $\bigvee X$ and $\bigvee Y$ exist for $X, Y \subseteq P$, then $\bigvee_{x \in X, y \in Y} xy$ exists and

$$\Big(\bigvee X\Big)\Big(\bigvee Y\Big) = \bigvee_{x \in X,\ y \in Y} xy.$$

(2) The residuals preserve all existing meets in the numerator, and convert existing joins to meets in the denominator, i.e., if $\bigvee X$ and $\bigwedge Y$ exist for $X, Y \subseteq P$, then for any $z \in P$, $\bigwedge_{x \in X} z/x$ and $\bigwedge_{y \in Y} y/z$ exist and

$$z\Big/\Big(\bigvee X\Big) = \bigwedge_{x \in X} z/x \quad \text{and} \quad \Big(\bigwedge Y\Big)\Big/z = \bigwedge_{y \in Y} y/z.$$

A *residuated lattice*, or a *residuated lattice-ordered monoid*, is an algebra

$$\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, 1 \rangle$$

such that $\langle L, \wedge, \vee \rangle$ is a lattice; $\langle L, \cdot, 1 \rangle$ is a monoid; and $\cdot$ is residuated, in the underlying partial order, with residuals $\backslash$ and $/$.

An *FL-algebra* $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, 1, 0\rangle$ is an algebra such that: (i) $\langle L, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ is a residuated lattice, and (ii) 0 is a distinguished element (nullary operation) of L.

We use the symbols $\mathcal{RL}$ and $\mathcal{FL}$ to denote the class of all residuated lattices and FL-algebras respectively. The following lemma collects basic properties of residuated lattices, most of which by now can be ascribed to the "folklore" of the subject.

Proposition 4.3. *The following equations (and their mirror images) hold in any residuated lattice (in particular, in any FL-algebra).*

(1) $x(y/z) \leq xy/z$
(2) $x/y \leq xz/yz$
(3) $(x/y)(y/z) \leq x/z$
(4) $x/yz \approx (x/z)/y$
(5) $x\backslash(y/z) \approx (x\backslash y)/z$
(6) $(1/x)(1/y) \leq 1/yx$
(7) $(x/x)x \approx x$
(8) $(x/x)^2 \approx x/x$

If a residuated lattice $\mathbf{L}$ has a bottom element $\perp$, then $\perp\backslash\perp$ is its top element $\top$. Moreover, for all $x \in P$, we have

$$x\perp = \perp = \perp x \quad \text{and} \quad \perp\backslash x = \top = x\backslash\top.$$

Sometimes it is useful to have $\perp$ and $\top$ in the signature. In particular, a *bounded FL-algebra* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, 1, 0, \perp, \top\rangle$ such that: (i) $\langle L, \wedge, \vee, \cdot, \backslash, /, 1, 0\rangle$ is an FL-algebra, and (ii) $\perp$ and $\top$ are, respectively, the bottom and top elements of L.

The next result, whose proof is left to the reader, provides a straightforward way to verify that $\mathcal{RL}$ and $\mathcal{FL}$ are equational classes.

Lemma 4.4. *An algebra* $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ *is a residuated lattice if and only if* $\langle L, \cdot, 1, \rangle$ *is a monoid,* $\langle L, \wedge, \vee\rangle$ *is a lattice, and for all* $a, b \in L$,

(1) *the maps* $x \mapsto ax$ *and* $x \mapsto xa$ *preserve finite joins*;
(2) *the maps* $x \mapsto a\backslash x$ *and* $x \mapsto x/a$ *are isotone*;
(3) $a(a\backslash b) \leq b \leq a\backslash ab$; *and*
(4) $(b/a)a \leq b \leq ba/a$.

Hence, we have the following equational characterization of $\mathcal{RL}$ and $\mathcal{FL}$:

Proposition 4.5. *The classes* $\mathcal{RL}$ *and* $\mathcal{FL}$ *are finitely based equational classes. Their defining equations consist of the defining equations for* *lattices and monoids together with the six equations given below:*

(1)	$x(y \vee z) \approx xy \vee xz$	$(y \vee z)x \approx yx \vee zx$
(2)	$x\backslash y \leq x\backslash(y \vee z)$	$y/x \leq (y \vee z)/x$
(3)	$x(x\backslash y) \leq y \leq x\backslash xy$	
(4)	$(y/x)x \leq y \leq yx/x$	

Two varieties of particular interest are the variety $\mathcal{CRL}$ of commutative residuated lattices and the variety $\mathcal{CFL}$ of commutative FL-algebras. These varieties satisfy the equation $xy \approx yx$, and hence the equation $x\backslash y \approx y/x$. In what follows, we use the symbol $\rightarrow$ to denote both the operations $\backslash$ and $/$. While we always think of these varieties as subvarieties of $\mathcal{RL}$ and $\mathcal{FL}$, respectively, we slightly abuse notation by listing only one occurrence of the operation $\rightarrow$ in describing their members.

Example 4.1. It is convenient sometimes to add an extra nullary operation 0 to the type of $\mathcal{RL}$, and think of $\mathcal{RL}$ as the subvariety of $\mathcal{FL}$ axiomatized, relative to $\mathcal{FL}$, by the equation $1 \approx 0$.

Example 4.2. The variety of Boolean algebras can be identified with the subvariety of $\mathcal{CFL}$, which we may again harmlessly call $\mathcal{BA}$, satisfying the additional equations $xy \approx x \wedge y, (x \rightarrow y) \rightarrow y \approx x \vee y$, and $x \wedge 0 \approx 0$. More specifically, every Boolean algebra $\mathbf{B} = \langle B, \wedge, \vee, \neg, 1, 0\rangle$ satisfies the equations above with respect to $\wedge, \vee, 0$, and Boolean implication $x \rightarrow y = \neg x \vee y$. Conversely, if a (commutative) residuated lattice $\mathbf{L}$ satisfies these equations and we define $\neg x = x \rightarrow 0$, then $\langle L, \wedge, \vee, \neg, 1, 0\rangle$ is a Boolean algebra. In stricter mathematical terms, the variety of Boolean algebras is *term-equivalent* to the subvariety of $\mathcal{CFL}$ satisfying the equations $xy \approx x \wedge y$, $(x \rightarrow y) \rightarrow y \approx x \vee y$, and $x \wedge 0 \approx 0$.

Likewise, the variety of Heyting algebras is term-equivalent to the subvariety of $\mathcal{CFL}$, which we again call $\mathcal{HA}$, satisfying the additional equations $xy \approx x \wedge y$ and $x \wedge 0 \approx 0$.

Example 4.3. Ring theory constitutes a historically remarkable source for residuated structures. (Refer to Subsection 4.6 for further details.) Let $\mathbf{R}$ be a ring with unit and let $\mathrm{I}(\mathbf{R})$ denote the lattice of two-sided ideals of $\mathbf{R}$. Then $\mathrm{I}(\mathbf{R}) = \langle \mathrm{I}(R), \cap, \vee, \cdot, \backslash, /, R, \{0\}\rangle$ is a (not necessarily commutative) FL-algebra, where, for $I, J \in \mathrm{I}(R)$,

$$IJ = \{\sum_{k=1}^{n} a_k b_k \mid a_k \in I; b_k \in J; n \geq 1\};$$

$$I\backslash J = \{x \in R \mid Ix \subseteq J\};\ \text{and}$$

$$J/I = \{x \in R \mid xI \subseteq J\}.$$

For a related interesting example, consider an integral domain $\mathbf{R}$ and its field of quotients $\mathbf{K}$. Let $\mathrm{L}(\mathbf{K})$ denote the lattice of $\mathbf{R}$-submodules of $\mathbf{K}$. Then $\mathrm{L}(\mathbf{K}) = \langle \mathrm{L}(K), \cap, \vee, \cdot, \backslash, /, R, \{0\}\rangle$ is an FL-algebra, where, for $I, J \in \mathrm{L}(K)$,

$$IJ = \{\textstyle\sum_{k=1}^{n} a_k b_k | a_k \in I; b_k \in J; n \geq 1\}.$$

Example 4.4. Lattice-ordered groups play a fundamental role in the study of algebras of logic. (Refer to Subsection 4.6 for a short account of their role in mathematics.) A *lattice-ordered group*, *ℓ-group* for short, is an algebra $\mathbf{G} = \langle G, \wedge, \vee, \cdot, ^{-1}, 1\rangle$ such that (i) $\langle G, \wedge, \vee\rangle$ is a lattice; (ii) $\langle G, \cdot, ^{-1}, 1\rangle$ is a group; and (iii) addition is order-preserving in each argument (equivalently, it satisfies the equation $x(y \vee z)w \approx (xyw) \vee (xzw)$). The variety of ℓ-groups is term-equivalent to the subvariety $\mathcal{LG}$ of $\mathcal{RL}$ defined by the additional equation $(1/x)x \approx 1$.

More specifically, if $\mathbf{G} = \langle G, \wedge, \vee, \cdot, ^{-1}, 1\rangle$ is an ℓ-group and we define $x/y = xy^{-1}$ and $y\backslash x = y^{-1}x$, then $\mathbf{G} = \langle G, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ is a residuated lattice satisfying the equation $(1/x)x \approx 1$. Conversely, if a residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ satisfies the last equation and we define $x^{-1} = 1/x$, then $\mathbf{L} = \langle L, \wedge, \vee, \cdot, ^{-1}, 1\rangle$ becomes an ℓ-group. Moreover, this correspondence is bijective.

Example 4.5. MV-algebras (refer to the discussion in Subsection 5.2) are the algebraic counterparts of the infinite-valued Łukasiewicz propositional logic. An *MV-algebra* is traditionally defined as an algebra $\mathbf{M} = \langle M, \oplus, \neg, 0\rangle$ of type $\langle 2, 1, 0\rangle$ that satisfies the following equations:

(MV1) $x \oplus (y \oplus z) \approx (x \oplus y) \oplus z$
(MV2) $x \oplus y \approx y \oplus x$
(MV3) $x \oplus 0 \approx x$
(MV4) $\neg\neg x \approx x$
(MV5) $x \oplus \neg 0 \approx \neg 0$
(MV6) $\neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x$

The variety of MV-algebras is term-equivalent to the subvariety, $\mathcal{MV}$, of $\mathcal{CFL}$ satisfying the extra equations $x \vee y \approx (x \to y) \to y$ and $x \wedge 0 \approx 0$.

In more detail, if $\mathbf{M} = \langle M, \oplus, \neg, 0\rangle$ is an MV-algebra and we define, for $x, y \in M$, $xy = \neg(\neg x \oplus \neg y)$, $x \wedge y = x(\neg x \oplus y)$, $x \vee y = x \oplus (\neg xy)$, $x \to y = \neg(x\neg y)$, and $1 = \neg 0$, then $\mathbf{M} = \langle M, \wedge, \vee, \cdot, \to, 1, 0\rangle$ is in $\mathcal{MV}$. Conversely, if $\mathbf{M} \in \mathcal{MV}$ and we define, for all $x, y \in M$, $\neg x = x \to 0$ and $x \oplus y = \neg(\neg x \neg y)$, then $\mathbf{M} = \langle M, \oplus, \neg, 0\rangle$ is an MV-algebra. We refer to [30] (Section 4.2) or [139] (Subsection 3.4.5) for the details of the proof.

Example 4.6. Many varieties of ordered algebras arising in logic – including Boolean algebras, Abelian ℓ-groups, and MV-algebras, but not Heyting algebras and ℓ-groups – are *semilinear*, that is, generated by their totally ordered members. An equational basis for the variety of semilinear residuated lattices $\mathcal{S}em\mathcal{RL}$ relative to $\mathcal{RL}$ consists of the equation (λ_z and ρ_w are defined in the next subsection)

$$\lambda_z(x/(x \vee y)) \vee \rho_w(y/(x \vee y)) \approx 1.$$

The proof of this result makes heavy use of the material of Subsection 4.5 (see [18] and [80]). A simplified equational basis given in [69] for the variety of commutative semilinear residuated lattices $\mathcal{CS}em\mathcal{RL}$ relative to $\mathcal{CRL}$ consists of the equations

$$[(x \to y) \vee (y \to x)] \wedge 1 \approx 1 \quad \text{and} \quad 1 \wedge (x \vee y) \approx (1 \wedge x) \vee (1 \vee y).$$

Semilinear varieties play a fundamental role in fuzzy logics. In particular, the varieties $\mathcal{UL}$ of commutative semilinear bounded FL-algebras and $\mathcal{MTL}$ of commutative semilinear bounded FL-algebras satisfying $1 \approx \top$ and $0 \approx \bot$ form algebraic semantics for *uninorm logic* ([99]) and *monoidal t-norm logic* ([47]) respectively (see Subsection 5.2).

4.3. **The class $\mathcal{RL}$ is an ideal variety.** The main result of this subsection is Theorem 4.12 below, first established in [18]. It shows that the congruences of members of $\mathcal{RL}$ are determined by their convex normal subalgebras (to be defined below). In particular, $\mathcal{RL}$, and hence $\mathcal{FL}$, is a 1-*regular variety*, that is, each congruence relation of an algebra in $\mathcal{RL}$ is determined by its equivalence class of 1. A more economical proof of 1-regularity for $\mathcal{RL}$ can be given by observing that this property is a *Mal'cev property*, meaning that one can establish if a variety has the property by checking whether it satisfies certain quasi-equations involving finitely many terms (two, in this special case). However, a concrete description of these equivalence classes is essential for developing the structure theory of residuated lattices and its applications to substructural logics.

If $\mathbf{L}$ is a residuated lattice, the set $L^- = \{a \in L \mid a \leq 1\}$ is called the *negative cone* of $\mathbf{L}$. Note that the negative cone is a submonoid of $\langle L, \cdot, 1\rangle$. As such, we will denote it by $\mathbf{L}^-$.

Let $\mathbf{L} \in \mathcal{RL}$. For each $a \in L$, define $\rho_a(x) = (ax/a) \wedge 1$ and $\lambda_a(x) = (a\backslash xa) \wedge 1$. We refer to ρ_a and λ_a respectively as *right and left conjugation* by a. An *iterated conjugation* map is a finite composition of right and left conjugation maps.

A subset $X \subseteq L$ is called (*order-*)*convex* if for any $x, y \in X$ and $a \in L$, $x \leq a \leq y$ implies $a \in X$; X is called *normal* if it is closed with respect to all iterated conjugations.

Let $\mathbf{L}$ be a residuated lattice. For $a, b \in L$ define $[a,b]_r = (ab/ba) \wedge 1$ and $[a,b]_l = (ba\backslash ab) \wedge 1$. We call $[a,b]_r$ and $[a,b]_l$ respectively the *right and left commutators* of a with b.

We will say that a subset X is *closed with respect to commutators* if for any $a \in L$ and $x \in X$, both the commutators $[a,x]_r$ and $[x,a]_l$ lie in X. Normality and "closure with respect to commutators" are identical properties for certain "nice" subsets as we show in the next two lemmas.

Lemma 4.6. *Let $\mathbf{H}$ be a convex subalgebra of $\mathbf{L}$. Then $\mathbf{H}$ is normal if and only if it is closed with respect to commutators.*

Proof. Suppose that $\mathbf{H}$ is normal. Then $1 \geq [a,h]_r = (ah/ha) \wedge 1 = ((ah/a)/h) \wedge 1 \geq (((ah/a) \wedge 1)/h) \wedge 1 = (\rho_a(h)/h) \wedge 1 \in H$ so that $[a,h]_r \in H$ by convexity. The proof that $[h,a]_l \in H$ is analogous.

Conversely, suppose that H is closed with respect to commutators. We have $[a,h]_r h \wedge 1 \in H$ and $[a,h]_r h \wedge 1 = ((ah/ha) \wedge 1)h \wedge 1 \leq (ah/ha)h \wedge 1 = ((ah/a)/h)h \wedge 1 \leq (ah/a) \wedge 1 = \rho_a(h) \leq 1$ so $\rho_a(h) \in H$ by convexity. The proof that $\lambda_a(h) \in H$ is analogous. □

The same result holds for convex submonoids of the negative cone of $\mathbf{L}$:

Lemma 4.7. *If $\mathbf{S}$ is a convex submonoid of $\mathbf{L}^-$, then $\mathbf{S}$ is normal if and only if S is closed with respect to commutators.*

Proof. Let $s \in S$ and $a \in L$ and suppose that $\mathbf{S}$ is normal. Then $1 \geq [a,s]_r = (as/sa) \wedge 1 = ((as/a)/s) \wedge 1 \geq (as/a) \wedge 1 = \rho_a(s) \in S$ where the last inequality above follows since $s \leq 1$. Similarly, $[s,a]_l \in S$. Conversely, if S is closed with respect to commutators, then $[a,s]_r s \in S$. But $[a,s]_r s = (((as/a)/s) \wedge 1)s \leq ((as/a)/s)s \wedge s \leq (as/a) \wedge s \leq (as/a) \wedge 1 = \rho_a(s) \leq 1$ and by convexity we have $\rho_a(s) \in S$. Similarly, $\lambda_a(s) \in S$. □

We often find it useful to convert one of the division operations into its dual. The following two equations, which are referred to as *switching equations* and can be verified by straightforward calculation, provide a means to do so in any residuated lattice:

$$z/y \leq py\backslash z \quad \text{, where } p = [z/y, y]_r \text{ , and}$$
$$x\backslash z \leq z/xq \quad \text{, where } q = [x, x\backslash z]_l.$$

Note that the above equations still hold if the "$\wedge 1$" factor is omitted from the commutators.

Lemma 4.8. *Let $\mathbf{L}$ be a residuated lattice and $\Theta \in Con(\mathbf{L})$. Then the following are equivalent:*

(1) $a \ \Theta \ b$
(2) $[(a/b) \wedge 1] \ \Theta \ 1$ *and* $[(b/a) \wedge 1] \ \Theta \ 1$
(3) $[(a\backslash b) \wedge 1] \ \Theta \ 1$ *and* $[(b\backslash a) \wedge 1] \ \Theta \ 1$

Proof. Suppose that $a \ \Theta \ b$. Then $(a/a) \ \Theta \ (b/a)$ so that

$$1 = [(a/a) \wedge 1] \ \Theta \ [(b/a) \wedge 1]$$

and the other relations in (2) and (3) follow similarly. Conversely, suppose that both $[(a/b) \wedge 1] \ \Theta \ 1$ and $[(b/a) \wedge 1] \ \Theta \ 1$. Setting $r = [(a/b) \wedge 1]b$ and $s = [(b/a) \wedge 1]a$, we have $r \ \Theta \ b$ and $s \ \Theta \ a$. Moreover, $r \leq (a/b)b \leq a$ and $s \leq (b/a)a \leq b$ so that $r = (a \wedge r) \ \Theta \ (a \wedge b)$ and $s = (b \wedge s) \ \Theta \ (b \wedge a)$ whence $b \ \Theta \ r \ \Theta \ (a \wedge b) \ \Theta \ \ s \ \Theta \ a$; we have shown (2) $\rightarrow$ (1). (3) $\rightarrow$ (1) is proved in an analogous manner. □

Lemma 4.9. *Let* Θ *be a congruence relation on a residuated lattice* $\mathbf{L}$*. Then* $[1]_\Theta = \{a \in A \mid a \ \Theta \ 1\}$ *is a convex normal subalgebra of* $\mathbf{L}$*.*

Proof. Since 1 is idempotent with respect to all the binary operations of $\mathbf{L}$, it immediately follows that $[1]_\Theta$ forms a subalgebra of $\mathbf{L}$. Convexity can be checked directly, or is a consequence of the well-known fact that the equivalence classes of any lattice congruence are convex. Finally, let $a \in [1]_\Theta$ and $c \in L$. Then

$$\lambda_c(a) = (c\backslash ac) \wedge 1 \ \Theta \ (c\backslash 1c) \wedge 1 = (c\backslash c) \wedge 1 = 1$$

so that $\lambda_c(a) \in [1]_\Theta$. Similarly, $\rho_c(a) \in [1]_\Theta$. □

Lemma 4.10. *Suppose that* $\mathbf{H}$ *is a convex normal subalgebra of* $\mathbf{L}$*. For any* $a, b \in L$*,*

$$(a/b) \wedge 1 \in H \ \Leftrightarrow \ (b\backslash a) \wedge 1 \in H.$$

Proof. Suppose that $(a/b) \wedge 1 \in H$. Since $\mathbf{H}$ is normal, we have

$$h = b\backslash(((a/b) \wedge 1)b) \wedge 1 \in H.$$

But $h \leq [b\backslash(a/b)b] \wedge 1 \leq (b\backslash a) \wedge 1 \leq 1 \in H$ so $(b\backslash a) \wedge 1 \in H$. The reverse implication is proved similarly. □

Next we characterize the congruence corresponding to a given convex normal subalgebra.

