Homotopical Fibring

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Fibring: Combining logics

$$
\begin{array}{ccc}
\langle \wedge, \vee, \neg | R_1, \ldots \rangle & \longrightarrow & \langle \wedge, \vee, \neg, \Box_1, \Diamond_1 | R_1, R_2 \ldots \rangle \\
\downarrow & & \downarrow \\
\langle \wedge, \vee, \neg, \Box_2, \Diamond_2 | R_1, R_2' \ldots \rangle & \longrightarrow & \langle \wedge, \vee, \neg, \Box_1, \Diamond_1, \Box_2, \Diamond_2 | R_1, R_2, R_2' \ldots \rangle
\end{array}
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\end{array}$$

Fibring is a pushout in a category of logics:
C. Caleiro, A. Sernadas, C. Sernadas, 1998 – "Fibring as a categorical construction"

Signatures

Definition:

- A signature is a sequence of sets $C = (C_k | k \in \mathbb{N})$

We fix a set V of variables.

- The language $L(C)$ over the signature C is the absolutely free algebra generated by V
- A morphism $f\colon C \to C'$ is a sequence of maps $f_k\colon C_k \to C'_k$

$\rightsquigarrow$ We get a category $\mathcal{S}ig$

- A logic is a pair $(C, \vdash)$ where $\vdash \subseteq \mathcal{P}(L(C)) \times L(C)$ is monotonous, increasing, idempotent, substitution invariant and finitary
- A morphism $f\colon (C, \vdash) \to (C', \vdash')$ of logics is $f\colon C \to C'$ such that

$$\Gamma \vdash \varphi \Rightarrow f(\Gamma) \vdash' f(\varphi)$$

$\rightsquigarrow$ We get a category $\mathcal{L}og$

Colimits of Logics

$\mathcal{L}og$ is fibred in posets over $\mathcal{S}ig$

Fact/Requirement: Consequence relations over a fixed signature (i.e. the fibers) form a complete lattice

Corollary: $\mathcal{L}og$ is as cocomplete as $\mathcal{S}ig$ is.

$$\begin{array}{c} \mathcal{L}og \\ \downarrow \\ \mathcal{S}ig \simeq Set^{\mathbb{N}} \end{array}$$

... i.e. cocomplete

A richer category of logics

Signatures: As before, $C = (C_k | k \in \mathbb{N})$
Morphisms of signatures: $(f_k \colon C_k \to L(C')_k | k \in \mathbb{N})$
(where $L(C')_k :=$ formulas with k variables)

New possibilities for combining and relating logics:

$$\begin{array}{ccc} CPL = \langle \wedge, \neg | \ldots \rangle & \longrightarrow & \langle \wedge, \neg, \Box | \ldots \rangle \\ \downarrow{\scriptstyle \neg\neg} & & \downarrow \\ INT = \langle \wedge, \vee, \neg, \rightarrow | \ldots \rangle & \longrightarrow & ? \end{array}$$

But: Now $\mathcal{S}ig$ is no longer cocomplete, hence neither is $\mathcal{L}og$

Equivalence of logics

Definition:

A morphism $f : (L, \vdash) \to (L', \vdash')$ of logics is an *equivalence* iff there exists a morphism $g : (L', \vdash') \to (L, \vdash)$ with $g \circ f(\phi) \dashv\vdash \phi \;\forall \phi \in L$ and $f \circ g(\phi) \dashv\vdash \phi \;\; \forall \phi \in L'$.

It is a *weak equivalence* iff $\Gamma \vdash \varphi \Leftrightarrow f(\Gamma) \vdash' f(\varphi)$ ("conservative translation") and for every $\phi \in L(C')$ there is a $\psi \in L(C)$ with $\phi \dashv' \vdash' f(\psi)$ ("dense")

Proposition: These two notions coincide.

Example: $CPL = \langle \wedge, \neg | ... \rangle \longrightarrow \langle \vee, \neg | ... \rangle = CPL$

Abstract Homotopy Theory

Setting: Category $\mathcal{C}$, $W \subseteq Mor\,\mathcal{C}$

From this can construct the homotopy category

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & Ho(\mathcal{C}) = \mathcal{C}[W^{-1}] \\ & \searrow & \vdots \\ & & \mathcal{D} \end{array}$$

