Two questions about canonical extensions

(Canonical extensions and universal properties)

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Two questions

(What are canonical extensions?)

A reminder

How do we define Scott-continuous maps on the canonical extension of a lattice?

Canonical extensions via dcpo presentations

What do canonical extensions have to do with topological lattice-based algebras?

Universal properties

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- provide an algebraic generalization of the representation theorem for Boolean algebras.
- are abstract completions, characterized up to isomorphism by order-theoretical properties.
- are duality-agnostic.

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Let $\mathbb{L}$ be a lattice and let $\mathbb{E}$ be a dcpo. We denote the canonical extension of $\mathbb{L}$ by $\mathbb{L}^\delta$.

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How can we see whether a map $f\colon \mathbb{L}^\delta \to \mathbb{E}$ is in $\mathbf{Dcpo}(\mathbb{L}^\delta, \mathbb{E})$, i.e. whether f is Scott-continuous?

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- We will see that $\mathbb{L}^\delta$ can be presented as a dcpo by generators P and relations $\sqsubseteq$ and $\triangleleft$.
- The elements of $\mathbf{Dcpo}(\mathbb{L}^\delta, \mathbb{E})$ now correspond to those maps $P \to \mathbb{E}$ which preserve the relations $\sqsubseteq$ and $\triangleleft$.

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Filters and ideals

Let $\mathbb{P}$ be a poset. By $\mathcal{F}\,\mathbb{P}$ and $\mathcal{I}\,\mathbb{P}$ we denote the filters and ideals of $\mathbb{P}$, respectively.

A dcpo presentation for the canonical extension

Let $\mathbb{L}$ be a lattice.

Definition

We define $\Delta(\mathbb{L})$ to be the dcpo presentation $\langle \mathcal{F}\,\mathbb{L}, \supseteq, \lhd_{\mathbb{L}} \rangle$, where for all $F \in \mathcal{F}\,\mathbb{L}$ and directed $\mathcal{S} \subseteq \mathcal{F}\,\mathbb{L}$,

$$F \lhd_{\mathbb{L}} \mathcal{S} \text{ iff } \forall I \in \mathcal{I}\,\mathbb{L}, \left[\forall F' \in \mathcal{S},\ F' \between I\right] \Rightarrow F \between I.$$

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Theorem (G. & V. 2011)

If $\mathbb{L}$ is a lattice, then $\Delta(\mathbb{L})$ is a dcpo presentation of $\mathbb{L}^{\delta}$, the canonical extension of $\mathbb{L}$.

Bonus: applications using dcpo algebras

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Theorem (G. & Harding 2001)

Let $\mathbb{A}$ be a lattice-based algebra and let $s \preccurlyeq t$ be an inequation. If for each operation ω occurring in s or t, $\omega_{\mathbb{A}}$ is an operator of which we take the lower canonical extension, then $\mathbb{A} \models s \preccurlyeq t$ implies $\mathbb{A}^{\delta} \models s \preccurlyeq t$.

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- We will see that canonical extensions have certain universal properties with respect to topological lattice-based algebras.
- Why is this the case?

Topological lattice-based algebras

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Note: (2) is not a special case of (1)!

Profinite algebras

We will view finite algebras as topological algebras with a discrete topology.
What does it mean for an algebra $\mathbb{A}$ to be profinite?

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See Johnstone (1982).

Examples of profinite algebras

Profiniteness gives a categorical characterizaton of some subcategories of categories of lattice-based algebras.

- Distributive lattices: $\mathbb{L}$ profinite iff $\mathbb{L}$ is (isomorphic to) a down-set lattice iff $\mathbb{L}$ is complete & bi-algebraic.

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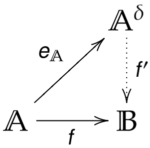
- Heyting algebras: $\mathbb{A}$ profinite iff $\mathbb{A}$ is (iso to) the down-set lattice of image-finite poset iff $\mathbb{A}$ is complete, bi-algebraic and residually finite. (Bezhanishvili 2008)

Canonical extensions and profinite algebras

Let $e_{\mathbb{A}} \colon \mathbb{A} \to \mathbb{A}^{\delta}$ be the canonical extension of a lattice-based algebra.

Theorem (V. 2010)

Let $f \colon \mathbb{A} \to \mathbb{B}$ be a homomorphism to a profinite lattice-based algebra $\mathbb{B}$. Then there exists a unique complete homomorphism $f' \colon \mathbb{A}^{\delta} \to \mathbb{B}$ such that $f' \circ e_{\mathbb{A}} = f$.



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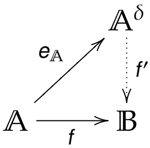

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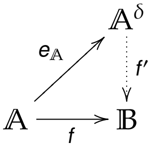

In many cases, the canonical extension $\mathbb{A}^{\delta}$ itself is not profinite: see V. (2010, §3.4.2) or Gouveia (2010).

Profinite lattices with (continuous) monotone operations

A profinite lattice with (continuous) monotone operations is a lattice-based algebra $\mathbb{A} = \langle A; \wedge, \vee, 0, 1, (\omega_{\mathbb{A}})_{\omega \in \Omega} \rangle$ such that:

- $\langle A; \wedge, \vee, 0, 1 \rangle$ is a profinite lattice;

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Complex algebras obtained from arbitrary Kripke frames (both intuitionistic and modal).

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Continuous example (V. 2010, §4.2)

Complex algebras of image-finite modal Kripke frames.

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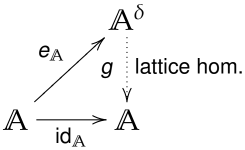
Let $\mathbb{A}$ be a profinite lattice with with monotone operations. Then there exists a unique complete lattice homomorphism $g\colon \mathbb{A}^\delta \to \mathbb{A}$ such that $g \circ e_{\mathbb{A}} = \mathrm{id}_{\mathbb{A}}$.



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1. $g\colon \mathbb{A}^\delta \to \mathbb{A}$ is an algebra homomorphism with respect to the full signature of $\mathbb{A}$;
2. all the operations of $\mathbb{A}$ are continuous, i.e. $\mathbb{A}$ is a Boolean topological algebra.

Example

Recall that profinite Boolean algebras with continuous modal operator correspond to image-finite Kripke frames. The last theorem now dualizes to the following folklore (?) result:

Fact

Let $\mathfrak{F}$ be a Kripke frame and let $\mathfrak{ue}\,\mathfrak{F}$ be its ultrafilter extension. The following are equivalent:

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1. $\mathfrak{F}$ embeds into $\mathfrak{ue}\,\mathfrak{F}$;
2. $\mathfrak{F}$ is image-finite.

Thank you!