Lemma 4.11. *Let* $\mathbf{H}$ *be a convex normal subalgebra of a residuated lattice* $\mathbf{L}$*. Then*

$$\begin{aligned} \Theta_{\mathbf{H}} &= \{(a,b) \mid \exists h \in H, ha \leq b \text{ and } hb \leq a\} \\ &= \{(a,b) \mid (a/b) \wedge 1 \in H \text{ and } (b/a) \wedge 1 \in H\} \\ &= \{(a,b) \mid (a\backslash b) \wedge 1 \in H \text{ and } (b\backslash a) \wedge 1 \in H\} \end{aligned}$$

is a congruence on $\mathbf{L}$*.*

Proof. First we show that the three sets defined above are indeed equal. That the second and third sets are identical follows from Lemma 4.10. If (a, b) is a member of the second set, then letting $h = (a/b) \wedge (b/a) \wedge 1$, we have $h \in H$, $ha \leq (b/a)a \leq b$ and $hb \leq (a/b)b \leq b$, so that (a, b) is a member of the first set. Conversely, if (a, b) is a member of the first set, then for some $h \in H$ we have $ha \leq b$ or $h \leq b/a$, and hence $h \wedge 1 \leq (b/a) \wedge 1 \leq 1$. By convexity, we get $(b/a) \wedge 1 \in H$. Similarly, $(a/b) \wedge 1 \in H$.

It is a simple matter to verify that $\Theta_{\mathbf{H}}$ is an equivalence relation. To prove that it is a congruence relation, we must establish its compatibility with respect to multiplication, meet, join, right division, and left division. We just verify compatibility for multiplication and right division:

Θ is compatible with multiplication
Suppose that $a \ \Theta \ b$ and $c \in L$. Then

$$(a/b) \wedge 1 \leq (ac/bc) \wedge 1 \leq 1$$

so $(ac/bc) \wedge 1 \in H$. Similarly, $(bc/ac) \wedge 1 \in H$ so $(ac) \ \Theta \ (bc)$. Next, using the normality of $\mathbf{H}$,

$$\rho_c((a/b) \wedge 1) = (c[(a/b) \wedge 1]/c) \wedge 1 \in H.$$

But $\rho_c((a/b) \wedge 1) \leq [c(a/b)/c] \wedge 1 \leq [ca/b/c] \wedge 1 = (ca/cb) \wedge 1 \leq 1 \in H$ so $(ca/cb) \wedge 1 \in H$. Similarly, $(cb/ca) \wedge 1 \in H$ so $(ca) \ \Theta \ (cb)$.

Θ is compatible with right division
Suppose that $a \ \Theta \ b$ and $c \in L$. Then

$$(a/b) \wedge 1 \leq [(a/c)/(b/c)] \wedge 1 \leq 1$$

so $[(a/c)/(b/c)] \wedge 1 \in H$. Similarly, $[(b/c)/(a/c)] \wedge 1 \in H$ so $(a/c) \ \Theta \ (b/c)$. Next,

$$(b/a) \wedge 1 \leq [(c/b) \backslash (c/a)] \wedge 1 \leq 1 \in H$$

so $[(c/b) \backslash (c/a)] \wedge 1 \in H$. Hence it follows by Lemma 4.10 that $[(c/b)/(c/a)] \wedge 1 \in H$. Similarly, $[(c/a)/(c/b)] \wedge 1 \in H$ so $(c/a) \ \Theta \ (c/b)$. $\square$

Theorem 4.12. *The lattice $\mathcal{CN}(\mathbf{L})$ of convex normal subalgebras of a residuated lattice $\mathbf{L}$ is isomorphic to its congruence lattice Con$(\mathbf{L})$. The isomorphism is given by the mutually inverse maps $\mathbf{H} \mapsto \Theta_{\mathbf{H}}$ and $\Theta \mapsto [1]_{\Theta}$.*

Proof. We have shown both that $\Theta_{\mathbf{H}}$ is a congruence and that $[1]_{\Theta}$ is a member of $\mathcal{CN}(\mathbf{L})$, and it is clear that the maps $\mathbf{H} \mapsto \Theta_{\mathbf{H}}$ and $\Theta \mapsto [1]_{\Theta}$

are isotone. It remains only to show that these two maps are mutually inverse, since it will then follow that they are lattice homomorphisms.

Given $\Theta \in \mathrm{Con}(\mathbf{L})$, set $H = [1]_\Theta$; we must show that $\Theta = \Theta_\mathbf{H}$. But this is easy; using Lemma 4.8,

$$a \;\Theta\; b \;\Leftrightarrow [((a/b) \wedge 1) \;\Theta\; 1 \text{ and } ((b/a) \wedge 1) \;\Theta\; 1] \Leftrightarrow$$

$$[((a/b) \wedge 1) \in H \text{ and } ((b/a) \wedge 1) \in H] \Leftrightarrow a \;\Theta_\mathbf{H}\; b.$$

Conversely, for any $\mathbf{H} \in \mathcal{C}N(\mathbf{L})$ we must show that $H = [1]_{\Theta_\mathbf{H}}$. But

$$h \in H \;\rightarrow\; [(h/1) \wedge 1 \in H \;\text{ and }\; (1/h) \wedge 1 \in H]$$

so $h \in [1]_{\Theta_\mathbf{H}}$. If $a \in [1]_{\Theta_\mathbf{H}}$, then $(a, 1) \in \Theta_\mathbf{H}$ and we use the first description of $\Theta_\mathbf{H}$ in Lemma 4.11 to conclude that there exists some $h \in H$ such that $ha \leq 1$ and $h = h1 \leq a$. Now it follows from the convexity of $\mathbf{H}$ that $h \leq a \leq h\backslash 1$ implies $a \in H$. □

We remark that in the event that $\mathbf{L}$ is commutative, then every convex subalgebra of $\mathbf{L}$ is normal. Thus the preceding theorem implies the following result of [69]:

Corollary 4.13. *The lattice $\mathcal{C}(\mathbf{L})$ of convex subalgebras of a commutative residuated lattice $\mathbf{L}$ is isomorphic to its congruence lattice Con$(\mathbf{L})$. The isomorphism is given by the mutually inverse maps $\mathbf{H} \mapsto \Theta_\mathbf{H}$ and $\Theta \mapsto [1]_\Theta$.*

4.4. **Convex normal submonoids and deductive filters.** In the previous subsection we saw that the congruences of a residuated lattice $\mathbf{L}$ correspond to its convex normal subalgebras. Here we show that these subalgebras in turn correspond to both the convex normal submonoids of $\mathbf{L}^-$ and the deductive filters of $\mathbf{L}$ (defined below). The original references for the first correspondence are [17], [18]. Deductive filters (under the name "filters") and their correspondence with congruences of residuated lattices were introduced in [15]. See also [80], [139], and [51].

The next lemma shows that a convex normal subalgebra is completely determined by its negative cone:

Lemma 4.14. *Let $\mathbf{S}$ be a convex normal submonoid of $\mathbf{L}^-$. Then $H_S := \{a \mid s \leq a \leq s\backslash 1,$ for some $s \in S\}$ is the universe of a convex normal subalgebra $\mathbf{H}_S$ of $\mathbf{L}$, and $S = H_S^-$. Conversely, if $\mathbf{H}$ is any convex normal subalgebra of $\mathbf{L}$ then, setting $S_H = H^-$, $\mathbf{S}_H$ is a convex normal submonoid of $\mathbf{L}^-$ and H can be recovered from S_H as described above. Moreover, the mutually inverse maps $\mathbf{H} \mapsto \mathbf{S}_H$ and $\mathbf{S} \mapsto \mathbf{H}_S$ establish a lattice isomorphism between $\mathcal{C}N(\mathbf{L})$ and $\mathcal{C}NM(\mathbf{L}^-)$.*

Proof. Given a convex, normal subalgebra $\mathbf{H}$ of $\mathbf{L}$, the assertions about S_H are easy to verify. Thus we turn our attention to the other direction. Let $\mathbf{S}$ be a convex normal submonoid of $\mathbf{L}^-$ and define H_S as above. It is easy to show that H_S is convex and normal. Moreover, it is immediate that $H_S^- = S$. It remains to prove closure under the binary operations. We just verify closure under left and right division. To this end, let $a, b \in H_S$. Then there are $s, t \in S$ so that $s \leq a \leq s\backslash 1$ and $t \leq b \leq t\backslash 1$.

Closure under left division
We have $a\backslash b \leq s\backslash(t\backslash 1) = (ts)\backslash 1$, but to find a lower bound for $a\backslash b$ is a little trickier. First notice that $t \leq b$ and $sa \leq 1$ imply that $tsa \leq b$. From this we derive $ats(ats\backslash tsa) \leq tsa \leq b$ and $ts(ats\backslash tsa) \leq a\backslash b$. Setting $p = (ats)\backslash(tsa)$ and $q = ts(p \wedge 1)$, we know that $p \wedge 1 = [ts, a]_l \in S$ and so $q \in S$. But now $q \leq tsp \leq a\backslash b$ and we have found the desired lower bound. Finally, setting $r = qts$, it follows that $r \leq a\backslash b \leq r\backslash 1$.

Closure under right division
Observe that $s \leq a$ and $tb \leq 1$ imply the inequalities $stb \leq a$ and $st \leq a/b$, but to find an upper bound requires extra work: $a/b \leq (s\backslash 1)/t \leq pt\backslash(s\backslash 1) = spt\backslash 1$, where $p = [(s\backslash 1)/t, t]_r$ as given by the switching equation. But $p \in S$ by the comments following Lemma 4.6 and we have found an appropriate upper bound. Finally, we can set $r = (st)(spt)$ and it follows that $r \leq a/b \leq r\backslash 1$.

We have shown that the maps between the two lattices are well-defined and mutually inverse. Since they are clearly isotone, the theorem is proved. □

A subset F of a residuated lattice is a *deductive filter* provided:

(DF1) $\uparrow\{1\} \subseteq F$,
(DF2) if $x, x\backslash y \in F$, then $y \in F$,
(DF3) if $x, y/x \in F$, then $y \in F$,
(DF4) if $x, y \in F$, then $x \wedge y \in F$,
(DF5) if $x \in F$ and $y \in L$, then $y\backslash(xy) \in F$ and $(yx)/y \in F$.

An alternative description of a deductive filter is provided by the following result of [139].

Lemma 4.15. *A subset F of a residuated lattice $\mathbf{L}$ is a deductive filter if and only if it is a non-empty order-filter of $\mathbf{L}$ closed under multiplication and conjugation.*

Let $\mathcal{CN}(\mathbf{L})$, $\mathcal{CNM}(\mathbf{L}^-)$, and $\mathcal{DF}(\mathbf{L})$ denote respectively the lattices under set-inclusion of convex normal subalgebras of $\mathbf{L}$, convex normal submonoids of $\mathbf{L}^-$, and deductive filters of $\mathbf{L}$. We have the following result (see [139] or [51]):

Proposition 4.16. *In a residuated lattice* $\mathbf{L}$*, the lattice* $\mathcal{CNM}(\mathbf{L}^-)$ *of convex normal submonoids of* $\mathbf{L}^-$ *is isomorphic to the lattice* $\mathcal{DF}(\mathbf{L})$ *of deductive filters of* $\mathbf{L}$*. The isomorphism is given by the mutually inverse maps* $M \mapsto \uparrow M$ *and* $F \mapsto F^-$*, for* $M \in \mathcal{CNM}(\mathbf{L}^-)$ *and* $F \in \mathcal{DF}(\mathbf{L})$*.*

4.5. **Convex normal subalgebra generation.** The original references for the results of this subsection are [17] and [18]. (See also [80].) They provide intrinsic descriptions of convex normal submonoids, convex normal subalgebras, and deductive filters. The local deduction theorem for the logic corresponding to the variety of commutative residuated lattices, Theorem 6.3 (1), and the parametrized local deduction theorem in [52] are the logical counterparts of and follow easily from Corollary 4.23 (1) and Proposition 4.19, respectively.

Lemma 4.17. *For all* $a_1, a_2, \ldots, a_n, b \in L$*, if* $a = \prod a_j$*, then*

$$\prod \rho_b(a_j) \leq \rho_b(a) \quad \text{and} \quad \prod \lambda_b(a_j) \leq \lambda_b(a).$$

Proof. We prove only the case $n = 2$; the proof can be completed by the obvious induction.

$$\begin{aligned}\rho_b(a_1)\rho_b(a_2) &= [(ba_1/b) \wedge 1][(ba_2/b) \wedge 1] \leq [(ba_1/b)(ba_2/b)] \wedge 1\\ &\leq [((ba_1/b)ba_2)/b] \wedge 1 \leq (ba_1a_2/b) \wedge 1 = \rho_b(a_1a_2).\end{aligned}$$

In the last two inequalities, we use Lemma 4.3 (5) and (4) respectively. The proof for λ_b is analogous. □

The next result provides an element-wise description of a convex normal submonoid of the negative cone generated by a subset.

Proposition 4.18. *Let* $\mathbf{L}$ *be a residuated lattice and* $S \subseteq L^-$*. An element* $x \leq 1$ *belongs to the convex normal submonoid of* $\mathbf{L}^-$ *generated by* S *iff there exist iterated conjugates* $\gamma_1(s_1), \ldots, \gamma_n(s_n)$ *of elements of* S *such that* $\gamma_1(s_1) \ldots \gamma_n(s_n) \leq x$*.*

Proof. Let $cnm(S)$ denote the set described in the statement of the proposition. It is clear that $1 \in M(S)$, that $cnm(S)$ is convex and closed under multiplication, and that any convex normal submonoid of $\mathbf{L}^-$ containing S must contain $cnm(S)$. Moreover, since $S \subseteq L^-$, $S \subseteq cnm(S)$. It only remains to show that $cnm(S)$ is normal. But this follows from Lemma 4.17 and the convexity of $cnm(S)$: if $x \in cnm(S)$, then for some iterated conjugates $\gamma_1(s_1), \ldots, \gamma_n(s_n)$ of elements of S, $\gamma_1(s_1) \ldots \gamma_n(s_n) \leq x \leq 1$. By Lemma 4.17, for all $a \in L, \rho_a(\gamma_1(s_1)) \ldots \rho_a(\gamma_n(s_n)) \leq \rho_a(x) \leq 1$, and similarly for $\lambda_a(x)$. □

For any subsets $S \subseteq L^-$ and $T \subseteq L$, we write $cnm(S)$ for the convex normal submonoid of $\mathbf{L}^-$ generated by S, $cn(T)$ for the convex normal subalgebra of $\mathbf{L}$ generated by T, and $df(T)$ for the deductive filter of $\mathbf{L}$ generated by T.

We have as a direct consequence of Lemma 4.14 and Proposition 4.18:

Proposition 4.19. *Let $\mathbf{L}$ be a residuated lattice and $S \subseteq L^-$. Then $x \in df(S)$ iff there exist iterated conjugates $\gamma_1(s_1), \dots, \gamma_n(s_n)$ of elements of S such that $\gamma_1(s_1) \dots \gamma_n(s_n) \leq x$.*

Likewise, Propositions 4.16 and 4.18 easily yield:

Proposition 4.20. *Let $\mathbf{L}$ be a residuated lattice and $S \subseteq L^-$. Then $x \in cn(S)$ iff there exist iterated conjugates $\gamma_1(s_1), \dots, \gamma_n(s_n)$ of elements of S such that $\gamma_1(s_1) \dots \gamma_n(s_n) \leq x \leq (\gamma_1(s_1) \dots \gamma_n(s_n))\backslash 1$.*

The natural question arises as to whether there are analogous descriptions of $cn(S)$ and $df(S)$ for arbitrary subsets S of L. Let us write $cnm(a)$ for $cnm(\{a\})$ (for $a \in L^-$), $df(a)$ for $df(\{a\})$, and $cn(a)$ for $cn(\{a\})$.

Lemma 4.21. *For any $a \in L$, $cn(a) = cn(a')$ and $df(a) = df(a')$, where $a' = a \wedge (1/a) \wedge 1$.*

Proof. Clearly $a' \in cn(a)$. On the other hand,

$$a' \leq a \leq (1/a)\backslash 1 \leq a'\backslash 1,$$

so $a \in cn(a')$, and likewise for $df(a)$. □

Thus we have the following corollary:

Corollary 4.22. *Let $S \subseteq L$ and set $S^* = \{s \wedge (1/s) \wedge 1 \mid s \in S\}$. Then:*

(1) *$x \in cn(S)$ iff there exist iterated conjugates $\gamma_1(s_1), \dots, \gamma_n(s_n)$ of elements of S^* such that*

$$\gamma_1(s_1) \dots \gamma_n(s_n) \leq x \leq x \leq (\gamma_1(s_1) \dots \gamma_n(s_n))\backslash 1.$$

(2) *$x \in df(S)$ iff there exist iterated conjugates $\gamma_1(s_1), \dots, \gamma_n(s_n)$ of elements of S^* such that*

$$\gamma_1(s_1) \dots \gamma_n(s_n) \leq x.$$

We close this subsection by noting that the above intrinsic descriptions become substantially simpler whenever $\mathbf{L}$ is a commutative residuated lattice. In this case, the convex normal submonoids of $\mathbf{L}^-$ are its convex submonoids, the convex normal subalgebras of $\mathbf{L}$ are the convex subalgebras, and the deductive filters of $\mathbf{L}$ are the non-empty order-filters closed under multiplication. Thus, we immediately get the following result from [69]:

Corollary 4.23. *Let* $\mathbf{L}$ *be a commutative residuated lattice, and let* $S \cup \{a\} \subseteq L^-$*. Then:*

(1) $x \in cnm(a)$ *iff there exists a natural number* n *such that* $a^n \leq x \leq 1$.
(2) $x \in cnm(S)$ *iff there exist* $s_1, \ldots, s_k \in S$ *and natural numbers* $n_1, \ldots, n_k$ *such that* ${s_1}^{n_1} \ldots {s_k}^{n_k} \leq x \leq 1$.
(3) $x \in cn(a)$ *iff there exists a natural number* n *such that* $a^n \leq x \leq a\backslash 1$.
(4) $x \in cn(S)$ *iff there exist* $s_1, \ldots, s_k \in S$ *and natural numbers* $n_1, \ldots, n_k$ *such that* ${s_1}^{n_1} \ldots {s_k}^{n_k} \leq x \leq ({s_1}^{n_1} \ldots {s_k}^{n_k})\backslash 1$.
(5) $x \in df(a)$ *iff there exists a natural number* n *such that* $a^n \leq x$.
(6) $x \in df(S)$ *iff there exist* $s_1, \ldots, s_k \in S$ *and natural numbers* $n_1, \ldots, n_k$ *such that* ${s_1}^{n_1} \ldots {s_k}^{n_k} \leq x$.

4.6. **Historical remarks.** In this section we attempt to summarily reconstruct the development of the concept of residuated structure in algebra. Instead of organizing this historical survey in strict chronological order, we prefer – for the sake of greater readability – to follow five separate thematic threads, each of which has in our opinion decisively contributed to shaping the contemporary notion of residuated structure.

A word of caution is in order here: although residuated maps are almost ubiquitous in mathematics, we circumscribe our survey to examples of residuation which bear a tighter connection to the main theme of this paper. Therefore, we will mostly confine ourselves to examining the historical development of residuated lattices, disregarding, e.g., the plentiful and certainly important examples of residuated pairs $(\varphi, \varphi_*) : \mathbf{P} \to \mathbf{Q}$, where $\mathbf{P}$ and $\mathbf{Q}$ are *different* posets. The content of the next subsection is the only exception to this policy, motivated by the great historical relevance of Galois theory as the first significant appearance of residuation in mathematics.

