Morphisms of Quantum Triads

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Sup-lattice – a complete join-semilattice L

Homomorphism $f\left(\bigvee x_i\right) = \bigvee f(x_i)$

Quantale – a sup-lattice Q with associative binary operation satisfying $q\left(\bigvee r_i\right) = \bigvee(qr_i)$ and $\left(\bigvee r_i\right)q = \bigvee(r_i q)$

Unital if Q has a multiplicative unit e

Homomorphism $f\left(\bigvee q_i\right) = \bigvee f(q_i)$ and $f(qr) = f(q)f(r)$

Examples

- Ideals of a ring
  (ideals generated by unions, ideal multiplication)
- Binary relations on a set
  (unions of relations, relation composition)
- Endomorphisms of a sup-lattice
  (pointwise suprema, mapping composition)
- Frame
  $\wedge$ as the binary operation
- Powerset of a semigroup
  $(A \cdot B = \{a \cdot b \mid a \in A,\ b \in B\})$
- $\mathcal{P}(X^+)$ – free quantale over X
  $\mathcal{P}(X^*)$ – free unital quantale over X

Right Q-module – a sup-lattice M with right action of the quantale satisfying $(\bigvee m_i)\, q = \bigvee (m_i q)$, $m\,(\bigvee q_i) = \bigvee (m q_i)$, $m(qr) = (mr)q$

Unital if Q is unital and $me = m$ for all m

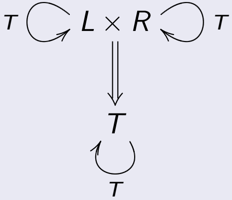
Homomorphism $f\,(\bigvee m_i) = \bigvee f(m_i)$, $f(mq) = f(m)q$

- A sub-sup-lattice of a quantale closed under right multiplication by quantale elements
- A sup-lattice with action of the quantale of its endomorphisms $f \cdot m = f(m)$ (a left module)
- Left-sided elements of a quantale (s.t. $ql \leq l$ for all $q \in Q$ $\iff 1l \leq l$)

Quantum triad (D. Kruml, 2008)

(L, T, R) such that

- Quantale T
- Left T-module L
- Right T-module R
- (T, T)-bimorphism (homomorphism of respective modules when fixing one component) $L \times R \to T$, satisfying associativities TLR, LRT



Example 1

L = right-sided elements of a quantale Q
R = right-sided elements of Q
T = two-sided elements of Q

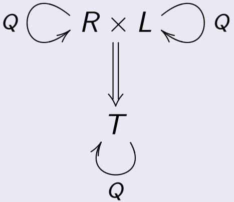
Example 2

Sup-lattice **2**-forms (P. Resende 2004) ($\sim$ Galois connections)
L, R sup-lattices
$T = \mathbf{2}$ (the 2-element frame)

Solution of the triad

Quantale Q such that

- L is a (T, Q)-bimodule
- R is a (Q, T)-bimodule
- there is a (Q, Q)-bimorphism $R \times L \to Q$ satisfying associativities QRL, RLQ, RTL, LQR, LRL, RLR



Example of right/left/two-sided elements: Q is a solution

Two special solutions

$$Q_0 = R \otimes_T L$$

- $(r_1 \otimes l_1) \cdot (r_2 \otimes l_2) = r_1(l_1 r_2) \otimes l_2$
- $l'(r \otimes l) = (l'r)l$
- $(r \otimes l)r' = r(lr')$

$$Q_1 = \{(\alpha, \beta) \in \mathbf{End}(L) \times \mathbf{End}(R) \mid \alpha(l)r = l\beta(r) \text{ for all } l \in L, r \in R\}$$

- $(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = (\alpha_2 \circ \alpha_1, \beta_1 \circ \beta_2)$
- $l'(\alpha, \beta) = \alpha(l')$
- $(\alpha, \beta)r' = \beta(r')$

Couple of solutions

There is a $\phi\colon Q_0 \to Q_1$, $\phi(r \otimes l) = ((- \cdot r)l, r(l \cdot -))$ which forms a so-called couple of quantales (Egger – Kruml 2008):

Q_0 is a (Q_1, Q_1)-bimodule with $\phi(q)r = qr = q\phi(r)$ for all $q, r \in Q_0$

All solutions Q of (L, T, R) then provide factorizations of the couple:

- There are maps $\phi_0\colon Q_0 \to Q$ and $\phi_1\colon Q \to Q_1$ s.t. $\phi_1 \circ \phi_0 = \phi$
- $\phi_0(\phi_1(k)q) = k\phi_0(q)$ and $\phi_o(q\phi_1(k)) = \phi_0(q)k$ (so ϕ_0 becomes a coupling map under scalar restriction along ϕ_1)

Example

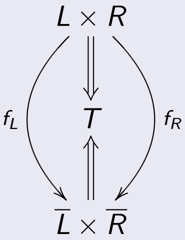
L is a sup-lattice, $R = T = \mathbf{2}$

$L \times \mathbf{2} \to \mathbf{2}$: $(0, y) \mapsto 0$, $(x, 0) \mapsto 0$, $(x, 1) \mapsto 1$

Then
$Q_0 = \mathbf{2} \otimes_{\mathbf{2}} L = L$ with $xy = y$,
$Q_1 = \{(x \mapsto 0, y \mapsto 0), (\mathrm{id}_L, \mathrm{id}_R)\} = \mathbf{2}$

Triad morphisms

Let (L, T, R) and $(\overline{L}, T, \overline{R})$ be triads over the same quantale T.
Module homomorphisms $f_L\colon L \to \overline{L}$ and $f_R\colon R \to \overline{R}$, that satisfy $f_L(l)f_R(r) = lr$ for every l, r, induce a quantale homomorphism $R \otimes_T L \to \overline{R} \otimes_T \overline{L}$.



In the context of **2**-forms: orthomorphisms

Both f_L and f_R are surjections $\implies$ $\overline{L} \otimes_T \overline{R}$ is a quantale quotient of $L \otimes_T R$.

Definition

A right Q-module M is flat if $M \otimes_Q - : Q\text{-}Mod \to SLat$ preserves monomorphisms (injective homomorphisms)

For unital modules (Joyal and Tierney 1984):
M flat $\iff$ M projective ($\mathrm{Hom}(M, -)$ preserves epimorphisms).

Both f_L and f_R are injections and R, $\overline{L}$ (or vice versa) are flat $\implies$ $\overline{L} \otimes_T \overline{R}$ is a subquantale of $L \otimes_T R$.

Projective modules (RŠ 2010)

- Infinitely 0-distributive (for all $x \in M$, $A \subseteq M$: $x \wedge a = 0$ for all $a \in A \implies x \wedge \bigvee A = 0$) and
- finitely spatial (every element is a join of join-irreducibles)

right Q-module M is projective $\iff$ $M \cong \prod Md_i$ where each d_i is an idempotent element of Q.

Example (Galatos – Tsinakis)

$\mathcal{P}(\mathbf{Fm})$ (sets of formulas), $\mathcal{P}(\mathbf{Eq})$ (sets of equations) are projective (cyclic) module over $\mathcal{P}((\Sigma)$ (sets of substitutions).

References

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Thank you for your attention!