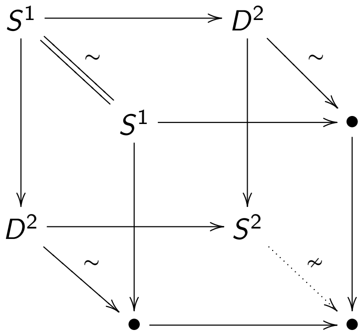
Want: Constructions respecting weak equivalences, e.g. homotopy pushouts
By this we mean a Kan extension

$$\begin{array}{ccc} \mathcal{C}^{\Gamma} & \xrightarrow{\textit{pushout}} & \mathcal{C} \\ \downarrow & \nearrow & \downarrow \\ & \nearrow & \\ Ho(\mathcal{C}^{\Gamma}) & \dashrightarrow & Ho(\mathcal{C}) \end{array}$$

where Γ denotes the category of pushout data $\begin{array}{cc} \bullet & \to \bullet \\ \downarrow & \\ \bullet & \end{array}$

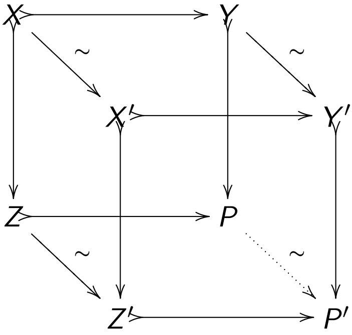
Topological Spaces

The pushout functor does not respect weak equivalences of diagrams:

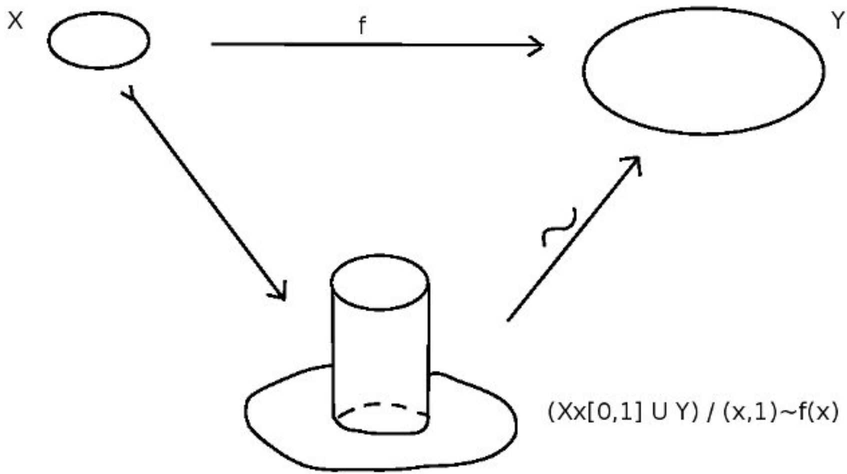


Constructing homotopy pushouts

The pushout functor *does* respect weak equivalences of diagrams *consisting of cofibrations*:



Can factorize any map of topological spaces by a cofibration, followed by a weak equivalence:



Topological Spaces

Recipe for constructing the homotopy pushout:

Replace a given diagram of pushout data by one consisting of cofibrations, then take the pushout:

$$\begin{array}{ccccc} Top^{\Gamma} & \longrightarrow & Top^{\Gamma} & \xrightarrow{\textit{pushout}} & Top \\ \downarrow & & & & \downarrow \\ Ho(Top^{\Gamma}) & & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\rightarrow & & Ho(Top) \end{array}$$

Theorem: The dotted arrow then is the desired Kan extension.

This solves two problems: Non-existence of colimits in $Ho(Top)$ and non-preservation of weak equivalences by pushout.

Actually the upper arrow has itself a universal property, and is also called the homotopy colimit.

Logics

Want to construct homotopy pushouts of logics in the same way. i.e. replace a given diagram of pushout data by one consisting of cofibrations, then take the pushout:

$$\begin{array}{ccccc} \mathcal{L}og^{\Gamma} & \longrightarrow & \mathcal{L}og^{\Gamma} & \xrightarrow{pushout} & \mathcal{L}og \\ \downarrow & & & & \downarrow \\ Ho(\mathcal{L}og^{\Gamma}) & & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\rightarrow & & Ho(\mathcal{L}og) \end{array}$$

Definition: A morphism $f\colon (S,\vdash) \to (S',\vdash')$ is a *cofibration* if the underlying signature morphism is given by injective maps $(f_k\colon C_k \to C'_k)$
Will show: The dotted arrow then is the desired Kan extension.

Actually the upper arrow has itself a universal property, and is also called the homotopy colimit. This will solve the problem of non-existence of colimits in $\mathcal{L}og$.

ABC Cofibration Categories

A left proper ABC Cofibration Category is a triple $(\mathcal{C}, Cof, W)$ where $\mathcal{C}$ is a category and $Cof, W \subseteq Mor(\mathcal{C})$ are classes of morphisms such that

(CC 1) Both Cof and W contain all isomorphisms of $\mathcal{C}$. For two maps $A \xrightarrow{f} B \xrightarrow{g} C$ if any two of f, g and $g \circ f$ are weak equivalences, then so is the third. Cof is closed under composition.