Galois theory. After the Italian Renaissance mathematicians Scipione del Ferro, Tartaglia, Cardano, and Ferrari had shown that cubic and quartic equations were solvable by radicals by means of a general formula, algebraists spent subsequent centuries striving to achieve a similar result for polynomial equations with rational coefficients of degree 5 or higher. These efforts came to an abrupt end in 1824, when Niels Abel (patching an earlier incorrect proof by another Italian, Paolo

Ruffini) showed that such equations have no general solution by radicals. Since, however, it was well-known that some *particular* equations of degree greater or equal than 5 could indeed be solved, the question remained open as to whether a general criterion was available to determine which polynomial equations were solvable by radicals and which ones were not.

In 1832, the French mathematician Évariste Galois (just before meeting his death in a tragical duel) found the right approach to settle the issue once and for all. He associated to each polynomial equation ε a permutation group (now called *Galois group* in honor of its inventor) consisting of those permutations of the set of all roots of ε having the property that every algebraic equation satisfied by the roots themselves is still satisfied after the roots have been permuted. As a simple example, let ε be the equation

$$x^2 - 4x + 1 \approx 0,$$

whose roots are $r_1 = 2 + \sqrt{3}$, $r_2 = 2 - \sqrt{3}$. It can be shown that *any* algebraic equation with rational coefficients in the variables x and y which is satisfied by $x = r_1$ and $y = r_2$ (for example, $xy \approx 1$ or $x + y \approx 4$) is also satisfied by $y = r_1$ and $x = r_2$. It follows that both permutations of the two-element set $\{r_1, r_2\}$ – the identity permutation and the permutation which exchanges r_1 with r_2 – belong to the Galois group of ε, which is therefore isomorphic to the cyclic group of order 2. More generally, Galois established that a polynomial equation is solvable by radicals if and only if its Galois group $\mathbf{G}$ is *solvable* – namely, if there exist subgroups $\mathbf{G}_0, ..., \mathbf{G}_n$ of $\mathbf{G}$ such that

$$\{1\} = \mathbf{G}_0 \subset \mathbf{G}_1 \subset ...\mathbf{G}_{n-1} \subset \mathbf{G}_n = \mathbf{G},$$

and, moreover, for all $i \leq n$, $\mathbf{G}_{i-1}$ is normal in $\mathbf{G}_i$ and $\mathbf{G}_i/\mathbf{G}_{i-1}$ is Abelian.

In the above example, permutations which respect algebraic equations satisfied by r_1 and r_2 can be seen as automorphisms of the quotient field $\mathbb{Q}(r_1, r_2)/\mathbb{Q}$, where $\mathbb{Q}(r_1, r_2)$ is nothing but the field one obtains from the field of rationals by adjoining the two roots of the given equation. This approach can be taken up in general, and it is indeed this abstract perspective that underlies present-day Galois theory (see, e.g., [81]), where Galois groups are seen as field automorphisms of a field extension $\mathbf{L}/\mathbf{F}$ of a given base field $\mathbf{F}$. Let us now stipulate that:

- $S(\mathbf{L}, \mathbf{F})$ is the set of all subfields of $\mathbf{L}$ that contain $\mathbf{F}$;
- for $\mathbf{M} \in S(\mathbf{L}, \mathbf{F})$, $\mathbf{Gal}_{\mathbf{M}}(\mathbf{L})$ is the group of all field automorphisms φ of $\mathbf{L}$ such that $\varphi|_M = id$;
- $Sg(\mathbf{Gal}_{\mathbf{F}}(\mathbf{L}))$ is the set of all subgroups of such a group.

Then the maps

$$\begin{aligned} f(\mathbf{M}) &= \mathbf{Gal}_{\mathbf{M}}(\mathbf{L}) \\ f_*(\mathbf{H}) &= \{a \in L \mid \varphi(a) = a \text{ for all } \varphi \in H\} \end{aligned}$$

induce a residuated pair (f, f_*) between $\mathbf{P} = \langle S(\mathbf{L}, \mathbf{F}), \subseteq \rangle$ and the order dual $\mathbf{Q}^{\partial}$ of $\mathbf{Q} = \langle Sg(\mathbf{Gal}_{\mathbf{F}}(\mathbf{L})), \subseteq \rangle$ (i.e., the poset obtained from $\mathbf{Q}$ by reversing its order)[4]. Consequently, Galois theory provides us with a first mathematically significant instance of residuation.

Ideal theory of rings. There is a another respect in which polynomial equations constitute a historically remarkable source for residuated structures. Around the middle of the XIX century, it was observed by Ernst Kummer that unique factorization into primes, true of ordinary integers in virtue of the fundamental theorem of arithmetic, fails instead for algebraic integers – namely, for roots of monic polynomials with integer coefficients. Let $\mathbb{Z}\left[\sqrt{-n}\right]$ denote the quadratic integer ring of all complex numbers of the form $a + b\sqrt{-n}$, with $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$. It turns out that unique factorization fails in $\mathbb{Z}\left[\sqrt{-n}\right]$ for several instances of n, although it holds in some special cases – e.g., for *Gauss integers* $(n = 1)$. Kummer tried to recover a weakened form of this fundamental property with his theory of *ideal numbers*, but the adoption of a modern abstract viewpoint on the issue, that would eventually lead to the birth of contemporary ring theory and to the ideal theory of rings, must be credited to Richard Dedekind's 1871 X Supplement to the second edition of Dirichlet's *Zahlentheorie*. There, Dedekind introduced the concepts of ring and ring ideal in what essentially is their modern usage (the term *ring*, however, was coined by Hilbert only much later) and proved that every ideal of the ring of algebraic integers is uniquely representable (up to permutation of factors) as a product of prime ideals. Unique factorization, therefore, is recovered at the level of ideals (see, e.g., [44]).

The ideal theory of commutative rings was intensively investigated early in the XX century by Lasker and Macaulay, who generalized the results by Dedekind to polynomial rings, and in the 1920's by Noether and Krull. In particular, Emmy Noether proved the celebrated theorem according to which, in any commutative ring whose lattice of ideals satisfies the ascending chain condition, ideals decompose into intersections of finitely many primary ideals.

[4]This example of residuated pair is actually so important that, in general, a residuated pair (f, f_*) between posets $\mathbf{P}$ and $\mathbf{Q}^{\partial}$ is often referred to as a *Galois correspondence* between $\mathbf{P}$ and $\mathbf{Q}$.

This theorem is typical of Noether's general approach. The system $I(\mathbf{R})$ of ideals of a commutative ring $\mathbf{R}$ is viewed as an instance of a lattice[5] endowed with an additional operation of multiplication (see Example 4.3). The same viewpoint was taken and further developed by Morgan Ward and his student R.P. Dilworth in a series of papers during the late 1930's (see, e.g., [38], [39], [40], [136], [137], [138]), whose focus is on another binary operation on ideals: the *residual* $J \to I$ of I with respect to J. Recall from Example 4.3 above that

$$J \to I = \{x \in R \mid xJ \subseteq I\}.$$

As Ward and Dilworth observed, $J \to I$ has the property that, for any ideal K of the ring $\mathbf{R}$, $JK \subseteq I$ iff $K \subseteq J \to I$. Ward and Dilworth introduced and investigated in detail, under the name of *residuated lattices*, some lattice-ordered structures with a multiplication which is abstracted from ideal multiplication, and with a residuation which is in turn abstracted from ideal residuation. They were thus in a position to extend to a purely lattice-theoretic setting some of the results obtained by Noether and Krull for lattices of ideals of commutative rings, including the above-mentioned Noether decomposition theorem.

Ward and Dilworth's papers did not have that much immediate impact, but began a line of research that would crop up again every so often in the following decades. Thus, the notion of residuated lattice re-emerged in the different contexts of the semantics for fuzzy logics (e.g., [65]) and substructural logics (e.g., [108]), and in the setting of studies with a more pronounced universal-algebraic flavor (e.g., [80]), with the latter two streams eventually converging into a single one (see, e.g., [51]).

Interestingly enough, neither Hájek's nor Ono's, nor our official definition of residuated lattice, given in Subsection 4.2 and essentially due to Blount and Tsinakis ([18]), exactly overlaps with the original definition given by Ward and Dilworth. Differences concern both the similarity type and, less superficially, the properties characterizing the respective algebras. The Hájek-Ono residuated lattices are invariably *bounded* as lattices and *integral* as partially ordered monoids, meaning that the top element of the lattice is the neutral element of multiplication. Moreover, multiplication is *commutative*. Residuated lattices as defined here are not necessarily bounded and, even if they are, they need not be integral; multiplication is not required to be commutative. What about the original Ward-Dilworth residuated lattices? Put in a

[5]Lattice theory underwent a remarkable development as a spin-off from Noether's abstract perspective, partly motivated by her desire to release ring theory from the concrete setting of polynomial rings.

very rough way, they lie somewhere in between the preceding concepts: in fact, Ward and Dilworth do not assume the existence of a top or bottom in the underlying lattice, but if there is a top, then it must be the neutral element of multiplication, which is supposed to be a commutative operation. It is only fair to observe that Dilworth also introduces in [39] a noncommutative variant of his notion of residuated lattice, abstracted from the residuated lattice of two-sided ideals of a noncommutative ring, but to the best of our knowledge this generalization was not taken up again until the noncommutative concept of residuated lattice was introduced and had already become established.

Boolean and Heyting algebras. One of the first *classes* of residuated lattices that received considerable attention in its own right was of course the class of Boolean algebras. As we have seen in Subsection 3.1, it is not historically correct to consider Boole as their inventor. A closer approximation to the modern understanding of Boolean algebras can instead be found in the writings of Ernst Schröder, especially his *Operationskreis des Logikkalkuls* (1854). There, he introduces something vaguely resembling an equational axiomatization of Boolean algebras, replacing Boole's partial aggregation operation by plain lattice join, and comes very close to realizing that Boolean algebras make clear-cut instances of residuated algebras: he observes, in fact, that the equation $x \wedge b = a$ is solvable iff $a \wedge \neg b = 0$ and possesses, when this condition is met, a smallest solution $x = a$ and a largest solution $x = a \vee \neg b$. What he apparently missed, however, is the fact that $a \vee \neg b$ is also a largest solution for the *inequation* $x \wedge b \leq a$, or, in other words, that such an operation residuates meet in Boolean algebras [46].

For a proper treatment of Boolean algebras as *algebras* one must wait until the turn of the century, when Edward V. Huntington provided the first equational basis for the variety, followed in the subsequent thirty years by more and more economical axiomatizations with different choices of primitive operations, due, e.g., to Bernstein and Sheffer. Most of the fundamental results of the structure theory for Boolean algebras were however established in the 1930's by Marshall Stone ([124]), *in primis*:

- the term-equivalence between Boolean rings (i.e., idempotent rings with identity) and Boolean algebras;
- the representation of Boolean algebras as algebras of sets, which can be seen as a corollary of a more comprehensive duality between Boolean algebras and a class of topological spaces (Boolean spaces, i.e., totally disconnected compact Hausdorff spaces.)

Both results are extremely powerful, the former because a class of algebras that had been introduced for the purpose of systematizing logical reasoning turns out to be nothing but a subclass of a class of structures whose centrality in "standard" mathematics can hardly be denied – and a very natural subclass, at that. Given a Boolean algebra $\mathbf{B} = \langle B, \wedge, \vee, \neg, 1, 0\rangle$, Stone associates to it an algebra $\mathbf{R_B} = \langle B, \cdot, +, -, 0, 1\rangle$, where for every $a, b \in B$

$$\begin{aligned} a \cdot b &= a \wedge b \\ a + b &= (a \wedge \neg b) \vee (b \wedge \neg a) \\ -a &= a. \end{aligned}$$

Conversely, given a Boolean ring $\mathbf{R} = \langle R, \cdot, +, -, 0, 1\rangle$, he constructs an algebra $\mathbf{B_R} = \langle R, \wedge, \vee, \neg, 1, 0\rangle$, where for every $a, b \in R$

$$\begin{aligned} a \wedge b &= a \cdot b \\ a \vee b &= a + b + a \cdot b \\ \neg a &= a + 1. \end{aligned}$$

He then shows that: (i) $\mathbf{R_B}$ is a Boolean ring; (ii) $\mathbf{B_R}$ is a Boolean algebra; (iii) $\mathbf{R_{B_R}} = \mathbf{R}$; (iv) $\mathbf{B_{R_B}} = \mathbf{B}$.

The latter result quoted above fleshes out a natural intuition about Boolean algebras as essentially algebras of sets. Given a Boolean algebra $\mathbf{B} = \langle B, \wedge, \vee, \neg, 1, 0\rangle$, a filter F of the lattice reduct of $\mathbf{B}$ is called an *ultrafilter* of $\mathbf{B}$ iff, for every $b \in B$, exactly one element in the set $\{b, \neg b\}$ belongs to F. Ultrafilters coincide with maximal filters of $\mathbf{B}$, i.e., filters that are not properly included in any proper filter of $\mathbf{B}$. Letting

$$U(\mathbf{B}) = \{F \subseteq B \mid F \text{ is an ultrafilter of } \mathbf{B}\},$$

Stone showed that the algebra of sets

$$\langle \wp(U(\mathbf{B})), \cap, \cup, -, U(\mathbf{B}), \emptyset\rangle$$

is a Boolean algebra and that $\mathbf{B}$ can be embedded into it via the map

$$f(a) = \{F \in U(\mathbf{B}) \mid a \in F\}.$$

Around the same time, due to the independent work of various researchers (in particular, Glivenko [124] and Heyting [72]), the algebraic counterpart of Brouwer's intuitionistic logic was found in *Heyting algebras*. As we have seen in Example 4.2, both Heyting algebras and Boolean algebras are term-equivalent to varieties of FL-algebras.

ℓ-groups. The name of Richard Dedekind is recurring quite often in these historical notes; in fact, he can legitimately be said to have invented, anticipated, or first investigated at the abstract level, many of the most fundamental notions in contemporary abstract algebra. We have already mentioned his contributions to ring theory in his X Supplement to the second edition of Dirichlet's *Zahlentheorie* (1871). In his Göttingen classnotes on algebra, written between 1858 and 1868 but only published more than a century later (see [118]), Dedekind investigates the abstract concept of group without confining himself to their concrete representations as groups of permutations. In a later paper (*Über Zerlegung von Zahlen durch ihre größten gemeinsamen Teiler*, 1897), he essentially introduces and investigates the notion of lattice, as well as combining the two concepts into what is now known as a *lattice-ordered group* (or *ℓ-group*) (cf. [46]). Dedekind's approach to lattices was algebraic rather than order-theoretic: he views lattices as sets equipped with two binary operations each satisfying associativity, commutativity, and idempotency, and linked together by the absorption law, rather than as posets where every finite set has a meet and a join. Moreover, since he was drawn to lattice theory mainly by his number-theoretic interests, the privileged examples of meet and join he has in mind are not the set-theoretic operations of intersection and union but the arithmetical operations of greatest common divisor and least common multiple ([82]).

As we have mentioned, Dedekind explicitly focused on ℓ-groups. The definition of ℓ-group given in Example 4.4 is, at least in the commutative case, implicit in Dedekind's paper. In a nutshell, he considers an Abelian group endowed with an additional semilattice operation, postulating that the group binary operation distributes over such a join. In other words, the resulting algebra $\mathbf{A} = \langle A, \cdot, \vee, ^{-1}, 1\rangle$ must satisfy the equation

$$x(y \vee z) \approx xy \vee xz.$$

Then he observes that the defined term operation $a \wedge b = ab(a \vee b)^{-1}$ turns the term reduct $\langle A, \wedge, \vee\rangle$ into a *distributive* lattice and the whole structure into a commutative ℓ-group. The same proof carries over to the noncommutative case if we only rephrase more carefully the above definition of meet as $a \wedge b = a(a \vee b)^{-1}b$, but Dedekind does not go as far as to observe this fact.

In the late 1920's and early 1930's, functional analysts like Riesz, Freudenthal, and Kantorovich developed conspicuous bits of ℓ-group theory in the context of their investigations into vector lattices. These different streams converged at last in Birkhoff's first edition (1940) of his *Lattice Theory* [11], containing a chapter on ℓ-groups where the concept is defined in full clarity and precision and Birkhoff's original contributions to the subject, as well as the main results that had been proved in other diverse research areas, are collected and systematized.

To the best of our knowledge, the first author who expressly noticed that in any ℓ-group multiplication is residuated by the two division operations was Jeremiah Certaine in his PhD thesis [26]. It was only much later, however ([17], [18]), that this observation was expanded to an explicit proof of the fact that the variety of ℓ-groups is term-equivalent to a variety of residuated lattices, as explained in Example 4.4.

Birkhoff's problem. Boolean algebras and ℓ-groups contributed towards the historical development of the theory of residuated structures also in several ways other than those reported on so far. In the already mentioned 1940 edition of his *Lattice Theory* [11, Problem 108], Garrett Birkhoff challenged his readers by suggesting the following project:

> Develop a common abstraction which includes Boolean algebras (rings) and lattice ordered groups as special cases.

Over the subsequent decades, several mathematicians tried their hands at Birkhoff's intriguing problem. A minimal requirement that has to be met by a class of structures to be considered an answer to Birkhoff's problem is including both Boolean algebras (or Boolean rings) and ℓ-groups as instances and it is clear that such a desideratum can be satisfied by concepts that are very different from one another. True to form, the list of suggestions advanced in response to Birkhoff's challenge includes such disparate items as classes of partial algebras ([141], [119], [120], [32]) or classes of structures with multi-valued operations ([102]).

Following the lead of the Indian mathematician K.L.N. Swamy ([125], [126], [127]), a number of authors observed that both Boolean algebras and ℓ-groups are residuated structures and formulated their common generalizations accordingly. Indeed, Swamy's dually residuated lattice-ordered semigroups, Rama Rao's direct products of Boolean rings and ℓ-groups ([116]), and Casari's lattice-ordered pregroups ([25]) all form varieties that are term-equivalent to some subvariety of $\mathcal{RL}$ or $\mathcal{FL}$.

5. Structural Proof Theory

Attempting to identify a precise border between ordered algebra and logic would be unwise. Nevertheless, we may safely say that while algebra focuses primarily on structures and their properties, logic (narrowly conceived) concerns itself more with syntax and deduction. Yet despite these differences in perspective, traditional Hilbert-style presentations of propositional logics as axiom systems typically enjoy a close relationship with classes of algebras, formalized, as we have seen, via the Lindenbaum-Tarski-inspired Blok-Pigozzi method of algebraization. In particular, theorems of classical or intuitionistic logic may be translated into equations holding in all Boolean or Heyting algebras and vice versa. To the algebraist, this may suggest that propositional logic is little more than "algebra in disguise." There is something to this point of view, though a logician may quickly respond, first, that some logics are not algebraizable and, second, that the case for an algebraic approach to first order logic is not so compelling. More pertinently for our present concerns, syntactic presentations offer an alternative perspective that pure semantics cannot provide. In particular, syntactic objects such as formulas, equations, and proofs, may be investigated themselves as first-class citizens using methods such as induction on formula complexity or height of a proof. This idea was first taken seriously by Hilbert who established the field of *proof theory* with the aim of proving the consistency of arithmetic and other parts of mathematics using only so-called "finitistic" methods (see, e.g., [73]).

The original goal of Hilbert's program was famously dashed in 1931 by Kurt Gödel's incompleteness theorems, but partially resurrected by Gerhard Gentzen in the mid-to-late 1930's ([54], [55]). Gentzen was able to show that the consistency of arithmetic is provable over the base theory of primitive recursive arithmetic extended with quantifier-free transfinite induction up to the ordinal ε_0. This was the first result of the area subsequently known as *ordinal analysis*. Our interest here lies, however, with the tools that Gentzen used to prove this result. A limitation of the early period of proof theory was the reliance on a rather rigid interpretation of the axiomatic method, that is, axiomatizations typically consisting of many axiom schemata and just a few rules, notably, modus ponens. The axiomatic approach is flexible but does not seem to reflect the way that mathematicians, or humans in general, construct and reason about proofs, and suffers from a lack of control over proofs as mathematical objects. These issues were addressed by Gentzen in [54] via the introduction of two new proof formalisms: *natural deduction* and the *sequent calculus*. In particular, he defined sequent calculi, LK and LJ, for first order classical logic and first order intuitionistic logic, respectively, giving birth to an area known now as *structural proof theory*. Since our interest lies here with ordered algebras, we will focus only on the propositional parts of Gentzen's systems, which, as we will see, correspond directly to Boolean algebras and Heyting algebras. *Substructural logics*, which themselves correspond to classes of the residuated lattices introduced in the previous section, are then obtained, very roughly speaking, by removing certain rules from these systems.

