Finite Embeddability Property of Distributive Lattice-ordered Residuated Groupoids with Modal Operators

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29-07-2011, TACL2011

Preliminaries

Associative Lambek Calculus L: (Lambek 1958) ($\Gamma \neq \varepsilon$)

$$(Id) \quad A \Rightarrow A$$

$$(\backslash L)\ \frac{\Gamma, B, \Delta \Rightarrow C \quad \Phi \Rightarrow A}{\Gamma, \Phi, A\backslash B, \Delta \Rightarrow C} \qquad (\backslash R)\ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A\backslash B}$$

$$(/L)\ \frac{\Gamma, B, \Delta \Rightarrow C \quad \Phi \Rightarrow A}{\Gamma, B/A, \Phi, \Delta \Rightarrow C} \qquad (/R)\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow B/A}$$

$$(\cdot L)\ \frac{\Gamma, A, B\Delta \Rightarrow C}{\Gamma, A \cdot B, \Delta \Rightarrow C} \qquad (\cdot R)\ \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \cdot B}$$

$$(CUT)\ \frac{\Gamma, A, \Delta \Rightarrow B \quad \Phi \Rightarrow A}{\Gamma, \Phi, \Delta \Rightarrow B}$$

Nonassociative Lambek Calculus NL: (Lambek 1961)
Fomula structures (trees): formulas, $\Gamma \circ \Delta$; Sequent: $\Gamma \Rightarrow A$

$$(\backslash L)\ \frac{\Delta \Rightarrow A \quad \Gamma[B] \Rightarrow C}{\Gamma[\Delta \circ A\backslash B] \Rightarrow C} \qquad (\backslash R)\ \frac{A \circ \Gamma \Rightarrow B}{\Gamma \Rightarrow A\backslash B}$$

$$(/L)\ \frac{\Gamma[A] \Rightarrow C \quad \Delta \Rightarrow B}{\Gamma[A/B \circ \Delta] \Rightarrow C} \qquad (/R)\ \frac{\Gamma \circ B \Rightarrow A}{\Gamma \Rightarrow A/B}$$

$$(\cdot L)\ \frac{\Gamma[A \circ B] \Rightarrow C}{\Gamma[A \cdot B] \Rightarrow C} \qquad (\cdot R)\ \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma \circ \Delta \Rightarrow A \cdot B}$$

$$(CUT)\ \frac{\Delta \Rightarrow A \quad \Gamma[A] \Rightarrow B}{\Gamma[\Delta] \Rightarrow B}$$

(CUT) is admissible in L and NL.

A residuated semigroup: $\mathcal{M} = (M, \leq, \cdot, \backslash, /)$ s.t. $(M, \leq)$ is a poset such that $(M, \cdot)$ is semigroup $\backslash, /$ are binary operations on M, respectively, satisfying the residuated law:

$$(RES) \quad a \cdot b \leq c \quad \text{iff} \quad b \leq a \backslash c \quad \text{iff} \quad a \leq c/b \tag{1}$$

A residuated groupoid: need not be associative

A valuation μ in $\mathcal{M}$ is a homomorphism from the formula into algebra $\mathcal{M}$. A sequent $\Gamma \Rightarrow A$ is true in the model ($\mathcal{M}$, μ), if $\mu(\Gamma) \leq \mu(A)$.

L is strongly complete w.r.t. residuated semigroups. NL is strongly complete w.r.t. residuated groupoids.

Lattice:

$$(\wedge L)\ \frac{\Gamma[A_i] \Rightarrow B}{\Gamma[A_1 \wedge A_2] \Rightarrow B} \quad (\wedge R)\ \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$(\vee L)\ \frac{\Gamma[A_1] \Rightarrow B \quad \Gamma[A_2] \Rightarrow B}{\Gamma[A_1 \vee A_2] \Rightarrow B} \quad (\vee R)\ \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}$$

Distributive axiom: (D) $A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$.

