A Canonical Model for Presheaf Semantics

Ivano A. Ciardelli

INRIA, LaBRI
Université de Bordeaux 1

What is this?

This is not

- a new result
- a result in categorical logic
- a brilliant idea

This is

- a totally natural construction which does not seem to appear anywhere
- a constructive completeness proof
- a logical proof, assuming no category theory

What is this?

This is not

- a new result
- a result in categorical logic
- a brilliant idea

This is

- a totally natural construction which does not seem to appear anywhere
- a constructive completeness proof
- a logical proof, assuming no category theory

What is this?

This is not

- a new result
- a result in categorical logic
- a brilliant idea

This is

- a totally natural construction which does not seem to appear anywhere
- a constructive completeness proof
- a logical proof, assuming no category theory

Presheaf semantics: models

Definition

A presheaf model M for $\mathcal{L}$ is a presheaf of first-order $\mathcal{L}-$structures over a Grothendieck site $(\mathcal{C}, \lhd)$:

- for any object u a first-order model M_u
- for any arrow $f : v \to u$ a homomorphism $\upharpoonright_f : M_u \to M_v$

satisfying the following extra conditions.

Separateness: if $u \lhd \mathcal{F}$ and $a\upharpoonright_f = b\upharpoonright_f$ for all $f \in \mathcal{F}$, then $a = b$.

Locality of atoms: if $u \lhd \mathcal{F}$ and $\vec{a}\upharpoonright_f \in R_{\mathrm{dom}(f)}$ for all $f \in \mathcal{F}$, $\vec{a} \in R_u$.

Presheaf semantics: Kripke-Joyal forcing

$$
\begin{array}{rcl}
u \Vdash_\nu R(\vec{t}) & \iff & [\vec{t}]_\nu \in R_u \\
u \Vdash_\nu t_1 = t_2 & \iff & [t_1]_\nu = [t_2]_\nu \\
u \Vdash_\nu \varphi \wedge \psi & \iff & u \Vdash_\nu \varphi \text{ and } u \Vdash_\nu \psi \\
u \Vdash_\nu \varphi \vee \psi & \iff & \text{there is } \mathcal{F} \rhd u \text{ s.t. for all } f \in \mathcal{F} \\
 & & \mathrm{dom}(f) \Vdash_\nu \varphi \text{ or } \mathrm{dom}(f) \Vdash_\nu \psi \\
u \Vdash_\nu \bot & \iff & u \lhd \emptyset \\
u \Vdash_\nu \varphi \rightarrow \psi & \iff & \text{for } f : v \rightarrow u, \text{ if } v \Vdash_\nu \varphi \text{ then } v \Vdash_\nu \psi \\
u \Vdash_\nu \forall x \varphi & \iff & \text{for } f : v \rightarrow u \text{ and } a \in M_v,\ v \Vdash_{\nu[x \mapsto a]} \varphi \\
u \Vdash_\nu \exists x \varphi & \iff & \text{there is } \mathcal{F} \rhd u \text{ and elts } a_f \in M_{\mathrm{dom}(f)} \\
 & & \text{s.t. } \mathrm{dom}(f) \Vdash_{\nu[x \mapsto a_f]} \varphi \text{ for any } f \in \mathcal{F}
\end{array}
$$

Completeness

Theorem (Intuitionistic completeness for presheaf semantics)

$$\models_{ps} \varphi \iff \vdash_{IQL} \varphi$$

Usually established by:

- equivalence with $\Omega-$models;
- construction of a canonical Kripke model.