5.1. **Gentzen's** LJ **and** LK**.** Hilbert-style axiom systems, and to a certain extent Gentzen's own natural deduction systems, are hindered by the fact that they deal directly with formulas. So-called *Gentzen systems* gain flexibility by considering more complicated structures. In particular, Gentzen introduced the notion of a *sequent*, ordered pairs of finite sequences of formulas, written:

$$\alpha_1, \ldots, \alpha_n \Rightarrow \beta_1, \ldots, \beta_m.$$

Intuitively, we might think of the disjunction of the formulas $\beta_1, \ldots, \beta_m$ "following from" the conjunction of the formulas $\alpha_1, \ldots, \alpha_n$, although since sequents are purely syntactic objects, any meaning ascribed to them follows only from the role that they play in the given proof system.

Sequent rules are typically written schematically using Γ and Δ to stand for arbitrary sequences of formulas, comma for concatenation, and an empty space for the empty sequence, and consist of *instances* with a finite set of premises and a single conclusion, rules with no premises being called *initial sequents*. A *sequent calculus* L is simply a set of sequent rules, and a *derivation* in such a system of a sequent S from a set of sequents X is a finite tree of sequents with root S such that each sequent is either a leaf and a member of X, or S is the conclusion and its children (if any) are the premises of an instance of a rule of the system. When such a tree exists, we say that "S is derivable from X in L" and write $X \vdash_{\mathrm{L}} S$.

Figure 1 displays an inessential variant of Gentzen's sequent calculus (propositional) LK in the same language as HCL that consists of simple initial sequents of the form $\alpha \Rightarrow \alpha$, a *cut rule* corresponding, like modus ponens, to the transitivity of deduction, and two distinguished collections of rules. The first collection contains rules that introduce occurrences of connectives on the left and right of the sequent arrow. Such *logical rules* may be thought of as defining (operationally) the meaning of the connectives, and roughly correspond to the elimination and introduction rules of Gentzen's natural deduction system. The second collection of *structural rules*, which also come in left/right pairs, simply manipulate the structure of sequents: *exchange rules*, *weakening rules*, and *contraction rules* allow formulas to be permuted, added, and combined, respectively.

Initial sequents

$$\frac{}{\alpha \Rightarrow \alpha}\ (\textsc{id})$$

Cut rule

$$\frac{\Gamma_2 \Rightarrow \alpha, \Delta_2 \quad \Gamma_1, \alpha, \Gamma_3 \Rightarrow \Delta_1}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta_1, \Delta_2}\ (\textsc{cut})$$

Left structural rules

$$\frac{\Gamma_1, \alpha, \beta, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \beta, \alpha, \Gamma_2 \Rightarrow \Delta}\ (\textsc{el})$$

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha, \Gamma_2 \Rightarrow \Delta}\ (\textsc{wl})$$

$$\frac{\Gamma_1, \alpha, \alpha, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha, \Gamma_2 \Rightarrow \Delta}\ (\textsc{cl})$$

Right structural rules

$$\frac{\Gamma \Rightarrow \Delta_1, \alpha, \beta, \Delta_2}{\Gamma \Rightarrow \Delta_1, \beta, \alpha, \Delta_2}\ (\textsc{er})$$

$$\frac{\Gamma \Rightarrow \Delta_1, \Delta_2}{\Gamma \Rightarrow \Delta_1, \alpha, \Delta_2}\ (\textsc{wr})$$

$$\frac{\Gamma \Rightarrow \Delta_1, \alpha, \alpha, \Delta_2}{\Gamma \Rightarrow \Delta_1, \alpha, \Delta_2}\ (\textsc{cr})$$

Left logical rules

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, 1, \Gamma_2 \Rightarrow \Delta}\ (1 \Rightarrow)$$

$$\frac{}{0 \Rightarrow}\ (0 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha, \Delta}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\Rightarrow)$$

$$\frac{\Gamma_1, \alpha, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha \wedge \beta, \Gamma_2 \Rightarrow \Delta}\ (\wedge\Rightarrow)_1$$

$$\frac{\Gamma_1, \beta, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha \wedge \beta, \Gamma_2 \Rightarrow \Delta}\ (\wedge\Rightarrow)_2$$

$$\frac{\Gamma_1, \alpha, \Gamma_2 \Rightarrow \Delta \quad \Gamma_1, \beta, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha \vee \beta, \Gamma_2 \Rightarrow \Delta}\ (\vee\Rightarrow)$$

$$\frac{\Gamma_2 \Rightarrow \alpha, \Delta_2 \quad \Gamma_1, \beta, \Gamma_3 \Rightarrow \Delta_1}{\Gamma_1, \Gamma_2, \alpha \to \beta, \Gamma_3 \Rightarrow \Delta_1, \Delta_2}\ (\to\Rightarrow)$$

Right logical rules

$$\frac{}{\Rightarrow 1}\ (\Rightarrow 1)$$

$$\frac{\Gamma \Rightarrow \Delta_1, \Delta_2}{\Gamma \Rightarrow \Delta_1, 0, \Delta_2}\ (\Rightarrow 0)$$

$$\frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\alpha}\ (\Rightarrow\neg)$$

$$\frac{\Gamma \Rightarrow \Delta_1, \alpha, \Delta_2}{\Gamma \Rightarrow \Delta_1, \alpha \vee \beta, \Delta_2}\ (\Rightarrow\vee)_1$$

$$\frac{\Gamma \Rightarrow \Delta_1, \beta, \Delta_2}{\Gamma \Rightarrow \Delta_1, \alpha \vee \beta, \Delta_2}\ (\Rightarrow\vee)_2$$

$$\frac{\Gamma \Rightarrow \Delta_1, \alpha, \Delta_2 \quad \Gamma \Rightarrow \Delta_1, \beta, \Delta_2}{\Gamma \Rightarrow \Delta_1, \alpha \wedge \beta, \Delta_2}\ (\Rightarrow\wedge)$$

$$\frac{\alpha, \Gamma \Rightarrow \beta, \Delta}{\Gamma \Rightarrow \alpha \to \beta, \Delta}\ (\Rightarrow\to)$$

Figure 1. The Sequent Calculus LK

Example 5.1. *Let us take a look at a derivation in* LK *of Peirce's law, noting that α and β can be any formulas:*

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\alpha \Rightarrow \alpha}\ (\textsc{id})}{\alpha \Rightarrow \beta, \alpha}\ (\textsc{wr})}{\Rightarrow \alpha \to \beta, \alpha}\ (\Rightarrow\to) \quad \cfrac{}{\alpha \Rightarrow \alpha}\ (\textsc{id})}{(\alpha \to \beta) \to \alpha \Rightarrow \alpha, \alpha}\ (\to\Rightarrow)}{(\alpha \to \beta) \to \alpha \Rightarrow \alpha}\ (\textsc{cr})}{\Rightarrow ((\alpha \to \beta) \to \alpha) \to \alpha}\ (\Rightarrow\to)$$

One of the most remarkable features of Gentzen's framework is that it also accommodates a calculus LJ for intuitionistic logic, obtained from LK simply by restricting sequents $\Gamma \Rightarrow \Delta$ so that Δ is allowed to contain at most one formula. In particular, LJ has no right exchange or right contraction rules, and right weakening is confined to premises with empty succedents (in a sense, LJ is the first example of a *substructural* sequent calculus). Hence, for instance, the derivation of Peirce's law in Example 5.1 is (rightly) blocked.

Let us show now that LK really is a sequent calculus corresponding to the Hilbert-style presentation HCL of classical propositional logic, and therefore also Boolean algebras (the same proof works also for LJ with respect to axiomatizations of intuitionistic logic and Heyting algebras). We let $\Box(\alpha_1, \dots, \alpha_n)$ stand for $\alpha_1 \Box \dots \Box \alpha_n$ for $\Box \in \{\wedge, \vee\}$ where $\wedge()$ is 1 and $\vee()$ is 0, and define:

$$\begin{array}{rcl} \tau(\alpha) & = & \{\Rightarrow \alpha\}; \\ \rho(\Gamma \Rightarrow \Delta) & = & \wedge\Gamma \to \vee\Delta. \end{array}$$

Theorem 5.1. *$X \vdash_{\mathrm{LK}} S$ if and only if $\{\rho(S') \mid S' \in X\} \vdash_{\mathrm{HCL}} \rho(S)$.*

Proof. The left-to-right direction is established by an induction on the height of a derivation in LK (straightforward, but requiring many tedious derivations in HCL). For the right-to-left direction, it is easily checked that for any axiom α of HCL, $\tau(\alpha)$ is derivable in LK. Moreover, if $\tau(\alpha)$ and $\tau(\alpha \to \beta)$ are derivable in LK, then so is $\tau(\beta)$, using (CUT) twice with the derivable sequent $\alpha, \alpha \to \beta \Rightarrow \beta$. Note also that for any sequent S: $S \vdash_{\mathrm{LK}} \tau(\rho(S))$ and $\tau(\rho(S)) \vdash_{\mathrm{LK}} S$. Hence if $\{\rho(S') \mid S' \in X\} \vdash_{\mathrm{HCL}} \rho(S)$, then $\{\tau(\rho(S')) \mid S' \in X\} \vdash_{\mathrm{LK}} \tau(\rho(S))$ and, as required, $X \vdash_{\mathrm{LK}} S$. $\Box$

However, it is worth asking at this point what advantages, if any, LJ and LK hold over Hilbert-style systems. Proof search in the latter is hindered by the need to guess formulas α and $\alpha \to \beta$ as premises when applying modus ponens. The same situation seems to occur for these sequent calculi: we have to guess which formula α to use when applying (CUT). Certainly finding derivations would be much simpler if we could do without this rule. Then we could just apply rules where formulas in the premises are subformulas of formulas in the conclusion. In fact, this is the case, as established by Gentzen for (first order) LJ and LK in his famous Hauptsatz. Indeed, Gentzen showed not only that (CUT) is not needed for deriving sequents from empty sets of assumptions, but also that there exists a *cut elimination* algorithm that transforms such derivations into cut-free derivations.

Let us consider briefly the ideas behind Gentzen's proof. Intuitively, the idea is to push applications of the cut rule upwards in derivations until they reach initial sequents and disappear. For example, suppose that we have a derivation in LJ ending

$$\dfrac{\dfrac{\vdots}{\Gamma_2 \Rightarrow \alpha} \quad \dfrac{\vdots}{\Gamma_1, \alpha, \Gamma_3 \Rightarrow \Delta}}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta}\;(\textsc{cut})$$

The cut-formula α occurs on the right in one premise, and on the left in the other. A natural strategy for eliminating this application of $(\textsc{cut})$ is to look at the derivations of these premises. If one of the premises is an instance of $(\textsc{id})$, then it must be $\alpha \Rightarrow \alpha$ and the other premise must be exactly the conclusion, derived with one fewer applications of $(\textsc{cut})$. Otherwise, we have two possibilities. The first is that one of the premises ends with an application of a rule where α is not the decomposed formula, e.g.,

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_2'' \Rightarrow \beta_1} \quad \dfrac{\vdots}{\Gamma_2', \beta_2, \Gamma_2''' \Rightarrow \alpha}}{\Gamma_2', \Gamma_2'', \beta_1 \to \beta_2, \Gamma_2''' \Rightarrow \alpha}\;(\to\Rightarrow) \quad \dfrac{\vdots}{\Gamma_1, \alpha, \Gamma_3 \Rightarrow \Delta}}{\Gamma_1, \Gamma_2', \Gamma_2'', \beta_1 \to \beta_2, \Gamma_2''', \Gamma_3 \Rightarrow \Delta}\;(\textsc{cut})$$

In this case, we can "push the cut upwards" in the derivation to get:

$$\dfrac{\dfrac{\vdots}{\Gamma_2'' \Rightarrow \beta_1} \quad \dfrac{\dfrac{\vdots}{\Gamma_2', \beta_2, \Gamma_2''' \Rightarrow \alpha} \quad \dfrac{\vdots}{\Gamma_1, \alpha, \Gamma_3 \Rightarrow \Delta}}{\Gamma_1, \Gamma_2', \beta_2, \Gamma_2''', \Gamma_3 \Rightarrow \Delta}\;(\textsc{cut})}{\Gamma_1, \Gamma_2', \Gamma_2'', \beta_1 \to \beta_2, \Gamma_2''', \Gamma_3 \Rightarrow \Delta}\;(\to\Rightarrow)$$

That is, we have a derivation where the left premise in the new application of $(\textsc{cut})$ has a shorter derivation than the application in the original derivation.

The second possibility is that the last application of a rule in both premises involves α as the decomposed formula, e.g.

$$\dfrac{\dfrac{\dfrac{\vdots}{\alpha_1, \Gamma_2 \Rightarrow \alpha_2}}{\Gamma_2 \Rightarrow \alpha_1 \to \alpha_2}\;(\Rightarrow\to) \quad \dfrac{\dfrac{\vdots}{\Gamma_1'' \Rightarrow \alpha_1} \quad \dfrac{\vdots}{\Gamma_1', \alpha_2, \Gamma_3 \Rightarrow \Delta}}{\Gamma_1', \Gamma_1'', \alpha_1 \to \alpha_2, \Gamma_3 \Rightarrow \Delta}\;(\to\Rightarrow)}{\Gamma_1', \Gamma_1'', \Gamma_2, \Gamma_3 \Rightarrow \Delta}\;(\textsc{cut})$$

Here we rearrange our derivation in a different way: we replace the application of $(\textsc{cut})$ with applications of $(\textsc{cut})$ with cut-formulas α_1

and α_2:

$$\dfrac{\dfrac{\vdots}{\Gamma_1'' \Rightarrow \alpha_1} \quad \dfrac{\dfrac{\vdots}{\alpha_1, \Gamma_2 \Rightarrow \alpha_2} \quad \dfrac{\vdots}{\Gamma_1', \alpha_2, \Gamma_3 \Rightarrow \Delta}}{\Gamma_1', \alpha_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta}\ (\textsc{cut})}{\Gamma_1', \Gamma_1'', \Gamma_2, \Gamma_3 \Rightarrow \Delta}\ (\textsc{cut})$$

We now have two applications of (CUT) but with cut-formulas of a smaller complexity than the original application.

This procedure, formalized using a double induction on cut-formula complexity and the combined height of derivations of the premises, eliminates applications of (CUT) for many sequent calculi. However, it encounters a problem with rules that contract formulas in one or more of the premises. Consider the following situation:

$$\dfrac{\dfrac{\vdots}{\Gamma_2 \Rightarrow \alpha} \quad \dfrac{\dfrac{\vdots}{\Gamma_1, \alpha, \alpha, \Gamma_3 \Rightarrow \Delta}}{\Gamma_1, \alpha, \Gamma_3 \Rightarrow \Delta}\ (\textsc{cl})}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta}\ (\textsc{cut})$$

In this case we need to perform several cuts simultaneously, e.g., making use of Gentzen's "mix" rule for LK,

$$\dfrac{\Gamma \Rightarrow \alpha, \Delta \quad \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma'_\alpha \Rightarrow \Delta', \Delta_\alpha}$$

where Γ' has at least one occurrence of α, and Γ'_α and Δ_α are obtained by removing all occurrences of α from Γ' and Δ, respectively.

Theorem 5.2 (Gentzen 1935)**.** *Cut-elimination holds for* LK *and* LJ

This result has many important applications. As an immediate consequence, for example, both LJ and LK are *consistent*: there cannot be any cut-free derivation of an arbitrary variable p for instance. Similarly, since any cut-free derivation of $\Rightarrow \alpha \vee \beta$ in LJ must necessarily involve a derivation of $\Rightarrow \alpha$ or $\Rightarrow \beta$, intuitionistic logic has the so-called *disjunction property*. Cut-elimination also facilitates easy proofs of decidability for the derivability of sequents for LK and LJ, and hence for checking validity in propositional classical or intuitionistic logic. Call two sequents *equivalent* if one can be derived from the other using the exchange and contraction rules. Then easily there are a finite number of non-equivalent sequents that can occur in a cut-free derivation of a sequent in LJ or LK, and hence, checking that equivalent sequents do not occur, the search for such a cut-free derivation must terminate. It follows that propositional classical logic and, more interestingly, propositional intuitionistic logic, are decidable. Algebraically of course this means that the equational theories of Boolean algebras and Heyting algebras are decidable, raising the question as to whether other classes of algebras can be proved decidable using similar methods. It is worth noting also that these calculi can also be used to prove complexity results for the respective logics and classes of algebras.

We remark finally that many variants of LJ and LK have appeared in the literature. In particular, the Finnish logician Oiva Ketonen [83] suggested a new version of LK where $(\wedge\Rightarrow)_1$, $(\wedge\Rightarrow)_2$, $(\Rightarrow\vee)_1$, and $(\Rightarrow\vee)_2$ are replaced by:

$$\frac{\Gamma_1, \alpha, \beta, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha \wedge \beta, \Gamma_2 \Rightarrow \Delta}\ (\wedge\Rightarrow)' \qquad \frac{\Gamma \Rightarrow \Delta_1, \alpha, \beta, \Delta_2}{\Gamma \Rightarrow \Delta_1, \alpha \vee \beta, \Delta_2}\ (\Rightarrow\vee)'$$

Also, Haskell B. Curry [35] later considered variants obtained by replacing $(\vee\Rightarrow)$ and $(\Rightarrow\wedge)$ with:

$$\frac{\Gamma_1, \alpha \Rightarrow \Delta_1 \quad \Gamma_2, \beta \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \alpha \vee \beta \Rightarrow \Delta_1, \Delta_2}\ (\vee\Rightarrow)' \qquad \frac{\Gamma_1 \Rightarrow \alpha, \Delta_1 \quad \Gamma_2 \Rightarrow \beta, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \alpha \wedge \beta, \Delta_1, \Delta_2}\ (\Rightarrow\wedge)'$$

It is an easy exercise to see that these rules are interderivable with the previous rules given for $\wedge$ and $\vee$, making crucial use of the structural rules of weakening, exchange, and contraction. In the absence of such rules, the connectives $\wedge$ and $\vee$ split into two. That is, the original rules define what are often called the *additive* or *lattice* connectives $\wedge$ and $\vee$, whereas Ketonen and Curry's rules define the so-called *multiplicative* or *group* connectives, renamed $\cdot$ and $+$. Moreover, in the absence of weakening rules, as we will see below, the constants 1 and 0 also split, as in the absence of exchange rules, does the implication connective $\to$.

5.2. **Substructural logics.** The expression "substructural logic" was suggested by Kosta Došen and Peter Schroeder-Heister at a conference in Tübingen in 1990 to describe a family of logics emerging (post-Gentzen) with a wide range of motivations from linguistics, algebra, set theory, philosophy, and computer science. Roughly speaking, the term "substructural" refers to the fact that these logics, which all live in a certain sense "below the surface" of classical logic, fail to admit one or more classically sound structural rules. Most convincingly, logics defined by sequent calculi obtained by removing weakening, contraction, or exchange rules from LJ or LK may be deemed substructural, although even in these cases, further logical rules may be added to capture connectives that (as remarked above) split when structural rules are removed. Other classically sound structural rules, such as "weaker" versions of weakening or contraction, may also be added, giving a family of logics characterized by cut-free sequent calculi. Nevertheless, there remain important classes of logics (e.g., relevant and fuzzy logics) typically accepted as substructural that do not fit comfortably into this framework, requiring more flexible formalisms such as hypersequents, display calculi, etc. More perplexing still, there are closely related logics (and classes of algebras) lacking structural rules for which no reasonable cut-free calculus is known. Are these also substructural?

A practical answer to this question, suggested by the authors of [51], is to define substructural logics by appeal to their algebraic semantics. That is, since most substructural logics correspond in some way to classes of *residuated lattices* (or slight variants thereof), this family could be identified with logics having these classes of algebras as equivalent algebraic semantics. Such a definition offers uniformity and clarity, although it may be objected that there exist both classes of algebras which have no corresponding logic and substructural logics which lack a corresponding class of residuated lattices. Here we deliberately refuse to say exactly what a substructural logic is, believing rather that an understanding of the richness of this family is best gained by a (necessarily brief, see [41], [117], [110], and [51] for a wealth of further material) historical survey of the most important candidates.