Full Lambek Calculus (FL) is strongly complete w.r.t lattice-ordered residuated semigroup. Full Nonassociative Lambek Calculus (FNL) is strongly complete w.r.t lattice-ordered residuated groupoid.

A distributive lattice-ordered residuated groupoid: $(G, \wedge, \vee, \cdot, \backslash, /)$ such that $(G, \wedge, \vee)$ is a distributive lattice and $(G, \cdot, \backslash, /)$ is a residuated groupoid, where the order is lattices order.

Distributive Full Nonassociative Lambek Calculus (DFNL) is strongly complete w.r.t distributive lattice-ordered residuated groupoid.

(CUT) is not admissible in system with (D).

Modalities(MOORTGAT 1996)

$$(\Diamond L)\ \frac{\Gamma[\langle A\rangle] \Rightarrow B}{\Gamma[\Diamond A] \Rightarrow B} \quad (\Diamond R)\frac{\Gamma \Rightarrow A}{\langle\Gamma\rangle \Rightarrow \Diamond A}$$

$$(\Box\downarrow L)\ \frac{\Gamma[A] \Rightarrow B}{\Gamma[\langle\Box\downarrow A\rangle] \Rightarrow B} \quad (\Box\downarrow R)\ \frac{\langle\Gamma\rangle \Rightarrow A}{\Gamma \Rightarrow \Box\downarrow A}$$

$$(4)\ \frac{\Gamma[\langle\Delta\rangle] \Rightarrow A}{\Gamma[\langle\langle\Delta\rangle\rangle] \Rightarrow A} \quad (T)\ \frac{\Gamma[\langle\Delta\rangle] \Rightarrow A}{\Gamma[\Delta] \Rightarrow A}$$

A distributive lattice-ordered residuated groupoid with $\mathrm{S4}$-operators ($\mathrm{S4}$-$dlrg$) is a structure $(G, \wedge, \vee, \cdot, \backslash, /, \Diamond, \Box\downarrow)$ such that $(G, \wedge, \vee)$ is a distributive lattice and $(G, \cdot, \backslash, /, \Diamond, \Box\downarrow)$ is a structure such that $\cdot, \backslash, /$ and $\Diamond$, $\Box\downarrow$ are binary and unary operations on G, respectively, satisfying the above conditions (1) and standard modal S4-axioms:

$$\mathrm{T} \quad a \leq \Diamond a, \quad 4 \quad \Diamond\Diamond a \leq \Diamond a \tag{2}$$

$$\mathrm{K} \quad \Diamond(a \wedge b) \leq \Diamond a \wedge \Diamond b \tag{3}$$

Remark: K is admissible in S4-$dlrg$. Here after we slip this axiom.

$\mathrm{DNFL}_{\mathrm{S4}}$ is strongly complete w.r.t S4-$dlrg$

A class of algebras $\mathcal{K}$ is said to have the finite embeddability property (FEP) if for every algebra $\mathcal{A}$ in $\mathcal{K}$ and every finite partial subalgebra $\mathcal{B}$ of $\mathcal{A}$, there exists a finite algebra $\mathcal{D}$ in $\mathcal{K}$ such that $\mathcal{B}$ embeds into $\mathcal{D}$.

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[8]. M. Farulewski, Finite embeddability property for residuated groupoids, Reports on Mathematical Logic, 43:25-42, 2008.

[4]. W. Buszkowski, and M. Farulewski, Nonassociative Lambek Calculus with Addi- tives and Context-Free Languages. In: O. Grumberg et al. (eds.),

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FEP of S4-$dlrg$s (Our results also state for $dlrg$s with modal operators satisfying 4 or T only).

A class $\mathcal{K}$ of algebras has Strong Finite Model Property (SFMP) if every Horn clause that fails to hold in $\mathcal{K}$ can be falsified in a finite member of $\mathcal{K}$.

Strong Finite Model Property (SFMP) of a formal system S: if $\vdash \phi \Rightarrow A$ does not hold in S, then there exist a finite model of S ($\mathcal{M}$, μ) such that all sequents from Φ are true, but $\Gamma \Rightarrow A$ is not in ($\mathcal{M}$, μ).

