Completions of semilattices

joint work with Hilary Priestley

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Finitely generated varieties of lattice-based algebras

Profinite completions

Canonical extensions

Natural extensions

Profinite completion of an algebra **A**

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- the natural homomorphisms $\mathbf{A} \to \mathbf{A}/\theta$ separate the elements of $\mathbf{A}$

  $\mathbf{A}$ embeds into $\hat{\mathbf{A}}$ via the embedding $a \mapsto (a/\theta)_{\theta \in \mathrm{Con}_f\, \mathbf{A}}$

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  i.e.

  $$\bigwedge e(F) \leqslant \bigvee e(I) \Leftrightarrow F \cap I \neq \emptyset$$

  for every filter F and ideal I of **A**

Gehrke and Harding 2001

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Harding 2006

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$$\begin{aligned} e\colon \mathbf{A} &\to \mathbf{M}^{\mathcal{V}(\mathbf{A},\mathbf{M})} \\ a &\mapsto e(a) : \mathcal{V}(\mathbf{A},\mathbf{M}) \to M \\ &\qquad\qquad h \mapsto h(a) \end{aligned}$$

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- the natural extension $n_{\mathcal{V}}(\mathbf{A})$ is the topological closure of $e(A)$ in the topological space $\mathbf{M}_{\mathcal{T}}^{\mathcal{V}(\mathbf{A},\mathbf{M})}$

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and so

$$\mathbf{A}^\delta \cong \hat{\mathbf{A}} \cong n_{\mathcal{V}}(\mathbf{A}) = \{R\text{-preserving maps } \mathcal{V}(\mathbf{A}, \mathbf{M}) \to M\}$$

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▷ $\hat{\mathbf{A}}$ is formed by all the $\{\vee, 0\}$-preserving maps $\mathcal{S}(\mathbf{A}, \mathbf{M})$ to $\mathbf{M}$

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- $\hat{\mathbf{A}}$ is the join-semilattice $\mathcal{S}(\mathcal{S}(\mathbf{A},\mathbf{M}),\mathbf{M})$

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$\mathbf{C}$ is a completion of $\mathbf{A}$ if $\mathbf{C}$ is a complete lattice and there exists an order embedding $e\colon \mathbf{A} \hookrightarrow \mathbf{C}$

C is compact if

$$\bigwedge e(F) \leqslant \bigvee e(I) \Leftrightarrow F \cap I \neq \emptyset$$

for every down-directed up-set F and every up-directed down-set I of **A**

C is dense if every element of C is a join of meets of down-directed up-sets of $e(A)$ and a meet of joins of up-directed down-sets of $e(A)$

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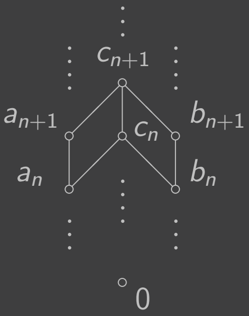
- The canonical extension of **A** is a compact and dense completion **C** relative to some order embedding $e\colon \mathbf{A} \hookrightarrow \mathbf{C}$
- Can we relate the canonical extension of **A** and its profinite completion $\hat{\mathbf{A}}$?

Example 1



A

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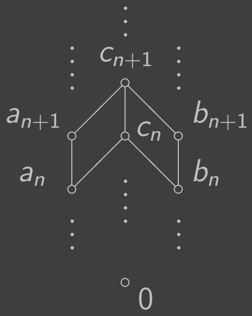

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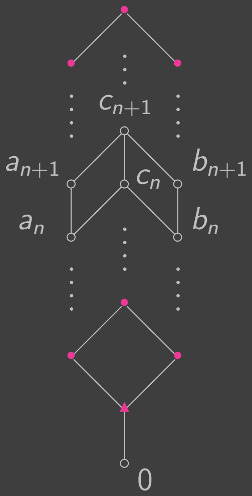

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- $\mathbf{C}_\vee(A)$ is dense

Theorem

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We denote the canonical extension of a semilattice $\mathbf{A} \in \mathcal{S}$ by $\mathbf{A}^{\delta}_{\vee}$.

Proposition

Let $\mathbf{A}$ be a semilattice in $\mathcal{S}$. Then $\mathbf{A}^{\delta}_{\vee}$ satisfies the $\bigvee\bigwedge$ restricted distributive law:

$$\bigvee\{\,\bigwedge e(Y) \mid Y \in \mathcal{Y}\,\} = \bigwedge\{\,\bigvee e(Z) \mid Z \subseteq A, \forall Y \in \mathcal{Y}\;\; Z \cap Y \neq \emptyset\,\},$$

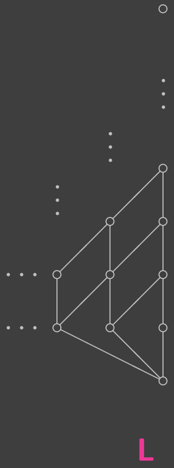
for every family $\mathcal{Y}$ of down-directed subsets of A.

Theorem

Let $\mathbf{L}$ be a lattice with 0 and let $\mathbf{A} = \mathbf{L}_{\vee}$ be its semilattice reduct in $\mathcal{S}$

The lattice $\mathbf{A}_{\vee}^{\delta}$ is the canonical extension of $\mathbf{L}$.

Example 2



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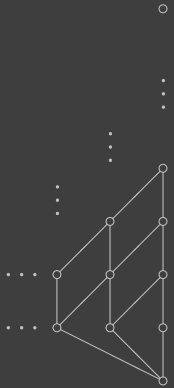
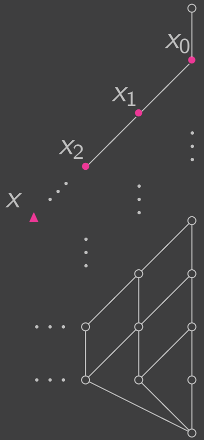


$\mathbf{L}$



$\widehat{\mathbf{A}}$

$\mathbf{A} = \mathbf{L}_\vee$

Proposition

Let $\mathbf{L}$ be a bounded distributive lattice.

Suppose the poset $\mathcal{I}_P(\mathbf{L})$ of prime ideals of $\mathbf{L}$ has finite width.

The canonical extension of $\mathbf{L}$ is isomorphic to the profinite completion $\widehat{\mathbf{L}_\vee}$ of its $\vee$-semilattice reduct $\mathbf{L}_\vee$.

Thank you!