The Lambek calculus and residuated lattices. Chronologically, the first substructural logic occurred in the field of linguistics. In a 1958 paper [88], Joachim Lambek made use of a substructural sequent calculus (which became known, naturally enough, as the Lambek calculus) to represent transformations on syntactic types of a formal grammar. Lambek's approach built on earlier work on *categorial grammar* in the 1930's by the Polish logician Kazimierz Ajdukiewicz, who aimed to develop an analysis of natural language by assigning syntactic types to linguistic expressions that describe their syntactic roles (e.g., verb, noun phrase, verb phrase, sentence). A naive approach to this task would consist of listing a number of lexical atoms (e.g., Joan, smiles, charmingly) and a number of mutually unrelated types (e.g., NP = noun phrase; V = verb; Adv = adverb; VP = verb phrase; S = sentence), and then tagging each lexical atom with the appropriate type:

$$\underline{\text{Joan}}\text{: NP; }\underline{\text{smiles}}\text{: V; }\underline{\text{charmingly}}\text{: Adv.}$$

However, Ajdukiewicz understood that the stock of basic grammatical categories can be substantially reduced by the use of *type-forming operators* $\backslash$ and $/$, where an expression v has type $\alpha\backslash\beta$ (respectively, β/α) if whenever the expression v' has type α, the expression $v'v$ (respectively, vv') has type β. Indeed, the whole apparatus of categorial grammar can then be constructed out of just two basic types, n (noun) and s (sentence).

Axioms

$$\frac{}{\alpha \Rightarrow \alpha}\ (\textsc{id})$$

Cut rule

$$\frac{\Gamma_1 \Rightarrow \alpha \quad \Gamma_2, \alpha, \Gamma_3 \Rightarrow \Delta}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta}\ (\textsc{cut})$$

Left logical rules

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, 1, \Gamma_2 \Rightarrow \Delta}\ (1 \Rightarrow)$$

$$\frac{}{0 \Rightarrow}\ (0 \Rightarrow)$$

$$\frac{\Gamma_2 \Rightarrow \alpha \quad \Gamma_1, \beta, \Gamma_3 \Rightarrow \Delta}{\Gamma_1, \beta/\alpha, \Gamma_2, \Gamma_3 \Rightarrow \Delta}\ (/ \Rightarrow)$$

$$\frac{\Gamma_2 \Rightarrow \alpha \quad \Gamma_1, \beta, \Gamma_3 \Rightarrow \Delta}{\Gamma_1, \Gamma_2, \alpha\backslash\beta, \Gamma_3 \Rightarrow \Delta}\ (\backslash \Rightarrow)$$

$$\frac{\Gamma_1, \alpha, \beta, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha \cdot \beta, \Gamma_2 \Rightarrow \Delta}\ (\cdot \Rightarrow)$$

$$\frac{\Gamma_1, \alpha, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha \wedge \beta, \Gamma_2 \Rightarrow \Delta}\ (\wedge\Rightarrow)_1$$

$$\frac{\Gamma_1, \beta, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha \wedge \beta, \Gamma_2 \Rightarrow \Delta}\ (\wedge\Rightarrow)_2$$

$$\frac{\Gamma_1, \alpha, \Gamma_2 \Rightarrow \Delta \quad \Gamma_1, \beta, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha \vee \beta, \Gamma_2 \Rightarrow \Delta}\ (\vee\Rightarrow)$$

Right logical rules

$$\frac{}{\Rightarrow 1}\ (\Rightarrow 1)$$

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (\Rightarrow 0)$$

$$\frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta/\alpha}\ (\Rightarrow /)$$

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta}\ (\Rightarrow \backslash)$$

$$\frac{\Gamma_1 \Rightarrow \alpha \quad \Gamma_2 \Rightarrow \beta}{\Gamma_1, \Gamma_2 \Rightarrow \alpha \cdot \beta}\ (\Rightarrow \cdot)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow\vee)_1$$

$$\frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow\vee)_2$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\Rightarrow\wedge)$$

Figure 2. The Full Lambek Calculus FL

For example, in English, John works is a sentence, but works John is not. The intransitive verb works has type $n\backslash s$: when applied to the right of an expression of type n, it yields an expression of type s. On the other hand, the adjective poor has type n/n: when applied to the left of an expression of type n, it yields another expression of type n (a complex noun phrase). We may write these transformations, respectively, as $n, n\backslash s \Rightarrow s$ and $n/n, n \Rightarrow n$. More generally, if α, β are types, the following transformations are permissible in Ajdukiewicz's categorial grammar:

$$\alpha, \alpha\backslash\beta \Rightarrow \beta \quad \text{and} \quad \beta/\alpha, \alpha \Rightarrow \beta.$$

Lambek extended the deductive power of categorial grammar by setting up a sequent calculus for permissible transformations on types, introducing a new type-forming operation $\cdot$ such that v has type $\alpha \cdot \beta$ whenever $v = v'v''$ with v' of type α and v'' of type β, and admitting, in addition to modus ponens, patterns of *hypothetical reasoning* corresponding to right introduction rules for the implications. Adding rules for the lattice connectives $\wedge$ and $\vee$, and the constants 1 and 0, gives the *Full Lambek Calculus* FL, displayed in Figure 2.

FL has come to play a distinguished role in the field of substructural logics. Just as classical logic is a candidate for the top element of the lattice of such logics, so FL is a candidate for the bottom element. That is, most other substructural logics may be obtained as extensions of FL (although, non-associative substructural logics have also been investigated, not least by Lambek himself.) In particular, Hiroakira Ono and colleagues have popularized the usage of FL_X where $X \subseteq \{e, c, w\}$ to denote the extension of FL with the appropriate grouping of exchange (e), contraction (c), and weakening rules (w), and InFL_X to denote the corresponding multiple-conclusion sequent calculus. In particular, $\mathrm{FL_{ewc}}$ and $\mathrm{InFL_{ewc}}$ correspond to LJ and LK with split connectives.

Not surprisingly, FL corresponds to the class of FL-algebras. Moreover, a sequent calculus RL for the class of residuated lattices is obtained by removing the rules for 0, as the following result makes precise. Let $\Box(\alpha_1, \dots, \alpha_n)$ stand for $\alpha_1 \Box \dots \Box \alpha_n$ for $\Box \in \{\cdot, +\}$ where $\cdot()$ is 1 and $+()$ is 0, and define:

$$\begin{aligned} \tau(\alpha \approx \beta) &= \{\alpha \Rightarrow \beta, \beta \Rightarrow \alpha\}; \\ \rho(\Gamma \Rightarrow \Delta) &= \{\cdot\Gamma \leq +\Delta\}. \end{aligned}$$

Theorem 5.3. *$X \vdash_{\mathrm{RL}} S$ if and only if $\{\rho(S') \mid S' \in X\} \vdash_{Eq(\mathcal{RL})} \rho(S)$.*

Proof. The left-to-right direction is proved by induction on the height of a derivation in RL. For the right-to-left direction, consider

$$\Sigma \cup \{(\alpha, \beta)\} \subseteq Fm^2$$

and define

$$\Sigma^{\Rightarrow} = \{\alpha' \Rightarrow \beta' | (\alpha', \beta') \in \Sigma\} \quad \text{and} \quad \Sigma^{\leq} = \{\alpha' \leq \beta' | (\alpha', \beta') \in \Sigma\}.$$

We will prove that $\Sigma^{\leq} \vdash_{Eq(\mathcal{RL})} \alpha \leq \beta$ implies $\Sigma^{\Rightarrow} \vdash_{\mathrm{RL}} \alpha \Rightarrow \beta$; the result then follows swiftly from the fact that for any sequent S, $S \vdash_{\mathrm{RL}} \tau(\rho(S))$ and $\tau(\rho(S)) \vdash_{\mathrm{RL}} S$.

Define the following binary relation on Fm:

$$\alpha \Theta_\Sigma \beta \qquad \text{iff} \qquad \Sigma^{\Rightarrow} \vdash_{\mathrm{RL}} \alpha \Rightarrow \beta \text{ and } \Sigma^{\Rightarrow} \vdash_{\mathrm{RL}} \beta \Rightarrow \alpha.$$

Θ_Σ is a congruence on $\mathbf{Fm}$. Clearly it is reflexive and symmetric, and for transitivity, if $\alpha\Theta_\Sigma\beta$ and $\beta\Theta_\Sigma\gamma$, then $\Sigma^{\Rightarrow} \vdash_{\mathrm{RL}} \{\alpha \Rightarrow \beta, \beta \Rightarrow \gamma\}$, so by (CUT), $\Sigma^{\Rightarrow} \vdash_{\mathrm{RL}} \alpha \Rightarrow \gamma$; similarly, $\Sigma^{\Rightarrow} \vdash_{\mathrm{RL}} \gamma \Rightarrow \alpha$, so $\alpha\Theta_\Sigma\gamma$ as required. Suppose, moreover, that $\alpha_1\Theta_\Sigma\alpha_2$ and $\beta_1\Theta_\Sigma\beta_2$. Then $\Sigma^{\Rightarrow} \vdash_{\mathrm{RL}} \{\alpha_1 \Rightarrow \alpha_2, \alpha_2 \Rightarrow \alpha_1, \beta_1 \Rightarrow \beta_2, \beta_2 \Rightarrow \beta_1\}$ and we can construct, e.g., derivations for $\Sigma^{\Rightarrow} \vdash_{\mathrm{RL}} \{\alpha_1\backslash\beta_1 \Rightarrow \alpha_2\backslash\beta_2, \alpha_1 \wedge \beta_1 \Rightarrow \alpha_2 \wedge \beta_2\}$ ending:

$$\dfrac{\dfrac{\alpha_2 \Rightarrow \alpha_1 \quad \beta_1 \Rightarrow \beta_2}{\alpha_2, \alpha_1\backslash\beta_1 \Rightarrow \beta_2}\,(\backslash\Rightarrow)}{\alpha_1\backslash\beta_1 \Rightarrow \alpha_2\backslash\beta_2}\,(\Rightarrow\backslash) \qquad \dfrac{\dfrac{\alpha_1 \Rightarrow \alpha_2}{\alpha_1 \wedge \beta_1 \Rightarrow \alpha_2}\,(\wedge\Rightarrow)_1 \quad \dfrac{\beta_1 \Rightarrow \beta_2}{\alpha_1 \wedge \beta_1 \Rightarrow \beta_2}\,(\wedge\Rightarrow)_2}{\alpha_1 \wedge \beta_1 \Rightarrow \alpha_2 \wedge \beta_2}\,(\Rightarrow\wedge)$$

Hence (using symmetry), Θ_Σ is compatible with $\backslash$ and $\wedge$, and similarly, with the other operations.

It follows easily that the quotient algebra $\mathbf{Fm}/\Theta_\Sigma$ with equivalence classes $[\alpha]$ for $\alpha \in Fm$ as elements is a residuated lattice. So if $\Sigma^{\leq}\vdash_{Eq(\mathcal{RL})}\alpha \leq \beta$, we can define the canonical evaluation $e(x) = [x]$ and prove by induction that $e(\gamma) = [\gamma]$ for all $\gamma \in Fm_{\mathrm{RL}}$. Since $\Sigma^{\Rightarrow} \vdash_{\mathrm{RL}} \alpha' \Rightarrow \beta'$ for all $(\alpha', \beta') \in \Sigma$, we have $e(\alpha') = [\alpha'] \leq [\beta'] = e(\beta')$ for all $(\alpha', \beta') \in \Sigma$. Hence $[\alpha] = e(\alpha) \leq e(\beta) = [\beta]$ and, as required, $\Sigma^{\Rightarrow} \vdash_{\mathrm{RL}} \alpha \Rightarrow \beta$. ☐

This equivalence extends easily to FL_X-algebras and the calculi FL_X for $X \subseteq \{e, c, w\}$, and also to classes of bounded FL-algebras with respect to calculi $\mathrm{FL}_X^{\mathrm{B}}$, obtained by adding to FL_X the rules:

$$\dfrac{}{\Gamma_1, \bot, \Gamma_2 \Rightarrow \Delta}\,(\bot\Rightarrow) \qquad \dfrac{}{\Gamma \Rightarrow \top}\,(\Rightarrow\top)$$

Relevance logics. An important source of substructural logics lacking weakening rules is the philosophy of logic and the long-standing debate over *entailment*. As is well-known, contemporary modal logic originated with C.I. Lewis' dissatisfaction and critical attitude towards Russell's classical propositional calculus, notably its – supposedly counterintuitive and repugnant to common sense – "paradoxes of material implication" such as the laws of *a fortiori* and *ex absurdo quodlibet*:

$$\alpha \to (\beta \to \alpha) \quad \text{and} \quad \neg\alpha \to (\alpha \to \beta).$$

In a series of writings culminating in his *Symbolic Logic*, coauthored with C.H. Langford ([90]), Lewis introduced calculi of *strict implication* to describe a tighter notion of implication intended to be true whenever it is impossible that the antecedent holds while the consequent does not. These calculi avoid Russell's material paradoxes, yet derive "paradoxes of strict implication" such as:

$$\alpha \to (\beta \vee \neg\beta) \quad \text{and} \quad (\alpha \wedge \neg\alpha) \to \beta.$$

The Lewis-Langford analysis was therefore deemed inadequate by many commentators since it fails to take into account the *relevance* connection between the antecedent and consequent of a logical implication.

On the other hand, Lewis showed how to derive his paradoxes using just a few seemingly unobjectionable modes of inference ([90], Ch.8), e.g., for *ex absurdo quodlibet*:

1.	$\alpha \wedge \neg\alpha$	assumption;
2.	α	1, simplification: from $\alpha \wedge \beta$ derive α;
3.	$\neg\alpha$	1, simplification: from $\alpha \wedge \beta$ derive β;
4.	$\alpha \vee \beta$	2, addition: from α derive $\alpha \vee \beta$;
5.	β	3,4, disjunctive syllogism: from $\neg\alpha, \alpha \vee \beta$ derive β.

Hence the relevant logician must show what is wrong with this reasoning. Several possible replies were devised in the 1930's, 1940's, and 1950's. Connexive logicians (e.g., [103]) worked out a notion of entailment on the basis of two principles: that α entails β just in case α is inconsistent with $\neg\beta$, and that α entails β only if α is consistent with β. Such a concept validates some classically falsifiable principles and falsifies some classical tautologies and valid inference rules – for example, the simplification and addition moves in Lewis' independent proof are not permissible. Analytic logicians (e.g., [112]) defended a Kantian-like view according to which the consequent of an implication should not contain concepts not already included in its antecedent. Given such a tenet, addition is clearly no good. Some philosophers ([53], [134], [122]), rather than questioning a specific step in Lewis' argument, put the blame on the possibility of freely chaining such inferential steps together. The resulting notions of entailment are only restrictedly transitive. Finally, a few commentators ([43], [22]) detected an equivocation in Lewis' use of "entails", while others ([45], [135]) accepted the independent proof, but denied the conclusion that it shows that every impossible proposition implies anything.

A completely different reply to Lewis came in the late 1950's from the American logicians Alan R. Anderson and Nuel D. Belnap, who, developing ideas of W. Ackermann ([1]), introduced the systems of *relevant logic* E and R ([3],[4]). Anderson and Belnap ([2]) suggest that the argument by Lewis is a *fallacy of ambiguity*: it equivocates over the meaning of disjunction. There is no single disjunction that allows both addition and the disjunctive syllogism; rather, there is an *intensional* relevant disjunction for which the disjunctive syllogism is valid but addition is not; and an *extensional* truth-functional disjunction for which addition, but not the disjunctive syllogism, holds. Although the connection was only implicit in Anderson and Belnap's paper, the extensional "or" roughly corresponds to additive disjunction, while the intensional "or" corresponds to multiplicative disjunction.

The placing of Anderson and Belnap's R within the framework of substructural logics is most apparent for the implication-negation fragment: it is the corresponding fragment of LK without the weakening rules [3] (or, identifying $\backslash$ and $/$ with $\to$, the calculus InFL_{ec}). However, the full logic corresponding to LK without weakening (InFL_{ec}) is not R but rather a weaker logic, studied intensively by Robert K. Meyer in his PhD thesis [101] and known in relevant circles as LR, that fails to derive the critical half of the distributive law:

$$\alpha \wedge (\beta \vee \gamma) \to (\alpha \wedge \beta) \vee (\alpha \wedge \gamma).$$

For a partly satisfactory solution to the problem of finding a suitable Gentzen-style formulation for R, instead, logicians had to wait until 1982, when Belnap ([10]) introduced the formalism of display logic.

An algebraic semantics for R in terms of involutive and distributive FL_{ec} algebras, suggested by J.M. Dunn, was already available in 1966; later, Urquhart introduced an operational semantics for the implication fragment of R and, finally, in the early 1970's, a series of papers by R. Routley and R.K. Meyer launched a long-awaited Kripke-style relational semantics for both R and an array of kindred systems (see [4] and [115]). This new research trend triggered the introduction of additional relevant systems, mostly weaker than E or R, motivated by natural semantical conditions. Since the Routley-Meyer evaluation clauses for conjunction and disjunction validate distribution, such non-distributive logics as LR have remained out of the limelight over the last decades as far as relevant logics are concerned.

Fuzzy logics. The first examples of logics failing to admit the structural rule of contraction occurred in the setting of many-valued logics. In the 1920's, Jan Łukasiewicz introduced logics with n truth values for every finite $n > 2$, as well as an infinite-valued logic Ł over the closed real unit interval $[0, 1]$, where 0 represents absolute falsity, 1 represents absolute truth, and values between 0 and 1 can be thought of as intermediate "degrees of truth." The truth functions corresponding to negation and implication are

$$\neg x = 1 - x \quad \text{and} \quad x \to y = \min(1, 1 - x + y).$$

In classical logic, there are several equivalent ways to define disjunction using negation and implication; for example, $\alpha \vee \beta$ is classically equivalent to both $\neg\alpha \to \beta$ and $(\alpha \to \beta) \to \beta$. However, in Ł:

$$x \oplus y = \neg x \to y = \min(1, x+y) \quad \text{and} \quad x \vee y = (x \to y) \to y = \max(x, y).$$

Similarly, there are two possibilities for conjunction:

$$x \cdot y = \max(0, x + y - 1) \quad \text{and} \quad x \wedge y = \min(x, y).$$

An axiomatization for Ł was proposed by Łukasiewicz, but although a completeness proof was obtained by Wajsberg in the 1930's, published proofs, by Rose and Rosser [114] and Chang [27] (introducing *MV-algebras*, see Example 4.5), appeared only in the late 1950's.

In subsequent years, the study of Łukasiewicz logic has become ever more intertwined with research into *fuzzy logics*. Following the approach promoted by Hájek in his influential monograph [65], the conjunction connective $\cdot$ of fuzzy logics is interpreted by a continuous t-norm (commutative associative increasing binary function on $[0, 1]$ with unit 1), the implication connective $\to$ by its residuum, and falsity constant 0 by 0. Other connectives, 1 interpreted by 1, $\wedge$ by min, and $\vee$ by max, are definable. Fundamental examples of continuous t-norms are the Łukasiewicz conjunction $\max(0, x+y-1)$, Gödel t-norm $\min(x, y)$, and the product t-norm xy (product of reals), which give rise, respectively, to Łukasiewicz logic Ł, Gödel-Dummett logic G ([60], [42]), as well as a relative newcomer, product logic P ([66]).

Common generalizations of these logics have included (in chronological order) Hájek's *basic logic* BL ([65]), Esteva and Godo's *monoidal t-norm logic* MTL ([47]), and Metcalfe and Montagna's *uninorm logic* UL ([99]). The variety of UL-algebras $\mathcal{UL}$ consists of semilinear bounded commutative FL-algebras, the variety $\mathcal{MTL}$ consists of UL-algebras satisfying $1 \approx \top$ and $0 \approx \bot$, and the variety $\mathcal{BL}$ of BL-algebras consists of MTL-algebras satisfying $x \wedge y \approx x \cdot (x \to y)$. The importance of these logics and their algebras is supported by the fact that BL has been shown to be the logic of continuous t-norms ([31]), MTL, the logic of left-continuous t-norms ([79]), and UL, the logic of left-continuous uninorms ([99]). Algebraically, this means that the varieties $\mathcal{BL}$, $\mathcal{MTL}$, and $\mathcal{BL}$, are generated not only by their totally ordered members but also by their so-called "standard" members of the form $\langle [0, 1], \min, \max, \cdot, \to, 1, 0, \bot, \top \rangle$.

