Categorical Duality between Point-Free and Point-Set Spaces

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Outline

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Rich phil. behind Stone dual. and point-free geom.

Whitehead's philosophy:

- Notions of points arise as limits of shrinking regions. Points are ideal. Regions are real (or can be perceived).
  - Analogy with "points as prime ideals" in duality theory. Region-based geom. is constructive. Points need a Zorn.
- Processes are more fundamental than things.

Hajime Tanabe (1885-1962) is a philosopher of Kyoto school.

- Individual-"less" sociology: societies are not collections of individuals; societies are more fundamental than indivi.
  - Tanabe was inspired by Brouwer's int.: spreads come first, and then real numbers appear as free choice sequences.

Witt.: "What makes it apparent that space is not a collection of points, but the realization of a law?" (*Phil. Remarks*, p.216).

Duality b/w Ontological and Epistemological Aspects

Duality seems to arise b/w ontological and epistemol. aspects.
But this distinction is relative.

	Ontological	Epistemological	Duality
Logic	Models	Theories	Stone
Logic	Alg.Sem.	Logics	Tarski?
Alg.Geom.	Varieties	Polynomials	Hilbert, Gro.
Gene.Top.	Points	Opens	Isbell, Papert
Conv.Geom.	Points	Convex Sets	Jacobs, M.
Harm.Anal.	Top. Grp.	Charact. Grp.	Pontry., Weil
Comp.Sci.	Denotations	Observ. Prop.	Abramsky
Comp.Sci.	Comp. Sys.	Its Properties	Coalg|Modal

These are related. Stone for BA $=$ Hilbert b/w idem. $\mathbb{F}_2$-algs. and affine varieties of arbi. dim. over $\mathbb{F}_2$ (more gene., $\mathbb{GF}(p^n)$).

Duality b/w Point-Set and Point-Free Spaces

This talk is concerned with:

- Duality between point-set spaces and point-free spaces that express the infinitary logic of those point-set spaces.
- General theory of such “infinitary" Stone-type dualities with an appl. to Scott’s continuous lat. and convexity spaces.

Point-Set Spaces $\fallingdotseq$ Spaces of Points:

- topo. spaces, measurable spaces, convexity spaces, etc.
- We use the notions of functor-struct. cat. and topo. axiom to discuss general point-set spaces (see the AHS book).

Point-Free Spaces $\fallingdotseq$ Logical Algebras of Regions:

- frames, σ-comp. Bool. algs., continuous lattices, etc.
- We use monads to discuss general point-free spaces.

Monads and Point-Free Spaces

Monads seem useful to discuss point-free spaces or infinitary logics of point-set spaces. **Frm** and σ**CBA** are monadic.

- The category of Scott's continuous lattices is monadic. Continuous lattices were first used for program semantics.
- They express the infinitary logic of convexity spaces (M.).
  - A convexity space := a set S with $\mathcal{C} \subset \mathcal{P}(S)$ that is closed under $\bigcap$ and directed $\bigcup$.
  - A cont. lat. is equiv. to a meet-complete poset with directed joins that distribute over meets.

Conv. sp. unifies conv. geom. of $\mathbb{R}^n$, Riem. manifolds, lattices, etc. (see: van de Vel, *Theory of Convex Structures*, North-Holland).

- We focus on monads on **Sets**, since we discuss “pure" algebra. Monads on **C** amount to “**C**-structured" algs.

General Duality Theories

Universal Algebraic Approach:

- “Natural dualities for the working algebraist" (Davey et al., CUP).
- It focuses on alg. with finitary operations, and is useless for our goal. Univ. Alg. is finitary, while Cat. Alg. is infinitary.

Categorical Approach:

- “Concrete dualities" by Porst-Tholen (1991). Of course, “Stone spaces" by Johnstone (1986).
  - “Enriched logical connections" by Kurz-Velebil (preprint).
- Our aim is to make Porst-Tholen adj. thm. specialize in duality b/w point-free and point-set spaces.
- Two cats. involved are symmetric in some cat. approaches. But they appear to be non-symmetric in practice, since one is of alg. nature and the other is of spatial nature.

Outline

Duality and Concreteness

Given two cats. **C**, **D**, and $\Omega \in$ **C**, **D** (like **2**), we want Hom functors

- $Hom_{\mathbf{C}}(\text{-}, \Omega)$ and $Hom_{\mathbf{D}}(\text{-}, \Omega)$ (to get a duality by them).

But, $Hom_{\mathbf{C}}(C, \Omega) \in$ **D**? No rel. b/w $Hom_{\mathbf{C}}(C, \Omega)$ and **D**. We want **D** to be based on **Set**, then *Hom* can be the base set of an obj. in **D**.