Canonical model construction: the underlying site

Definition (Canonical site)

- Category: we take the Lindenbaum-Tarski algebra $\overline{\mathcal{L}}$
  - Objects: equivalence classes $\overline{\varphi}$ of formulas
  - Arrows: $\overline{\varphi} \leq \overline{\psi} \iff \varphi \vdash \psi$
- Grothendieck topology: $\overline{\varphi} \lhd \mathcal{F} \iff \overline{\varphi} = \bigvee \mathcal{F}$

Observation

Spelling out, $\overline{\varphi} \lhd \{\overline{\psi}_i \mid i \in \mathcal{I}\}$ means: for any χ

$$\varphi \vdash \chi \iff (\psi_i \vdash \chi \text{ for all } i \in \mathcal{I})$$

Canonical model construction: the underlying site

Definition (Canonical site)

- Category: we take the Lindenbaum-Tarski algebra $\overline{\mathcal{L}}$
  - Objects: equivalence classes $\overline{\varphi}$ of formulas
  - Arrows: $\overline{\varphi} \leq \overline{\psi} \iff \varphi \vdash \psi$
- Grothendieck topology: $\overline{\varphi} \lhd \mathcal{F} \iff \overline{\varphi} = \bigvee \mathcal{F}$

Observation

Spelling out, $\overline{\varphi} \lhd \{\overline{\psi}_i \mid i \in \mathcal{I}\}$ means: for any χ

$$\varphi \vdash \chi \iff (\psi_i \vdash \chi \text{ for all } i \in \mathcal{I})$$

Canonical model construction: the underlying site

Definition (Canonical site)

- Category: we take the Lindenbaum-Tarski algebra $\overline{\mathcal{L}}$
  - Objects: equivalence classes $\overline{\varphi}$ of formulas
  - Arrows: $\overline{\varphi} \leq \overline{\psi} \iff \varphi \vdash \psi$
- Grothendieck topology: $\overline{\varphi} \lhd \mathcal{F} \iff \overline{\varphi} = \bigvee \mathcal{F}$

Observation

Spelling out, $\overline{\varphi} \lhd \{\overline{\psi}_i \mid i \in \mathcal{I}\}$ means: for any χ

$$\varphi \vdash \chi \iff (\psi_i \vdash \chi \text{ for all } i \in \mathcal{I})$$

Canonical model construction: the presheaf

- Put $t \equiv_\varphi t'$ in case $\varphi \vdash t = t'$.
- Denote by t^φ the equivalence class of t modulo $\equiv_\varphi$.

Definition (Canonical presheaf)

- Model $M_{\overline{\varphi}}$:
  1. Universe $|M_{\overline{\varphi}}|$: set of equivalence classes t^φ of closed terms;
  2. Function symbols: $f_{\overline{\varphi}}(\vec{t}^{\varphi}) = f(\vec{t})^{\varphi}$;
  3. Relation symbols: $\vec{t}^{\varphi} \in R_{\overline{\varphi}} \iff \varphi \vdash R(\vec{t})$.
- Restriction. If $t^\psi \in M_{\overline{\psi}}$ and $\overline{\varphi} \leq \overline{\psi}$, put $t^\psi \upharpoonright_{\overline{\varphi}} = t^\varphi$.

Canonical model construction: the presheaf

- Put $t \equiv_\varphi t'$ in case $\varphi \vdash t = t'$.
- Denote by t^φ the equivalence class of t modulo $\equiv_\varphi$.

Definition (Canonical presheaf)

- Model $M_{\overline{\varphi}}$:
  1. Universe $|M_{\overline{\varphi}}|$: set of equivalence classes t^φ of closed terms;
  2. Function symbols: $f_{\overline{\varphi}}(\vec{t}^\varphi) = f(\vec{t})^\varphi$;
  3. Relation symbols: $\vec{t}^\varphi \in R_{\overline{\varphi}} \iff \varphi \vdash R(\vec{t})$.
- Restriction. If $t^\psi \in M_{\overline{\psi}}$ and $\overline{\varphi} \leq \overline{\psi}$, put $t^\psi \upharpoonright_{\overline{\varphi}} = t^\varphi$.

Canonical model construction: the presheaf

- Put $t \equiv_\varphi t'$ in case $\varphi \vdash t = t'$.
- Denote by t^φ the equivalence class of t modulo $\equiv_\varphi$.