Perhaps surprisingly given their origins, it has emerged that many of these logics and their accompanying classes of algebras have an elegant presentation as Gentzen-style proof systems. The catch is that instead of sequents, they are formulated using *hypersequents*, introduced by Avron in [5] and consisting of finite multisets of sequents. The monograph [100] provides a comprehensive account of hypersequent (and other) calculi for fuzzy logics and their applications.

Set theory. A particularly intriguing motivation for dropping contraction rules arises from Curry's 1942 proof that the law of absorption, corresponding to the sequent $\Rightarrow (\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$, plays a crucial role in set-theoretic paradoxes ([34]). At the turn of the last century, Russell showed that naive set theory yields, via unrestricted comprehension, a formula provably equivalent to its own negation. This is a contradiction if the underlying logic derives the law of excluded middle and other classical principles. Some mathematicians, including Brouwer, began therefore to nurture the belief that an intuitionistically correct set theory would escape Russell's paradox. However, as Curry demonstrated, the following variant of Russell's paradox follows from intuitionistically acceptable principles. Let α be an arbitrary sentence in the language of the theory, and let $C = \{x : x \in x \to \alpha\}$. Then:

1.	$C \in C \to (C \in C \to \alpha)$	definition of C;
2.	$(C \in C \to \alpha) \to C \in C$	definition of C;
3.	$(C \in C \to (C \in C \to \alpha)) \to (C \in C \to \alpha)$	law of absorption;
4.	$C \in C \to \alpha$	1, 3, modus ponens;
5.	$C \in C$	2, 4, modus ponens;
6.	α	4, 5, modus ponens.

Hence any set theory which contains an unrestricted comprehension axiom and the (intuitionistically correct) law of absorption is bound to be trivial. From a substructural point of view, however, the excluded middle and the law of absorption are equally vicious: the former requires a use of right contraction, while the latter presupposes an application of left contraction. Intuitionistic logic is not substructural enough to accommodate naive set theory.

The first logician who explored the possibility of reconstructing set theory on a nonclassical basis was Skolem, who devoted a series of papers to the subject in the late 1950's and early 1960's (see, e.g., [121]). Skolem thought that infinite-valued Łukasiewicz logic Ł was a plausible candidate: Russell's paradoxical sentence is equivalent to its own negation, but this causes no problem in Ł where any formula whose value is 0.5 has this property. Moreover, absorption and other contraction-related principles do not hold for Ł.

However, weakening also plays a role in producing the set-theoretic paradoxes. A result by Grishin ([64]), in fact, indicates that if we add the extensionality axioms to classical logic without contraction (a weaker system than Ł), contraction can be recovered. Subsequent research on logical bases for naive set theories has therefore focused on systems in the vicinity of linear logic although the systems FL_{ew} and

InFL_{ew}, interesting for being decidable at the first order level, have also been investigated extensively by Ono and Komori in [109].

Linear logic. Further motivation for dropping structural rules arose from the constructive approach to logic. Since Heyting, followers of this approach have focused on the notion of proof, stressing, however, that what matters is not *whether* a given formula is provable, but *how* it is proved. A formula may therefore be identified with its set of proofs, so that a proof of α from the assumptions $\alpha_1, ..., \alpha_n$ – seen as a method for converting proofs of $\alpha_1, ..., \alpha_n$ into a proof of α – amounts to a function $f(x_1, ..., x_n)$ which associates to elements $a_i \in A_i$, the element $f(a_1, ..., a_n) \in A$.

This idea is already implicit in the so-called Brouwer-Heyting-Kolmogorov interpretation of intuitionistic logic. In the 1960's, W.A. Howard ([78]) added to this interpretation the identification of intuitionistic natural deduction proofs with terms of typed lambda calculus. A proof of the formula α is associated with a term of type α, and it then becomes possible to spell out the computational content of the inference rules in the $\{\wedge, \to\}$-fragment of the intuitionistic natural deduction calculus:

- if t and s are terms of respective types α and β, then $\langle t, s\rangle$ (the *pairing* of t and s) is a term of type $\alpha \wedge \beta$;
- if t is a term of type $\alpha \wedge \beta$, then $\pi_1(t)$ and $\pi_2(t)$ (the first and second *projections* of t) are terms of respective types α and β;
- if x is a variable of type α and t is a term of type β, then $\lambda x.t$ (the *abstraction* of t w.r.t. x) is a term of type $\alpha \to \beta$;
- if t and s are terms of respective types $\alpha \to \beta$ and α, then ts (the *application* of t to s) is a term of type β.

The ensuing correspondence between intuitionistic natural deduction proofs and terms in the lambda calculus with projection and pairing functors can be seen as a fully-fledged isomorphism (and is indeed referred to as the *Curry-Howard isomorphism*) in that there is a perfect match between the notions of conversion, normality, and reduction introduced in the two frameworks.

In the light of the Curry-Howard isomorphism, it was acknowledged that the problem of finding a "semantics of proofs" for a given constructive logic and the problem of providing lambda calculus (or, for that matter, functional programming) with a semantic interpretation were two sides of the same coin. In Dana Scott's *domain theory*, a first attempt to accomplish this task, a type α was interpreted by means of a particular topological space. On the other hand, Jean-Yves Girard ([57]) introduced for this purpose in the mid 1980's, the notion of a *coherent space* – a set A equipped with a reflexive and symmetric relation $R^{\mathbf{A}}$, called the *coherence relation* of the space. Now, a type (*alias* formula) α can be associated with a coherent space $\mathbf{A} = \langle A, R^{\mathbf{A}} \rangle$, and a term t of type α (*alias* a proof of α) with a *clique* of $\mathbf{A}$, i.e., with a subset $B \subseteq A$ of pairwise coherent elements of A. Similarly, compound formulas and their proofs can be interpreted using more complex coherent spaces. This semantics of proofs allows for substructural distinctions; for example:

- The space $\mathbf{A} \cdot \mathbf{B}$ is a coherent space whose universe is the cartesian product $A \times B$, and whose coherence relation is given by:
$$(a, b) R^{\mathbf{A} \cdot \mathbf{B}} (a', b') \quad \text{iff} \quad a R^{\mathbf{A}} a' \text{ and } b R^{\mathbf{B}} b'.$$
- The space $\mathbf{A} \wedge \mathbf{B}$ is a coherent space whose universe is the disjoint union $A \uplus B = \{(a, 0) : a \in A\} \cup \{(b, 1) : b \in B\}$, and whose coherence relation is given by:
$$\begin{array}{l} (a,0) R^{\mathbf{A} \wedge \mathbf{B}} (a', 0) \quad \text{iff} \quad a R^{\mathbf{A}} a'; \\ (b,1) R^{\mathbf{A} \wedge \mathbf{B}} (b', 1) \quad \text{iff} \quad b R^{\mathbf{B}} b'; \\ (a,0) R^{\mathbf{A} \wedge \mathbf{B}} (b, 1) \text{ for any } a \in A, b \in B. \end{array}$$

Omitting details, let us just mention that Girard introduces a coherent space $\mathbf{A} \rightarrow \mathbf{B}$ corresponding to intuitionistic implication and a coherent space $\mathbf{A} \multimap \mathbf{B}$ corresponding to a new kind of implication, which he terms *linear* implication. If we let $!\mathbf{A}$ be the space whose universe is the set of finite cliques of $\mathbf{A}$, and whose coherence relation is given by
$$c R^{!\mathbf{A}} c' \quad \text{iff} \quad \text{there exists a clique } c'' \text{ of } \mathbf{A} \text{ such that } c, c' \subseteq c'',$$
then we get the fundamental property that the space $\mathbf{A} \multimap \mathbf{B}$ is isomorphic to the space $!\mathbf{A} \rightarrow \mathbf{B}$; in other words, the semantics of coherent spaces yields a decomposition of intuitionistic implication into a new kind of implication, linear implication, and a new unary operator, $!$. A new kind of logic, *linear logic*, had been born.

How can we make intuitive sense of this logic? One option is to view formulas as concrete resources that once consumed in a deduction to get some conclusion, cannot be recycled or reused. Formulas of the form $!\alpha$, on the other hand, represent "ideal" resources that can be reused at will. Thus, while the availability of an intuitionistic implication $\alpha \rightarrow \beta$ means that using as many α's as I might need I can get one β, the availability of a linear implication $\alpha \multimap \beta$ expresses the fact that using just one α I can get one β – something that squares perfectly with the coherent space isomorphism pointed out above. We can also view the other compound formulas of linear logic as concrete resources: for example, $\alpha \cdot \beta$ expresses the availability of *both* resource α and resource β, while $\alpha \wedge \beta$ expresses the availability of *any* one of these resources.

In his seminal 1987 paper, Girard introduces a sequent calculus for this new logic that corresponds (with a quite different syntax) to $\mathrm{InFL}_e^{\mathrm{B}}$ (i.e., LK without weakening and contraction, with the split connectives) extended with the following rules for the unary connective ! (*of course!*) (? (*why not?*) can be defined dually as $?\alpha = \neg!\neg\alpha$):

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{!\alpha, \Gamma \Rightarrow \Delta}\,(!\Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta}{!\alpha, \Gamma \Rightarrow \Delta}\,(!\textsc{wl}) \quad \frac{!\alpha, !\alpha, \Gamma \Rightarrow \Delta}{!\alpha, \Gamma \Rightarrow \Delta}\,(!\textsc{cl}) \quad \frac{!\Gamma \Rightarrow \alpha}{!\Gamma \Rightarrow !\alpha}\,(\Rightarrow!)$$

where $!\Gamma$ is obtained by prefixing ! to all formulas in Γ.

The importance of the exponentials is fully realized if we take into account the fact that Girard was not interested in setting up a logic weaker than classical or intuitionistic logic: he rather wanted a logic that permits a better analysis of proofs through a stricter control of structural rules. Exponentials are there precisely to recapture the deductive power of weakening and contraction, an aim that is attained – in a sense – by showing that both classical logic and intuitionistic logic can be embedded into linear logic.

6. The interplay of algebra and logic

As we have seen in earlier sections, although ordered algebras and logics have typically emerged from distinct traditions, they may nevertheless be related via algebraization. In this final section, we illustrate the benefits of this correspondence with some examples where techniques in one field may be used to solve problems in the other. Needless to say, we do not aim for completeness, not even for the few topics covered here; rather our intention is to share some of the core ideas involved in the proofs and provide a pointer to key references.

6.1. **Completeness.** One of the most surprising and revealing features of recent work on the correspondence between ordered algebras and logic is the encroachment of algebraic methods on that typically most syntactic of endeavors: establishing cut elimination for Gentzen systems. To be more precise (since cut elimination in proof-theoretic parlance usually implies giving an algorithm for removing cuts from derivations), these methods, originating independently in the work of Maehara ([93]) and Okada ([104],[105],[106]), establish the admissibility of cut for the cut-free system. Or to put matters yet another way, completeness is proved for the cut-free system with respect to some class of algebras. Such results do not supplant constructive proofs (where the elimination algorithm may be fundamental), but suffice for the kind of benefits, such as establishing decidability or interpolation (see below), that a cut-free Gentzen system affords a class of ordered algebras.

The algebraic approach to completeness has undoubtedly made Gentzen systems more attractive to algebraists, and promises to lead – through the combination with other algebraic techniques – to insights and proofs unobtainable solely by syntactic means. Indeed, algebraic methods have been employed by Terui ([131]) to give a semantic characterization of extensions of the sequent calculus RL for residuated lattices with (certain forms of) structural rules that admit cut elimination. Also, Ciabattoni, Galatos, and Terui ([29]) have combined algebraic and syntactic techniques to obtain an algorithm that converts Hilbert-style axioms of a certain form into structural rules for sequent and hypersequent calculi that preserve cut elimination.

Here our aim will be more modest. We will give the main ideas of the proof of the completeness of cut-free RL with respect to residuated lattices, taking elements of our presentation from [80], [9], and [131]. As a starting point, let us consider a construction of residuated lattices that has proved invaluable in several contexts. Let $\mathbf{M} = \langle M, \cdot, 1\rangle$ be a monoid, and for each $X, Y \subseteq M$, define:

$$\begin{aligned} X \cdot Y &= \{x \cdot y | x \in X \text{ and } y \in Y\}; \\ X\backslash Y &= \{y \in M | X \cdot \{y\} \subseteq Y\}; \\ Y/X &= \{y \in M | \{y\} \cdot X \subseteq Y\}. \end{aligned}$$

Then the "powerset algebra" $\wp(\mathbf{M}) = \langle \wp(M), \cap, \cup, \cdot, \backslash, /, \{1\}\rangle$ is easily seen to be a residuated lattice.

This is just the first step of the construction, however. Next, a special kind of map on $\wp(M)$ is used to refine this algebra according to the job at hand. A *nucleus* on the powerset $\wp(M)$ is a map $\gamma : \wp(M) \to \wp(M)$ satisfying $X \subseteq \gamma(X)$, $\gamma(\gamma(X)) \subseteq \gamma(X)$, $X \subseteq Y$ implies $\gamma(X) \subseteq \gamma(Y)$, and $\gamma(X) \cdot \gamma(Y) \subseteq \gamma(X \cdot Y)$.

Lemma 6.1. *If* $\mathbf{M}$ *is a monoid and* γ *is a nucleus on* $\wp(M)$*, then*

$$\wp(\mathbf{M})_\gamma = \langle \gamma(\wp(M)), \cap, \cup_\gamma, \cdot_\gamma, \backslash, /, \gamma(\{1\})\rangle$$

is a complete residuated lattice with $X \cup_\gamma Y = \gamma(X \cup Y)$ *and* $X \cdot_\gamma Y = \gamma(X \cdot Y)$*.*

Now we turn our attention to the sequent calculus RL and residuated lattices. The idea is to construct a special example of the latter such that (similar to the Lindenbaum-Tarski algebra construction), validity in this algebra corresponds to cut-free derivability in RL. Let $\mathbf{Fm}^*$ be the free monoid generated by the formulas of RL; that is, the elements Fm^* of $\mathbf{Fm}^*$ are finite sequences of formulas, multiplication is concatenation, and the unit element is the empty sequence. Intuitively, we build our algebra from sets of sequences of formulas that "play the same role" in cut-free derivations in RL. We define:

$$\begin{array}{rcl}[\Gamma_1\_\Gamma_2 \Rightarrow \alpha] & = & \{\Gamma \in Fm^* | \Gamma_1, \Gamma, \Gamma_2 \Rightarrow \alpha \text{ is cut-free derivable in RL}\};\\ \mathcal{D} & = & \{[\Gamma_1\_\Gamma_2 \Rightarrow \alpha] | \Gamma_1, \Gamma_2 \in Fm^* \text{ and } \alpha \in Fm\};\\ \gamma(X) & = & \bigcap\{Y \in \wp(Fm^*) | X \subseteq Y \subseteq \mathcal{D}\}.\end{array}$$

Then γ is a nucleus on $\wp(Fm^*)$ and hence the algebra $\wp(\mathbf{Fm}^*)_\gamma$ is a residuated lattice.

We define an evaluation for this algbera by $e(p) = \gamma(\{p\})$ and prove by induction on formula complexity that for each $\alpha \in Fm$:

$$\alpha \in e(\alpha) \subseteq [\_ \Rightarrow \alpha].$$

Now consider a sequent $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ such that $\alpha_1 \cdot \ldots \cdot \alpha_n \leq \beta$ holds in all residuated lattices. In particular, it holds in $\wp(\mathbf{Fm}^*)_\gamma$, so $e(\alpha_1) \cdot \ldots \cdot e(\alpha_n) \subseteq e(\beta)$. But then, since $\alpha_i \in e(\alpha_i)$ for $i = 1 \ldots n$ and $e(\beta) \subseteq [\_ \Rightarrow \beta]$,

$$\alpha_1 \cdot \ldots \cdot \alpha_n \in [\_ \Rightarrow \beta].$$

I.e., $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ is cut-free derivable in RL.

The core idea of proofs of the form outlined above is to show the completeness of a system by establishing the admissibility of a particular rule for that system: in this case, cut. This idea applies also to other completeness results and gives corresponding generation results for classes of algebras. In particular, elimination of the "density rule" of Takeuti and Titani ([128]) has been used by Metcalfe and Montagna ([99], see also [100]) to show the completeness of certain fuzzy logics with respect to algebras based on the rational unit interval $[0, 1] \cap \mathbb{Q}$, or, reformulating this algebraically, the generation of certain varieties of semilinear commutative bounded FL-algebras by their dense totally ordered members. First it is shown that adding the density rule to any axiomatic extension of a Hilbert system UL for semilinear commutative bounded FL-algebras gives a system that is complete with respect to the dense totally ordered members of the corresponding variety. For the second step, axiomatizations are reformulated as hypersequent calculi and it is shown that in certain cases, applications of the hypersequent version of the density rule can be eliminated (constructively, similarly to cut elimination) from derivations.

6.2. **Decidability.** Decidability problems – determining whether there exists an effective method for checking membership of some class – have long played a prominent role in both logic and algebra, bridging the gap between abstract presentations and computational methods. Perhaps most significant are the validity problem for a logic L – can we decide whether $\vdash_{\mathrm{L}} \alpha$ holds for any formula α? – and, for a class of algebras $\mathcal{K}$, the decidability of the equational theory – can we decide whether $\vdash_{Eq(\mathcal{K})} \alpha \approx \beta$ holds for any equation $\alpha \approx \beta$? Of course, decidability of the validity problem for an algebraizable logic implies decidability of the equational theory for the corresponding class of algebras and vice versa.

Intriguingly, when tackling these problems for substructural logics and classes of residuated lattices, methods from both fields, logic and algebra, appear to be essential. Let us consider first a strategy for logics that makes use of cut-free Gentzen systems. We have already seen that cut elimination for LJ and LK facilitates a simple proof of decidability of the validity problem for propositional classical logic and propositional intuitionistic logic, and consequently, the equational theory of Boolean algebras and Heyting algebras. The proof for the sequent calculus RL and neighboring systems admitting cut elimination such as FL, FL_e, FL_w, and FL_{ew} is even easier. We simply observe that sequents occurring in any cut-free derivation for these systems must get smaller as we progress upwards in the tree, so proof search is finite. The proof for FL_{ec} (a reworking of a proof for InFL_{ec} by Meyer [101]) is more complicated, but follows a similar pattern: we introduce a restricted version of the calculus and show that the number of sequents occurring in cut-free derivations for the restricted system must be finite (see, e.g., [51] for details).

Cut-free sequent calculi are powerful tools but there remain many substructural logics, not too mention interesting classes of algebras, that are (at least so far) lacking in this department. In some cases, however, more complicated structures can be used to obtain decidability results. For example, display calculi, mentioned in the remarks on relevance logics, with more structural connectives have been used to establish decidability for the equational theory of various varieties of distributive residuated lattices ([20], [21], [86]). Also, hypersequent calculi, mentioned above in connection with fuzzy logics, have been used to establish decidability for various semilinear varieties (see [100] for details).

A complementary algebraic approach to establishing decidability stems from the familiar observation that if a finitely axiomatizable logic is complete with respect to a class of finite algebras – the so-called *finite model property* – then we can check whether a given formula holds in the one-element members, then the two-element members, and so on, and at the same time search for a derivation of height one, height two, etc. Hence the validity problem is decidable. From an algebraic perspective, we say that a class of algebras has the *finite model property* FMP if every equation that fails to hold in the class, fails in a finite member of the class. However, when the forthcoming algebraic method applies, we actually get something stronger, the *strong finite model property* SFMP: any *quasi-equation* that fails to hold in the class, fails in a finite member of the class. If a finitely axiomatizable class of algebras has the FMP, then its equational theory is decidable, and if it has the SFMP, then its quasi-equational theory (and in fact, for quasivarieties of residuated lattices, its universal theory) is decidable.