- **D** (and **C**) should be concrete cat. A concrete cat. **C** over **Sets** is defined as (**C**, U : **C** $\rightarrow$ **Sets**) with U faithful.
  - Porst-Tholen and Johnstone follow the same idea.
- It is essential how to make $Hom_{\mathbf{C}}(C, \Omega)$ be in **D**.
  - Porst-Tholen uses initial lifting conditions. We use Stone-Zariski-like topology and Harmony Condition.

This is another reason why we discuss monads on **Sets**, whose algebras form conc. cats. (but **Sets** may be replaced with a conc. cat. and this is crucial, e.g., for DCPO).

Topo. axioms in functor-structured categories

Let ($\mathbf{C}$, $U : \mathbf{C} \to \mathbf{Sets}$) be a conc. cat.. Define a cat. $\mathbf{Spa}(U)$:

- An object of $\mathbf{Spa}(U)$ is $(C, \mathcal{O})$ s.t. $C \in \mathbf{C}$ and $\mathcal{O} \subset U(C)$.
- An arrow of $\mathbf{Spa}(U)$ from $(C, \mathcal{O})$ to $(C', \mathcal{O}')$ is an arrow $f : C \to C'$ of $\mathbf{C}$ such that $U(f)[\mathcal{O}] \subset \mathcal{O}'$.

A functor-costructured cat. is a cat. of the form $(\mathbf{Spa}(U))^{\mathrm{op}}$.

- A topo. coaxiom in $(\mathbf{C}, U)$ is $p : C \to C'$ in $\mathbf{C}$ s.t. $U(C) = U(C')$, and $U(p)$ is the identity on $U(C)$.

$C \in \mathbf{C}$ satisfies a topo. coaxiom $p : D' \to D$ in $(\mathbf{C}, U)$ iff

- $\forall f : C \to D$ in $\mathbf{C}$, $\exists f' : C \to D'$ in $\mathbf{C}$ s.t. $U(f) = U(f')$.

Let X be a class of topological coaxioms in a conc. cat. $\mathbf{C}$.

- A full subcat. $\mathbf{D}$ of $\mathbf{C}$ is definable by X in $\mathbf{C}$ iff the objects of $\mathbf{D}$ coincide with those objects of $\mathbf{C}$ that satisfy any $p \in X$.

Bool. Topo. Coaxiom

We introduce a new concept of Bool. topo. coaxiom.

- A Bool. topo. coaxiom in $(\mathbf{Spa}(U))^{\mathrm{op}}$ is a topo. coaxiom $p : (C, \mathcal{O}) \to (C', \mathcal{O}')$ in $(\mathbf{Spa}(U))^{\mathrm{op}}$ s.t.
    - Any element of $\mathcal{O} \setminus \mathcal{O}'$ can be expressed as a (possibly infinitary) Boolean combination of elements of $\mathcal{O}'$.

Let $\mathcal{Q} : \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Sets}$ be the contravariant power-set functor.

- Any of **Top**, **Meas**, and **Conv** can be expressed as a fullsubcat. of $\mathbf{Spa}(\mathcal{Q})^{\mathrm{op}}$ definable by Bool. topo. coaxioms.

Motivation: $(S, \mathcal{O})$ with Bool. closure conditions on $\mathcal{O}$.

- **Top** is definable by: $1_S : (S, \{\emptyset, S\}) \to (S, \emptyset)$; $1_S : (S, \{X, Y, X \cap Y\}) \to (S, \{X, Y\})$; $1_S : (S, \mathcal{O} \cup \{\bigcup \mathcal{O}\}) \to (S, \mathcal{O})$.

Outline

The Idea of Schizophrenic Obejct

- Lawvere: a potential duality arises when a single object Ω lives in two different categories.
    - Lawvere credited this to Isbell (Barr et al. “Isbell duality").
- Such an Ω is called a schizophrenic object (H. Simmons).
    - Porst-Tholen gave a definition of a schizophrenic object via (symmetric) initial lifting conditions.
    - Our theory contains more details about how the lifting becomes possible, though its scope is more restricted.
- The notion of adj. induced by schizo. obj. is inappropriate in some “quantum" cases. But it seems useful in the case of duality b/w point-free and point-set sp.

$(\mathbf{Alg}, \mathbf{Spa}, \Omega)$

We define

- **Alg** $:=$ the E-M cat. of a monad T on **Sets**.
- **Spa** $:=$ a full subcat. of $\mathbf{Spa}(\mathcal{Q})^{\mathrm{op}}$ definable by Bool. topo. coaxioms. **Spa** is a topological category.