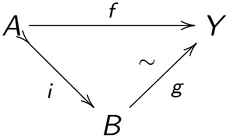
(CC 2) For a cofibration $i : A \rightarrowtail B$ and a map $f : A \to Y$ there exists a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle \bar{i}} \\ B & \xrightarrow[\bar{f}]{} & B \coprod_A Y \end{array}$$

in which $\bar{i}$ is a cofibration. If f is a weak equivalence, then so is $\bar{f}$.

ABC Cofibration Categories

(CC 3) For any morphism $f : A \to Y$ there exists a commutative diagram



in which i is a cofibration and g a weak equivalence.

(CC 4) For each object Y there is a trivial cofibration $Y \overset{\sim}{\rightarrowtail} RY$ with RY a fibrant object. Here an object R is called fibrant if every trivial cofibration $R \overset{\sim}{\rightarrowtail} Y$ has a retraction $r : Y \to R$ with $r \circ i = id_R$.

(CC 5) Coproducts of (trivial) cofibrations are (trivial) cofibrations

(CC 6) Colimits of sequences of cofibrations exist, their transfinite composition is a cofibration. If all are weak equivalences, then so is their transfinite composition.

Theorem (A. Rădulescu-Banu, 2006)

- *Homotopy pushouts in an ABC cofibration category can be constructed as above.*
- *All homotopy colimits exist*

Theorem

With the given classes of weak equivalences and cofibrations $\mathcal{L}og$ is an ABC cofibration category

Corollary: Usual fibring preserves weak equivalences.

About the proof

(CC 3)

$$\begin{array}{ccc} (C,\vdash) & \xrightarrow{\quad f \quad} & (C',\vdash') \\ & \searrow \qquad \nearrow_{\sim} & \\ & \langle C \coprod C' \mid \ \vdash, \vdash', \varphi \dashv\vdash f(\varphi) \rangle & \end{array}$$

(CC 2)

$$\begin{array}{ccc} (C,\vdash) & \xrightarrow{\quad f \quad} & (C'',\vdash'') \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle \bar{i}} \\ (C',\vdash') & \longrightarrow & \langle C'' \coprod (C' \setminus C) \mid ... \rangle \end{array}$$

Preservation of Properties

Definition: By *hofibring* we mean homotopy pushouts in the category $\mathcal{L}og$.

Proposition: If a property is preserved under fibring and under weak equivalences, then it is preserved under hofibring

Proposition: The following are such properties:

- Existence of implicit connectives
- Metatheorem of deduction
- Being protoalgebraizable, algebraizable, equivalential, ...
- Craig interpolation

Completeness Preservation

- C-interpretation structure: $\mathcal{B} = (B, \leq, \top)$
- Interpretation system: $\mathcal{I} = (C, \mathcal{A})$
- $\Gamma \vDash_{\mathcal{I}} \varphi :\Leftrightarrow \forall \mathcal{B} \in \mathcal{A} \forall v : L(C) \to \mathcal{B} : \; v(\Gamma) = \{\top\} \Rightarrow v(\varphi) = \top$
- $(C, \vdash)$ is complete w.r.t. $\mathcal{I} \; :\Leftrightarrow \; \vdash = \vDash_{\mathcal{I}}$

Theorem

The category of interpretation structures is a left proper ABC cofibration category

Theorem

Completeness w.r.t. an interpretation system is preserved under weak equivalences. Hence hofibring preserves completeness.

Variations

Variations of the setup also leading to ABC cofibration categories:

- can vary the properties required from consequence relations (but we need substitution invariance)
- can use sorted signatures, coloured operads
- fibring of institutions via c-parchments
- undercategories
- (conjecture:) into the sorted version include "provisos" $\rightsquigarrow$ 1st and higher order logic
- (conjecture:) "logical spaces"
- ...

General approach to fibring

1. Choose abstract model for logics
2. Choose notion of translation
3. Say when a translation is a weak equivalence

Then: Choose a framework for $(\infty, 1)$-categories;
e.g. quasicategories, simplicial categories, relative categories, complete Segal spaces always work!

If one can find more structure, even better; e.g. I-categories, categories with cylinder functor, Quillen model categories, ...

Then construct homotopy colimits.

Vista

Homotopical view point on categories of logics

A. Homotopy limits; possible translation semantics

B. Find other models for given categories of logics (i.e. equivalent, but differently presented, $(\infty, 1)$-categories)

C. Homotopical versions of properties of categories of logics; e.g. homotopically locally presentable

D. Directed Homotopy Theory

Construction: Directed Classifying Space of a logic
$\rightsquigarrow$ Invariants of logics; e.g. fundamental category, directed homology
Other nerve-like constructions...

Thank you!