The SFMP corresponds in turn to an embedding property known to researchers such as McKinsey and Tarski in the 1940's ([98]), explored in particular by Evans ([48]), and developed extensively for residuated lattices and related structures by Blok and Van Alten ([14], [16], [133]). Given an algebra $\mathbf{A} = \langle A, \langle f_i^{\mathbf{A}} | i \in I\rangle\rangle$ of any type and $B \subseteq A$, a *partial subalgebra* $\mathbf{B}$ of $\mathbf{A}$ is the partial algebra $\langle B, \langle f_i^{\mathbf{B}} | i \in I\rangle\rangle$ where for $i \in I$, k-ary f_i, and $b_1, \ldots, b_k \in B$,

$$f_i^{\mathbf{B}}(b_1, \ldots, b_k) = \begin{cases} f_i^{\mathbf{B}}(b_1, \ldots, b_k) & \text{if } f_i^{\mathbf{B}}(b_1, \ldots, b_k) \in B \\ \text{undefined} & \text{otherwise.} \end{cases}$$

An *embedding* of a partial algebra $\mathbf{B}$ into an algebra $\mathbf{A}$ of the same type is a 1-1 map $\varphi : B \to A$ such that $\varphi(f_i^{\mathbf{B}}(b_1, \ldots, b_k)) = f_i^{\mathbf{A}}(\varphi(b_1), \ldots, \varphi(a_k))$ whenever $f_i^{\mathbf{B}}(b_1, \ldots, b_k)$ is defined.

A class $\mathcal{K}$ of algebras of the same type has the *finite embeddability property* (FEP for short) if every finite partial subalgebra of some member of $\mathcal{K}$ can be embedded into some finite member of $\mathcal{K}$. It is easily seen that if a (quasi)variety $\mathcal{K}$ has the FEP, then $\mathcal{K}$ has the SFMP. Moreover, for quasivarieties of finite type such as (quasi)varieties of residuated lattices we have an equivalence between the two properties.

The FEP is easily established for the variety of Heyting algebras $\mathcal{HA}$ (the subvariety of $\mathcal{CFL}$ satisfying the additional equations $xy \approx x \wedge y$ and $x \wedge 0 \approx 0$; see Example 4.2). Following McKinsey and Tarski's proof ([98]), let $\mathbf{B}$ be a finite partial subalgebra of some $\mathbf{A} \in \mathcal{HA}$. Then the lattice $\mathbf{D}$ generated by $B \cup \{0, 1\}$ is a finitely generated distributive lattice and hence finite, even though this might not be true of the Heyting algebra finitely generated by $\mathbf{B}$. Since the meet operation in any finite distributive lattice is residuated, $\mathbf{D}$ can be made into a Heyting algebra. Moreover, the partially defined residuum operation of $\mathbf{B}$ coincides (where defined) with the residuum of the meet of $\mathbf{D}$, so $\mathbf{B}$ can be embedded into this algebra.

This proof, however, relies on the fact that the lattice reduct of a member of the variety is distributive, and that the meet coincides with the product. A more complicated construction has been introduced by Blok and Van Alten ([14], [16], [133]) that establishes the FEP for numerous subvarieties of $\mathcal{FL}$ (and many other classes of algebras) obeying some kind of integrality or idempotency property. In particular, this construction was used by Ono (private communication, see also [85]) to establish the decidability of various semilinear varieties corresponding to fuzzy logics; a simplified presentation may be found in [28].

For varieties of residuated lattices such as $\mathcal{RL}$ and $\mathcal{CRL}$ that lack integrality and idempotence properties, (versions of) the following algebra based on the integers provides a good candidate for a counterexample:

$$\mathbf{Z} = \langle \mathbb{Z}, \min, \max, +, \to, 0 \rangle,$$

where $x \to y = -x + y$. (Refer to Example 4.4.) Consider the quasi-equation:

$$(1 \leq x \ \ \& \ \ x \cdot y \approx 1) \quad \Rightarrow \quad (x \approx 1).$$

It is easy to see that this holds in all finite residuated lattices, but fails in **Z**. So the SFMP and hence the FEP fails for $\mathcal{RL}$ and $\mathcal{CRL}$.

Finally, what of classes of algebras for which these algebraic or syntactic methods do not suffice? Proofs that the universal theory of (commutative) residuated lattices are undecidable may be lifted from the corresponding proofs for full linear logic of [91]. Urquhart proved that certain varieties of distributive residuated lattices are undecidable ([132]). Finally, many interesting problems are open. In particular, it is unknown whether the variety of cancellative residuated lattices (even adding commutativity and/or integrality) is decidable, or the variety of semilinear (commutative) residuated lattices. A selection of results with references to the first (perhaps implicit) proof is given in Table 2, omitting references when the result is folklore.

6.3. **Amalgamation and interpolation.** *Amalgamation* is an important categorical property of classes of algebras (and more generally in model theory, structures) that guarantees that under certain conditions, two members of the class that contain a common subalgebra can be regarded as subalgebras of a third member so that their intersection contains the common subalgebra. Close relationships between amalgamation and fundamental logical properties, including the *Robinson property*, *Beth definability*, and various forms of *interpolation*, are well known and much studied ([113], [6], [94], [95], [96], [92], [36]). Indeed, a broad and quite bewildering number of notions have been introduced to match exactly algebraic and logical concepts occurring in this area.

Variety	Name	Equational Theory	Universal Theory
Residuated lattices	$\mathcal{RL}$	decidable	undecidable [80]
Commutative $\mathcal{RL}$	$\mathcal{CRL}$	decidable	undecidable [91]
Distributive $\mathcal{RL}$	$\mathcal{DRL}$	decidable [86]	undecidable [50]
Distributive $\mathcal{CRL}$	$\mathcal{CDRL}$	decidable [21]	undecidable [50]
Idempotent $\mathcal{CDRL}$	$\mathcal{CIdDRL}$	undecidable [132]	undecidable [132]
Integral $\mathcal{RL}$	$\mathcal{IRL}$	decidable	decidable [14]
Integral $\mathcal{CRL}$	$\mathcal{CIRL}$	decidable	decidable [14]
Semilinear $\mathcal{RL}$	$\mathcal{SemRL}$		
Semilinear $\mathcal{CRL}$	$\mathcal{CSemRL}$		
MTL-algebras	$\mathcal{MTL}$	decidable [85]	decidable [85]
Cancellative $\mathcal{RL}$	$\mathcal{CanRL}$		
Cancellative $\mathcal{CRL}$	$\mathcal{CCanRL}$		
ℓ-groups	$\mathcal{LG}$	decidable [77]	undecidable [58]
MV-algebras	$\mathcal{MV}$	decidable	decidable
Abelian ℓ-groups	$\mathcal{AbLG}$	decidable [76]	decidable [76]
Heyting algebras	$\mathcal{HA}$	decidable [54]	decidable [54]
Boolean algebras	$\mathcal{BA}$	decidable	decidable

Table 2. (Un)decidability of some subvarieties of $\mathcal{RL}$ and $\mathcal{FL}$

We refer to [51] and [84] for a guide to some of the choices on offer. Here we prefer to illustrate this quite general relationship between logic and algebra by focussing on one particularly useful example: a connection between amalgamation and the *deductive interpolation property*.

A variety $\mathcal{V}$ has the *amalgamation property* AP if for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{V}$ and embeddings i and j of $\mathbf{A}$ into $\mathbf{B}$ and $\mathbf{C}$, respectively, there exist $\mathbf{D} \in \mathcal{V}$ and embeddings h, k of $\mathbf{B}$ and $\mathbf{C}$, respectively, into $\mathbf{D}$ such that $h \circ i = k \circ j$.

Let us write $var(K)$ for the variables occurring in some expression (formula, equation, set of equations, etc.) K. A variety $\mathcal{V}$ is said to have the *Deductive Interpolation Property* DIP if whenever $\Sigma \vdash_{Eq(\mathcal{V})} \varepsilon$, there exists a set of equations Π with $var(\Pi) \subseteq var(\Sigma) \cap var(\varepsilon)$ such that $\Sigma \vdash_{Eq(\mathcal{V})} \Pi$ and $\Pi \vdash_{Eq(\mathcal{V})} \varepsilon$.

Theorem 6.2. *A variety of commutative residuated lattices has the* AP *iff it has the* DIP*.*

The preceding result is stated and proved in [52]. However, it is a consequence of the general fact that for varieties with at least one nullary operation and all operations of finite arity, the AP and the DIP are equivalent in the presence of the congruence extension property. This being said, the result paves the way for an intriguing approach to proving amalgamation for varieties of commutative residuated lattices: namely, we show that a related interpolation property holds for the corresponding logic. Let us say that a logic L of a variety of commutative residuated lattices has the *Craig interpolation property* CIP if whenever $\vdash_{\mathrm{L}} \alpha \to \beta$, there exists a formula γ with $var(\gamma) \subseteq var(\alpha) \cap var(\beta)$ such that $\vdash_{\mathrm{L}} \alpha \to \gamma$ and $\vdash_{\mathrm{L}} \gamma \to \beta$. In fact, in the context of commutative residuated lattices, the CIP is a strictly stronger property, a consequence of the *local deduction theorem* (part (1) of the following theorem).

Theorem 6.3. *Suppose that a variety $\mathcal{V}$ of commutative residuated lattices is an equivalent algebraic semantics for a logic* L*. Then:*

(1) $T \cup \{\alpha\} \vdash_{\mathrm{L}} \beta$ iff $T \vdash_{\mathrm{L}} (\alpha \wedge 1)^n \to \beta$ for some $n \in \mathbb{N}$.

(2) If L *has the* CIP*, then $\mathcal{V}$ has the* DIP *and hence the* AP*.*

Proof. (1) is provable either by a simple induction on the height of a derivation of $T \cup \{\alpha\} \vdash_{\mathrm{L}} \beta$ or as an immediate consequence of Corollary 4.23 (1). For (2), it is suffices by algebraizability to prove the logical counterpart of the DIP for L. Suppose that $T \vdash_{\mathrm{L}} \alpha$. Then $\{\beta_1 \wedge \ldots \wedge \beta_n\} \vdash_{\mathrm{L}} \alpha$ for some $\{\beta_1, \ldots, \beta_n\} \subseteq T$ and by (1), $\vdash_{\mathrm{L}} (\beta_1 \wedge \ldots \wedge \beta_n \wedge 1)^n \to \alpha$ for some $n \in \mathbb{N}$. If L has the CIP, then $\vdash_{\mathrm{L}} (\beta_1 \wedge \ldots \wedge \beta_n \wedge 1)^n \to \gamma$ and $\vdash_{\mathrm{L}} \gamma \to \alpha$ for some formula γ with $var(\gamma) \subseteq var(\beta_1 \wedge \ldots \wedge \beta_n) \cap var(\alpha)$. But then, again by (1), $\{\beta_1 \wedge \ldots \wedge \beta_n\} \vdash_{\mathrm{L}} \gamma$ and $\{\gamma\} \vdash_{\mathrm{L}} \alpha$ as required. ☐

Theorem 6.4 ([107])**.** FL_e *has the* CIP*.*

Proof. For convenience (basically to reduce the number of cases considerably), we will make use of a slightly different calculus. Let us use (just for this proof) Γ and Δ to denote finite *multisets* rather than sequences of formulas; Γ, Δ and $()$ now denote the multiset union of Γ and Δ, and the empty multiset, respectively. Sequents are then ordered pairs of finite multisets of formulas and we obtain a calculus – easily seen to prove the same sequents as FL_e – simply by reinterpreting the calculus FL with this new definition. Let us make a couple more cosmetic changes. We remove the redundant exchange rules, write $\alpha \to \beta$ for $\alpha \backslash \beta$ and drop the rules for $/$, to obtain a calculus that we denote FL_e^m. It then suffices to prove the following:

If $\vdash_{\mathrm{FL}_e^m} \Gamma, \Delta \Rightarrow \alpha$, then there exists a formula β with $var(\beta) \subseteq var(\Gamma) \cap var(\Delta, \alpha)$ such that $\vdash_{\mathrm{FL}_e^m} \Gamma \Rightarrow \beta$ and $\vdash_{\mathrm{FL}_e^m} \Delta, \beta \Rightarrow \alpha$.

We proceed by induction on the height of a cut-free derivation d of $\Gamma, \Delta \Rightarrow \alpha$ in FL_e^m. Suppose first for the base case that $\Gamma, \Delta \Rightarrow \alpha$ is an instance of (ID). If Γ is (α) and Δ is $()$, let β be α. If Γ is $()$ and Δ is

(α), let β be 1. Also, if $\Gamma, \Delta \Rightarrow \alpha$ is an instance of $(\Rightarrow 1)$, then Γ and Δ are () and α is 1, so let β be 1. The case for 0 is similar.

For the inductive step, we must consider the last application of a rule in d. Let us just treat the paradigmatic case of implication. Suppose first that α is $\alpha_1 \to \alpha_2$ and d ends with:

$$\dfrac{\dfrac{\vdots}{\Gamma, \Delta, \alpha_1 \Rightarrow \alpha_2}}{\Gamma, \Delta \Rightarrow \alpha_1 \to \alpha_2}\ (\Rightarrow\to)$$

Then by the induction hypothesis, there exists β with $var(\beta) \subseteq var(\Gamma) \cap var(\Delta, \alpha_1 \to \alpha_2)$ such that $\Gamma \Rightarrow \beta$ and $\Delta, \beta, \alpha_1 \Rightarrow \alpha_2$ are derivable in FL_e^m. Hence also, by $(\Rightarrow\to)$, the sequent $\Delta, \beta \Rightarrow \alpha_1 \to \alpha_2$ is derivable in FL_e^m.

Suppose now that d ends with:

$$\dfrac{\dfrac{\vdots}{\Gamma_1, \Delta_1 \Rightarrow \gamma_1} \quad \dfrac{\vdots}{\Gamma_2, \Delta_2, \gamma_2 \Rightarrow \alpha}}{\Gamma_1, \Gamma_2, \gamma_1 \to \gamma_2, \Delta_1, \Delta_2 \Rightarrow \alpha}\ (\to\Rightarrow)$$

There are two subcases. First, suppose that Γ is Γ_1, Γ_2 and Δ is $\gamma_1 \to \gamma_2, \Delta_1, \Delta_2$. Then by the induction hypothesis twice, there exist β_1, β_2 such that the following sequents are derivable in FL_e^m:

$$\Gamma_1 \Rightarrow \beta_1 \qquad \Gamma_2 \Rightarrow \beta_2 \qquad \Delta_1, \beta_1 \Rightarrow \gamma_1 \qquad \Delta_2, \gamma_2, \beta_2 \Rightarrow \alpha$$

where $var(\beta_1), var(\beta_2) \subseteq var(\Gamma) \cap var(\Delta, \alpha)$. Let β be $\beta_1 \cdot \beta_2$. Then we have the following required derivations:

$$\dfrac{\dfrac{\vdots}{\Gamma_1 \Rightarrow \beta_1} \quad \dfrac{\vdots}{\Gamma_2 \Rightarrow \beta_2}}{\Gamma_1, \Gamma_2 \Rightarrow \beta_1 \cdot \beta_2}\ (\Rightarrow \cdot) \qquad \dfrac{\dfrac{\dfrac{\vdots}{\Delta_1, \beta_1 \Rightarrow \gamma_1} \quad \dfrac{\vdots}{\Delta_2, \gamma_2, \beta_2 \Rightarrow \alpha}}{\Delta_1, \Delta_2, \gamma_1 \to \gamma_2, \beta_1, \beta_2 \Rightarrow \alpha}\ (\to\Rightarrow)}{\Delta_1, \Delta_2, \gamma_1 \to \gamma_2, \beta_1 \cdot \beta_2 \Rightarrow \alpha}\ (\cdot \Rightarrow)$$

Now suppose that Γ is $\Gamma_1, \Gamma_2, \gamma_1 \to \gamma_2$ and Δ is Δ_1, Δ_2. Here we must be a bit more careful. Considering the derivable sequent $\Gamma_1, \Delta_1 \Rightarrow \gamma_1$, we associate Γ_1 with γ_1, and obtain by the induction hypothesis, a formula β_1 with $var(\beta_1) \subseteq var(\Gamma) \cap var(\Delta, \alpha)$ such that the following sequents are FL_e^m-derivable:

$$\Delta_1 \Rightarrow \beta_1 \qquad\qquad \Gamma_1, \beta_1 \Rightarrow \gamma_1.$$

For the derivable sequent $\Gamma_2, \Delta_2, \gamma_2 \Rightarrow \alpha$, we apply the induction hypothesis to obtain β_2 with $var(\beta_2) \subseteq var(\Gamma) \cap var(\Delta, \alpha)$ such that the following sequents are FL_e^m-derivable:

$$\Gamma_2, \gamma_2 \Rightarrow \beta_2 \qquad\qquad \Delta_2, \beta_2 \Rightarrow \alpha.$$

Variety	Name	CIP	DIP	AP
Residuated lattices	$\mathcal{RL}$	yes	?	?
Commutative $\mathcal{RL}$	$\mathcal{CRL}$	yes	yes	yes
Integral $\mathcal{CRL}$	$\mathcal{CIRL}$	yes	yes	yes
Semilinear $\mathcal{CRL}$	$\mathcal{CS}em\mathcal{RL}$	no	?	?
MTL-algebras	$\mathcal{MTL}$	no	?	?
ℓ-groups	$\mathcal{LG}$	no	no	no
MV-algebras	$\mathcal{MV}$	no	yes	yes
Abelian ℓ-groups	$\mathcal{A}b\mathcal{LG}$	no	yes	yes
Heyting algebras	$\mathcal{HA}$	yes	yes	yes
Boolean algebras	$\mathcal{BA}$	yes	yes	yes

Table 3. Amalgamation and interpolation properties for some subvarieties of $\mathcal{RL}$ and $\mathcal{FL}$

Let β be $\beta_1 \to \beta_2$. Then we have the following required derivations:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1, \beta_1 \Rightarrow \gamma_1} \quad \dfrac{\vdots}{\Gamma_2, \gamma_2 \Rightarrow \beta_2}}{\Gamma_1, \Gamma_2, \gamma_1 \to \gamma_2, \beta_1 \Rightarrow \beta_2}\,(\to\Rightarrow)}{\Gamma_1, \Gamma_2, \gamma_1 \to \gamma_2 \Rightarrow \beta_1 \to \beta_2}\,(\Rightarrow\to) \qquad \dfrac{\dfrac{\vdots}{\Delta_1 \Rightarrow \beta_1} \quad \dfrac{\vdots}{\Delta_2, \beta_2 \Rightarrow \alpha}}{\Delta_1, \Delta_2, \beta_1 \to \beta_2 \Rightarrow \alpha}\,(\to\Rightarrow) \qquad \square$$

Corollary 6.5. *$\mathcal{CRL}$ admits the* DIP *and therefore the* AP*.*

This proof method works also for proving amalgamation for $\mathcal{CIRL}$ and related varieties, as well as $\mathcal{BA}$ and $\mathcal{HA}$, but fails in the absence of a suitable Gentzen system. Indeed, we remark that there are many open problems regarding interpolation and amalgamation to be resolved for classes of residuated lattices. In particular, although $\mathcal{RL}$ has the Craig interpolation property, it is unknown whether it has the amalgamation or deductive interpolation properties. Table 3 summarizes the known landscape for a selection of subvarieties of $\mathcal{RL}$ and $\mathcal{FL}$.

References

[1] W. Ackermann, *Begrundung einer strengen Implikation*, J. Symbolic Logic **21, 2** (1956), 113-128.

[2] A.R. Anderson and N.D. Belnap, *Tautological entailments*, Philosophical Studies **13** (1962), 9-24.

[3] ———, *Entailment. The logic of relevance and necessity*, Princeton University Press, 1975.

[4] A.R. Anderson, N.D. Belnap, and J.M. Dunn, *Entailment: The Logic of Relevance and Necessity II*, Princeton University Press, 1992.

[5] A. Avron, *A Constructive Analysis of RM*, J. Symbolic Logic **52** (1987), no. 4, 939–951.

[6] P. D. Bacsish, *Amalgamation properties and interpolation theorems for equational theories*, Algebra Universalis **5** (1975), 45–55.

[7] R. Balbes and P. Dwinger, *Distributive Lattices*, University of Missouri Press, 1974.

[8] J. Barnes, *Logical form and logical matter*, Logica, Mente e Persona (A. Alberti, ed.), Olschki, Firenze, 1990, pp. 7–119.

[9] F. Belardinelli, P. Jipsen, and H. Ono, *Algebraic aspects of cut elimination*, Studia Logica **77** (2004), 209-240.