We assume Ω:

- there is Ω both in **Alg** and in **Spa**, i.e., there are Ω in **Sets**, h_Ω s.t. $(\Omega, h_\Omega) \in \mathbf{Alg}$, and $\mathcal{O}_\Omega$ s.t. $(\Omega, \mathcal{O}_\Omega) \in \mathbf{Spa}$.

Equip $\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ with the “topology" generated in **Spa** by

- $\{\langle a \rangle_O \ ; \ a \in A \text{ and } O \in \mathcal{O}_\Omega\}$ where $\langle a \rangle_O := \{v \in \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega) \ ; \ v(a) \in O\}$.
    - This is inspired by Stone and Zariski topology.

Thus, we have $\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega) \in \mathbf{Spa}$.

Harmony Condition

$(\mathbf{Alg}, \mathbf{Spa}, \Omega)$ is said to satisfy the harmony condition iff, for each $S \in \mathbf{Spa}$, there is

$$h_S : T(\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)) \to \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$$

such that, for any $s \in S$,

$$\begin{array}{ccc} T(\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)) & \xrightarrow[h_S]{} & \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega) \\ \Big\downarrow{\scriptstyle T(p_s)} & & \Big\downarrow{\scriptstyle p_s} \\ T(\Omega) & \xrightarrow[h_\Omega]{} & \Omega \end{array} .$$

By this, we have $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega) \in \mathbf{Alg}$. The harmony cond. is easy to verify in concrete cases. In Isbell duality, it amounts to the fact: opens are closed under $\bigcup$ and $\cap$.

Dual Adjunction Theorem

The induced contravariant Hom-functors

- $\mathrm{Hom}_{\textbf{Alg}}(\text{-}, \Omega) : \textbf{Alg} \to \textbf{Spa}$ and $\mathrm{Hom}_{\textbf{Spa}}(\text{-}, \Omega) : \textbf{Spa} \to \textbf{Alg}$

can be shown to form a dual adjunction b/w **Alg** and **Spa**:

Theorem

$\mathrm{Hom}_{\textbf{Alg}}(\text{-}, \Omega)$ *is left adjoint to* $\mathrm{Hom}_{\textbf{Spa}}(\text{-}, \Omega)^{\mathrm{op}}$*.*

This thm. encompasses:

- dual adj. b/w frames and topo. sp.; dual adj. b/w σ-comp. BA and meas. sp.; dual adj. b/w cont. lat. and conv. sp.; many Stone-type adj. for logics.

But it does not encompass Pontryagin duality for compact abelian groups, since our focus is on duality b/w “pure" algebras and “pure" spaces. It can also be subsumed by replacing **Sets** with a concrete cat. in our framework.

$\mathbf{Spa}^{\mathrm{op}}$ is quasi-monadic

A category is called quasi-monadic iff it is a regular-epi-reflective subcategory of a monadic category.

- Barr et al. 1996: $\mathbf{Top}^{\mathrm{op}}$ is quasi-monadic.
- Thus, $\mathbf{Top}^{\mathrm{op}}$ is a quasi-variety.
    - In fact, $\mathbf{Top}^{\mathrm{op}}$ is a quasi-equationally definable category of some grids (i.e., frames with certain unary operations).

Theorem

$\mathbf{Spa}^{\mathrm{op}}$ *is quasi-monadic. Hence,* $\mathbf{Top}^{\mathrm{op}}$*,* $\mathbf{Meas}^{\mathrm{op}}$*, and* $\mathbf{Conv}^{\mathrm{op}}$ *are all quasi-monadic.*

In such a way, the concept of Bool. topo. coaxiom enables a uniform treatment of various point-set spaces.

Conclusions

Alg $:=$ the E-M cat. of a monad on **Sets**. **Spa** $:=$ a full subcat. of **Spa**$(\mathcal{Q})^{\mathrm{op}}$ definable by Bool. topo. coaxioms. Results:

- Dual adjunction b/w **Alg** and **Spa** under the assumptions of $\Omega \in$ **Alg**, **Spa** and the harmony condition.
  - This subsums: adj. b/w frames and topo. sp.; adj. b/w σ-comp. BA and meas. sp.; adj. b/w cont. lat. and conv. sp.; adj. b/w com. rings and topo. sp; many others.
- **Spa**$^{\mathrm{op}}$ is quasi-monadic.
  - **Spa** enables a uniform treatment of point-set spaces.
- Adj. b/w cont. lat. and conv. sp. is refined into equiv. b/w spatial (or alg.) cont. lat. and sober conv. sp.
  - A conv. sp. is sober iff any polytope is the convex hull of a unique point. Polytopes are like irr. varieties in alg. geom.
  - M., Fundamental results for pointfree convex geometry, *Ann. Pure Appl. Logic* 161 (2010) 1486-1501.