Definition (Canonical presheaf)

- Model $M_{\overline{\varphi}}$:
  1. Universe $|M_{\overline{\varphi}}|$: set of equivalence classes t^φ of closed terms;
  2. Function symbols: $f_{\overline{\varphi}}(\vec{t}^\varphi) = f(\vec{t})^\varphi$;
  3. Relation symbols: $\vec{t}^\varphi \in R_{\overline{\varphi}} \iff \varphi \vdash R(\vec{t})$.
- Restriction. If $t^\psi \in M_{\overline{\psi}}$ and $\overline{\varphi} \leq \overline{\psi}$, put $t^\psi \upharpoonright_{\overline{\varphi}} = t^\varphi$.

Completeness theorem

Truth Lemma

For any formula φ and sentence ψ,

$$\overline{\varphi} \Vdash \psi \quad \Longleftrightarrow \quad \varphi \vdash \psi$$

Proof By induction on ψ. The two directions of each inductive step amount to the introduction and elimination rules for the given logical constant. Let us look at the case of the existential quantifier.

Completeness theorem

$\Rightarrow$

- Suppose $\overline{\varphi} \Vdash \exists x \psi(x)$.
- There is a family $\{\overline{\varphi_i} \mid i \in \mathcal{I}\}$ and elts $t_i^{\varphi_i} \in M_{\overline{\varphi_i}}$ such that $\overline{\varphi_i} \Vdash_{[x \mapsto t_i^{\varphi_i}]} \psi(x)$ for all $i \in \mathcal{I}$.
- Since $[t] = t^{\varphi_i}$ for closed t at $\overline{\varphi_i}$, this is $\overline{\varphi_i} \Vdash \psi(t_i)$.
- By induction hypothesis amounts to $\varphi_i \vdash \psi(t_i)$.
- By rule $(\exists i)$, for any $i \in \mathcal{I}$ we have $\varphi_i \vdash \exists x \psi(x)$.
- Since $\overline{\varphi} \lhd \{\overline{\varphi_i} \mid, i \in \mathcal{I}\}$, by the meaning of $\lhd$ we have $\varphi \vdash \exists x \psi(x)$.

Completeness theorem

$\Leftarrow$

- Suppose $\varphi \vdash \exists x \psi(x)$.
- We must provide a covering of $\overline{\varphi}$ and local witnesses.
- For any constant c, define $\varphi_c = \varphi \wedge \psi(c)$.
- Since $\varphi_c \vdash \psi(c)$, by induction hypothesis $\overline{\varphi_c} \Vdash \psi(c)$.
- Since $[c] = c^{\varphi_c}$ at $\overline{\varphi_c}$, also $\overline{\varphi_c} \Vdash_{[x \mapsto c^{\varphi_c}]} \psi(x)$, i.e. the element c^{φ_c} is a witness for the existential at $\overline{\varphi_c}$.
- It remains to be seen that $\overline{\varphi} \lhd \{\overline{\varphi_c} \mid c \text{ a constant}\}$.

Completeness theorem

$\Leftarrow$, continued

- Suppose ξ is derivable from $\varphi \wedge \psi(c)$ for any constant c.
- Let c^* be a constant that occurs neither in φ nor in ξ.
- In particular, $\varphi \wedge \psi(c^*) \vdash \xi$, that is, $\varphi, \psi(c^*) \vdash \xi$.
- But since c^* occurs neither in φ nor in ξ, by the rule $(\exists e)$ we have $\varphi, \exists x \psi(x) \vdash \xi$.
- Thus by the assumption $\varphi \vdash \exists x \psi(x)$ we also have $\varphi \vdash \xi$.
- This shows that $\overline{\varphi} \lhd \{\overline{\varphi_c} \mid c \text{ a constant}\}$.
- Hence we conclude $\overline{\varphi} \Vdash \exists x \psi(x)$.

Completeness theorem

Theorem (Completeness)

If $\models_{ps} \varphi$ *then* $\vdash \varphi$

Proof

If $\models_{ps} \varphi$ then in the given canonical model we must have $\overline{\top} \Vdash \varphi$, whence by truth lemma $\top \vdash \varphi$, i.e. $\vdash \varphi$.

Thank you for your attention