[10] N.D. Belnap, *Display logic*, Journal of Philosophical Logic **11** (1982), 375-417.

[11] G. Birkhoff, *Lattice Theory*, 1st ed., American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, R.I., 1940.

[12] W.J. Blok and D. Pigozzi, *Algebraizable Logics*, Memoirs of the AMS, American Mathematical Society, Providence, Rhode Island, 1989.

[13] W.J. Blok and J. Rebagliato, *Algebraic semantics for deductive systems*, Studia Logica **73** (2003), 1-29.

[14] W.J. Blok and C.J. van Alten, *The finite embeddability property for residuated lattices, pocrims and BCK-algebras*, Algebra Universalis **48** (2002), 253–271.

[15] ———, *Residuated Ordered Structures in Algebra and Logic*, Talk by W. Blok presented at Vanderbilt University, 2002.

[16] ———, *On the finite embeddability property for residuated ordered groupoids*, Trans. Amer. Math. Soc. **357** (2005), no. 10, 4141–4157.

[17] K. Blount, *On the structure of residuated lattices*, Ph.D. thesis, Vanderbilt University, Nashville, Tennessee, 1999.

[18] K. Blount and C. Tsinakis, *The structure of residuated lattices*, International Journal of Algebra and Computation **13** (2003), no. 4, 437–461.

[19] G. Boolos, *Frege's theorem and the Peano postulates*, Bulletin of Symbolic Logic **1** (1995), 317-326.

[20] R.T. Brady, *The Gentzenization and decidability of RW*, Journal of Philosophical Logic **19** (1990), 35–73.

[21] ———, *The Gentzenization and decidability of some contractionless relevant logics*, Journal of Philosophical Logic **20** (1991), 97–117.

[22] D.J. Bronstein, *The meaning of implication*, Mind **45, 178** (1936), 157-180.

[23] S. Burris and H.P. Sankappanavar, *A Course in Universal Algebra*, Graduate Texts in Mathematics, Springer, 1981, available online.

[24] A. Cantini, *I Fondamenti della Matematica* (1979).

[25] E. Casari, *Comparative logics and Abelian l-groups*, Logic Colloquium '88 (R. Ferro et al., ed.), North Holland, Amsterdam, 1989, pp. 161-190.

[26] J. Certaine, *Lattice-Ordered Groupoids and Some Related Problems*, Ph.D. thesis, Harvard University, Cambridge, MA, 1945.

[27] C.C. Chang, *A new proof of the completeness of the Lukasiewicz axioms*, Transactions of the AMS **93** (1959), 74–90.

[28] A. Ciabattoni, G. Metcalfe, and F. Montagna, *Algebraic and proof-theoretic characterizations of truth stressers for MTL and its extensions*, Fuzzy Sets and Systems **161** (2010), no. 3, 369–389.

[29] A. Ciabattoni, N. Galatos, and K. Terui, *From axioms to analytic rules in nonclassical logics*, Proceedings of LICS 2008, 2008, pp. 229-240.

[30] R. Cignoli, I. D'Ottaviano, and D. Mundici, *Algebraic Foundations of Many-valued Reasoning*, Trends in Logic, Kluwer, Dordrecht, 2000.

[31] R. Cignoli, F. Esteva, L. Godo, and A. Torrens, *Basic Fuzzy Logic is the logic of continuous t-norms and their residua*, Soft Computing **4** (2000), 106–112.

[32] C. Constantinescu, *Some properties of spaces of measures*, Atti Sem. Mat. Fis. Univ. Modena **35** (1989), 1-286.

[33] J. Corcoran, *Aristotle's natural deduction system*, Ancient Logic and Its Modern Interpretations (J. Corcoran, ed.), Reidel, Dordrecht, 1974, pp. 85–131.

[34] H.B. Curry, *The inconsistency of certain formal logics*, J. Symbolic Logic **7** (1942), 115-117.

[35] ———, *The inferential approach to logical calculus*, Logique et Analyse **n.s., 3** (1960), 119-136.

[36] J. Czelakowski and D. Pigozzi, *Amalgamation and interpolation in abstract algebraic logic*, Models, Algebras, and Proofs (Bogota 1995) (X. Caicedo and C. H. Montenegro, ed.), Lecture Notes in Pure and Applied Mathematics, vol. 203, Marcel Dekker, Inc., 1999, pp. 187–265.

[37] B.A. Davey and H.A. Priestley, *Introduction to lattices and order*, 2nd ed., Cambridge University Press, New York, 2002.

[38] R.P. Dilworth, *Abstract residuation over lattices*, Bulletin of the American Mathematical Society **44** (1937), 262-268.

[39] ———, *Non-commutative residuated lattices*, Transactions of the American Mathematical Society **46** (1939), 426–444.

[40] ———, *The structure and arithmetical theory of non-commutative residuated lattices*, PhD thesis, California Institute of Technology, 1939.

[41] K. Došen, *A historical introduction to substructural logics*, Substructural Logics (K. Došen and P. Schroeder-Heister, eds.), Oxford Univ. Press, 1993, pp. 1–30.

[42] M. Dummett, *A propositional calculus with denumerable matrix*, J. Symbolic Logic **24** (1959), 97–106.

[43] A.E. Duncan-Jones, *Is strict implication the same as entailment?*, Analysis **2** (Unknown Month 1934), 1970-1978.

[44] H.M. Edwards, *The genesis of ideal theory*, Archive for History of Exact Sciences **23** (1980), 321-378.

[45] A.F. Emch, *Implication and deducibility*, J. Symbolic Logic **1** (1936), 26-35.

[46] M. Erné, *Adjunctions and Galois connections: Origins, history and development*, Galois Connections and Applications (K. Denecke, M. Ern, and S.L. Wismath, eds.), Kluwer, Dordrecht, 2004, pp. 1-138.

[47] F. Esteva and L. Godo, *Monoidal t-norm based logic: towards a logic for left-continuous t-norms*, Fuzzy Sets and Systems **124** (2001), no. 3, 271–288.

[48] T. Evans, *Some connections between residual finiteness, finite embeddability and the word problem*, Journal of London Mathematics Society **2** (1969), no. 1, 399–403.

[49] M. Friedman, *Kant's theory of geometry*, Philosophical Review **94** (1985), 455–506.

[50] N. Galatos, *The undecidability of the word problem for distributive residuated lattices*, Ordered algebraic structures (Jorge Martinez, ed.), Kluwer, Dordrecht, 2002, pp. 231–243.

[51] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono, *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, Studies in Logic and the Foundations of Mathematics, Elsevier, Amsterdam, 2007.

[52] N. Galatos and H. Ono, *Algebraization, parametrized local deduction theorem and interpolation for substructural logics over* **FL**, Studia Logica **83** (2006), 279–308.

[53] P.T. Geach, *Entailment*, Proceedings of the Arist. Society **s.v. 32** (1958), 157-172.

[54] G. Gentzen, *Untersuchungen über das logische Schließen I, II*, Mathematische Zeitschrift **39** (1935), no. 1, 176–210, 405–431.

[55] ______, *Die Widerspruchfreiheit der reinen Zahlentheorie*, Mathematische Annalen **112** (1936), 493–565.

[56] G. Gierz, K.H. Hofmann, K. Keimel, J. Lawson, M. Mislove, and D. Scott, *A compendium of continuous lattices*, Springer-Verlag, 1980.

[57] J-Y Girard, *Linear logic*, Theoretical Computer Science **50** (1987), 1–102.

[58] A.M.W. Glass and Y. Gurevich, *The word problem for lattice-ordered groups*, Trans. Amer. Math. Soc. **280** (1983), no. 1, 127–138.

[59] V. Glivenko, *Sur quelques points de la logique de M. Brouwer*, Bulletin Academie des Sciences de Belgique **15** (1929), 183–188.

[60] K. Gödel, *Zum intuitionistischen Aussagenkalkl*, Anzeiger der Akad. der Wiss. Wien **32** (1932), 65-66.

[61] I. Grattan-Guinness, *The Search for Mathematical Roots 1870-1940* (2000).

[62] G. Grätzer, *General lattice theory*, 2nd ed., Birkhäuser Verlag, Basel, 1998. New appendices by the author with B.A. Davey, R. Freese, B. Ganter, M. Greferath, P. Jipsen, H.A. Priestley, H. Rose, E. T. Schmidt, S. E. Schmidt, F. Wehrung and R. Wille.

[63] ______, *Universal Algebra (paperback)*, 2nd ed., Springer, 2008.

[64] V.N. Grišin, *Predicate and set-theoretical calculi based on logic without the contraction rule*, Mathematical USSR Izvestiya **18** (1982), 41–59; English transl., Izvéstia Akademii Nauk SSSR **45** (1981), 47–68.

[65] P. Hájek, *Metamathematics of Fuzzy Logic*, Trends in Logic, vol. 4, Kluwer, Dordrecht, 1998.

[66] P. Hájek, L. Godo, and F. Esteva, *A complete many-valued logic with product-conjunction*, Archive for Mathematical Logic **35** (1996), 191–208.

[67] T. Halperin, *Boole's Logic and Probability*, Studies in Logic and the Foundations of Mathematics, vol. 85, North Holland, Amsterdam, 1986.

[68] ______, *Boole's algebra isn't Boolean algebra*, A Boole Anthology (J. Gasser, ed.), Kluwer, Dordrecht, 2000.

[69] J. Hart, L. Rafter, and C. Tsinakis, *The Structure of Commutative Residuated Lattices*, International Journal of Algebra and Computation **12** (2002), no. 4, 509–524.

[70] T.L. Heath, *A History of Greek Mathematics*, Oxford University Press, Oxford, 1921.

[71] R. Heck, *The development of arithmetic in Frege's* Grundgesetze der Arithmetik, J. Symbolic Logic **58** (1993), no. 2, 579–601.

[72] A. Heyting, *Die formalen Regeln der intuitionistischen Logik*, Sitzungsberichte der Preuischen Akademie der Wissenschaften, Phys.-mathem. Klasse (1930), 42-56.

[73] D. Hilbert, *Die Grundlagen der Mathematik*, Abhandlungen aus dem Seminar der Hamburgischen Universität **6** (1928), 65–85.

[74] D. Hilbert and P. Bernays, *Grundlagen der Mathematik*, Springer, 1934 (first volume), 1939 (second volume).

[75] J. Hintikka, *Kant on the mathematical method*, Kant Studies Today (L.W. Beck, ed.), Open Court, La Salle, 1969, pp. 117-140.

[76] N.G. Hisamiev, *Universal theory of lattice-ordered Abelian groups*, Algebra i Logika Sem. **5** (1966), no. 3, 71–76.

[77] W. C. Holland and S. H. McCleary, *Solvability of the word problem in free lattice-ordered groups*, Houston Journal of Mathematics **5** (1979), no. 1, 99–105.

[78] W.A. Howard, *The formulae-as types notion of construction*, To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus, and Formalism (J.R. Hindley and J. Seldin, eds.), Academic Press, New York, 1980, pp. 479-490.

[79] S. Jenei and F. Montagna, *A Proof of Standard Completeness for Esteva and Godo's Logic MTL*, Studia Logica **70** (2002), no. 2, 183–192.

[80] P. Jipsen and C. Tsinakis, *A survey of residuated lattices*, Ordered Algebraic Structures (Jorge Martinez, ed.), Kluwer, Dordrecht, 2002, pp. 19–56.

[81] I. Kaplanski, *Fields and Rings*, Chicago Lectures in Mathematics, Chicago University Press, Chicago, 1965.

[82] K. Keimel, *Some trends in lattice-ordered groups and rings*, Lattice Theory and its Applications (K.A. Baker and R. Wille, eds.), Heldermann Verlag, Berlin, 1995, pp. 131-161.

[83] O. Ketonen, *Untersuchungen zum Prdikatenkalkl*, Annales Academiae Scientiarum Fennicae **I, 23** (1944).

[84] H. Kihara and H. Ono, *Interpolation Properties, Beth Definability Properties and Amalgamation Properties for Substructural Logics*, to appear in *Journal of Logic and Computation.*

[85] T. Kowalski and H. Ono, *Fuzzy logics from substructural perspective*, Fuzzy Sets and Systems **161** (2010), no. 3, 301–310.

[86] M. Kozak, *Distributive full Lambek calculus has the finite model property*, Studia Logica **91** (2009), no. 2, 201–216.

[87] G. Kreisel, *Perspectives in the philosophy of pure mathematics*, Logic, Methodology and Philosophy of Science IV (Suppes P. et al., ed.), North Holland, Amsterdam, 1973, pp. 255-277.

[88] J. Lambek, *The mathematics of sentence structure*, American mathematical Monthly **65** (1958), 154–170.

[89] J. Lear, *Aristotle and Logical Theory*, Cambridge University Press, Cambridge, 1999.

[90] C.I. Lewis and C.H. Langford, *Symbolic Logic*, The Century Co., New York, 1932.

[91] P. Lincoln, J. Mitchell, A. Scedrov, and N. Shankar, *Decision problems for propositional linear logic*, Ann. Pure Appl. Logic **56** (1992), no. 1-3, 239–311.

[92] J. Madarász, *Interpolation and amalgamation: Pushing the limits. Part I*, Studia Logica **61** (1998), 311–345.

[93] S. Maehara, *Lattice-valued representation of the cut-elimination theorem*, Tsukuba Journal of Mathematics **15** (1991), 509–521.

[94] L.L. Maksimova, *Craig's theorem in superintuitionistic logics and amalgamable varieties of pseudo-Boolean algebras*, Algebra i Logika **16** (1977), 643–681.

[95] ———, *Interpolation properties of superintuitionistic logics*, Studia Logica **38** (1979), 419–428.

[96] ———, *Interpolation theorems in modal logics and amalgamable varieties of topological Boolean algebras*, Algebra i Logika **18** (1979), 556–586.

[97] R.N. McKenzie, G.F. McNulty, and W.F. Taylor, *Algebras, Lattices, Varieties*, Vol. 1, Wadsworth & Brooks/Cole, Monterey, California, 1987.

[98] J. C. C. McKinsey and Alfred Tarski, *On closed elements in closure algebras*, Ann. of Math. (2) **47** (1946), 122–162. MR0015037 (7,359e)

[99] G. Metcalfe and F. Montagna, *Substructural Fuzzy Logics*, J. Symbolic Logic **72** (2007), no. 3, 834–864.

[100] G. Metcalfe, N. Olivetti, and D. Gabbay, *Proof Theory for Fuzzy Logics*, Applied Logic, vol. 36, Springer, 2008.

[101] R.K. Meyer, *Topics in modal and many-valued logic*, Doctoral dissertation, University of Pittsburgh, 1966.

[102] T. Nakano, *Rings and partly ordered systems*, Mathematische Zeitschrift **99** (1967), 355-376.

[103] E.J. Nelson, *Intensional relations*, Mind **39** (1930), 440-453.

[104] M. Okada, *Phase semantics for higher order completeness, cut-elimination and normalization proofs (extended abstract)*, Electronic Notes in Theoretical Computer Science **3** (1996), 22 pp. (electronic).

[105] ———, *Phase semantic cut-elimination and normalization proofs of first- and higher-order linear logic*, Theoretical Computer Science **227** (1999), 333–396.

[106] M. Okada and K. Terui, *The finite model property for various fragments of intuitionistic linear logic*, J. Symbolic Logic **64** (1999), 790–802.

[107] H. Ono, *Proof-theoretic methods for nonclassical logic: an introduction*, Theories of Types and Proofs (M. Takahashi, M. Okada, and M. Dezani-Ciancaglini, eds.), MSJ Memoirs, vol. 2, Mathematical Society of Japan, 1998, pp. 207–254.

[108] ———, *Logics without contraction rule and residuated lattices I*, Festschrift on the occasion of R.K. Meyer's 65th birthday. forthcoming.

[109] H. Ono and Y. Komori, *Logics without the contraction rule*, J. Symbolic Logic **50** (1985), 169–201.

[110] F. Paoli, *Substructural Logics: A Primer*, Trends in Logic, vol. 13, Kluwer, Dordrecht, 2002.

[111] F. Paoli, F. Spinks, and R. Veroff, *Abelian logic and the logic of pointed lattice-ordered varieties*, Logica Universalis **2** (2008), 209-233.

[112] W.T. Parry, *Ein Axiomensystem fr eine neue Art von Implikation*, Ergebnisse eines Mathematischen Kolloquiums (1933), 5-6.

[113] D. Pigozzi, *Amalgamations, congruence-extension, and interpolation properties in algebras*, Algebra Universalis **1** (1972), 269–349.

[114] A. Rose and J.B. Rosser, *Fragments of many-valued statement calculi*, Transactions of the American Mathematical Society **87** (1958), 1-53.

[115] R. Routley, V. Plumwood, R.K. Meyer, and R.T. Brady, *Relevant Logics and Their Rivals*, Ridgeview, Atascadero, 1982.

[116] V.V. Rama Rao, *On a common abstraction of Boolean rings and lattice-ordered groups I*, Monatshefte fr Mathematik **73** (1969), 411-421.

[117] G. Restall, *Displaying and deciding substructural logics 1*: *Logics with contraposition*, J. Philosophical Logic **27** (1998), 179-216.

[118] W. Scharlau, *Richard Dedekind 1831-1981, Festschrift 1981*, Vieweg, Braunschweig, 1982.

[119] K.D. Schmidt, *A common abstraction of Boolean rings and lattice ordered groups*, Compositio Mathematica **54** (1985), 51-62.

[120] ______, *Minimal clans: a class of ordered partial semigroups including Boolean rings and partially ordered groups*, Semigroups: Theory and Applications, Springer, Berlin, 1988, pp. 300-341.

[121] T. Skolem, *Studies on the axiom of comprehension*, Notre Dame Journal of Formal Logic **4, 3** (1963), 162-179.

[122] T.J. Smiley, *Entailment and deducibility*, Proceedings of the Arist. Society **s.v. 33** (1959), 233-249.

[123] ______, *What is a syllogism?*, Journal of Philosophical Logic **2** (1973), 136-154.

[124] M. H. Stone, *The theory of representations for Boolean algebras*, Trans. Amer. Math. Soc. **40** (1936), no. 1, 37–111.

[125] K.L.N. Swamy, *Dually residuated lattice-ordered semigroups*, Mathematische Annalen **159** (1965), 105-114.

[126] ______, *Dually residuated lattice-ordered semigroups II*, Mathematische Annalen **160** (1965), 64-71.

[127] ______, *Dually residuated lattice-ordered semigroups III*, Mathematische Annalen **167** (1966), 71-74.

[128] G. Takeuti and T. Titani, *Intuitionistic fuzzy logic and intuitionistic fuzzy set theory*, J. Symbolic Logic **49** (1984), no. 3, 851–866.

[129] A. Tarski, *Foundations of the calculus of systems (Engl. transl.)*, Logic, Semantics, Metamathematics, Oxford University Press, New York, 1956, pp. 342-383.

[130] ______, *On the concept of logical consequence (Engl. transl.)*, Logic, Semantics, Metamathematics, Oxford University Press, New York, 1956, pp. 409-420.

[131] K. Terui, *Which structural rules admit cut elimination? — an algebraic criterion*, J. Symbolic Logic **72** (2007), no. 3, 738–754.

[132] A. Urquhart, *The undecidability of entailment and relevant implication*, J. Symbolic Logic **49** (1984), no. 4, 1059–1073.

[133] C.J. van Alten, *The finite model property for knotted extensions of propositional linear logic*, J. Symbolic Logic **70** (2005), no. 1, 84–98.
[134] G.H. von Wright, *Logical Studies*, Routledge, London, 1957.
[135] P.G.J. Vredenduijn, *A system of strict implication*, J. Symbolic Logic **4, 2** (1939), 73-76.
[136] M. Ward, *Structure residuation*, Annals of Mathematics **39** (1938), 558-568.
[137] M. Ward and R.P. Dilworth, *Residuated lattices*, Proceedings of the National Academy of Sciences **24** (1938), 162-164.
[138] ______, *Residuated lattices*, Transactions of the AMS **45** (1939), 335–354.
[139] A.M. Wille, *Residuated Structures with Involution*, Shaker Verlag, Aachen, 2006.
[140] R. Wójcicki, *Theory of Logical Calculi*, Kluwer, 1988.
[141] O. Wyler, *Clans*, Compositio Mathematica **17** (1965), 172-189.

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