If a formal system $\mathcal{S}$ is strongly complete with respect to $\mathcal{K}$, then it yields, actually, an axiomatization of the Horn theory of $\mathcal{K}$; hence SFMP for S with respect to $\mathcal{K}$ yields SFMP for $\mathcal{K}$.

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Theorem

If a class of algebras $\mathcal{K}$ is closed under (finite) products, then SFMP for $\mathcal{K}$ is equivalent to FEP for $\mathcal{K}$

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SFMP for $\mathrm{DNFL_{S4}}$(FEP of S4-$dlrg$s)

Linguistic analysis of modalities and additives

L or NL enriched with modalities or additive can be used to analysis some linguistic phenomenon like feature agreement, feature description, parasitic gap and so on.

Let me show some very easy example:

$\Box\downarrow_{sing} np$ denote singular noun phrase and $\Box\downarrow_{pl} np$ denote plural noun phrase

1. *walks* $\rightarrow \Box\downarrow_{sing} np\backslash s$
2. *walk* $\rightarrow \Box\downarrow_{pl} np\backslash s$
3. *walked* $\rightarrow np\backslash s$
4. *John* $\rightarrow \Box\downarrow_{sing} np$
5. *the Beatles* $\rightarrow \Box\downarrow_{pl} np$
6. *the Chinese* $\rightarrow \Box\downarrow_{sing} \Box\downarrow_{pl} np$

John walks. *John walked*. *John walk*.

$$\frac{\Box\downarrow_{sing} np \Rightarrow \Box\downarrow_{sing} np \quad s \Rightarrow s}{\Box\downarrow_{sing} np \circ \Box\downarrow_{sing} np\backslash s \Rightarrow s}\ (\backslash L) \qquad \frac{\dfrac{\dfrac{np \Rightarrow np}{\langle \Box\downarrow_{sing} np\rangle \Rightarrow np}\ (\Box^{\downarrow} L)}{\Box\downarrow_{sing} np \Rightarrow np}\ (\mathbf{T}) \quad s \Rightarrow s}{\Box\downarrow_{sing} np \circ np\backslash s \Rightarrow s}\ (\backslash L)$$

$$\frac{\Box\downarrow_{sing} np \Rightarrow \Box\downarrow_{pl} np \quad s \Rightarrow s}{\Box\downarrow_{sing} np \circ \Box\downarrow_{pl} np\backslash s \Rightarrow s}\ (\backslash L) \quad \text{(not derivable)}$$

The Chinese walk. The Chinese walks.

$$\dfrac{\dfrac{\dfrac{np \Rightarrow np}{\langle \Box \downarrow_{pl}\ np\rangle \Rightarrow np}\ (\Box^{\downarrow}L),(\mathrm{T})}{\Box \downarrow_{sing} \Box \downarrow_{pl}\ np \Rightarrow \Box \downarrow_{sing}\ np}\ (\Box^{\downarrow}L),(\Box^{\downarrow}R) \quad s \Rightarrow s}{\Box \downarrow_{sing} \Box \downarrow_{pl}\ np \circ \Box \downarrow_{sing}\ np\backslash s \Rightarrow s}\ (\backslash L)$$

$$\dfrac{\dfrac{\Box \downarrow_{pl}\ np \Rightarrow \Box \downarrow_{pl}\ np}{\langle \Box \downarrow_{sing} \Box \downarrow_{pl}\ np\rangle \Rightarrow \Box \downarrow_{pl}\ np}\ (\Box^{\downarrow}L),(\mathrm{T}) \quad s \Rightarrow s}{\Box \downarrow_{sing} \Box \downarrow_{pl}\ np \circ \Box \downarrow_{sing}\ np\backslash s \Rightarrow s}\ (\backslash L)$$

1. *become* $\rightarrow vp/np \vee ap$
2. *wealthy* $\rightarrow ap$
3. *and* $\rightarrow (ap \vee np\backslash ap \vee np)/ap \vee np$
4. *a professor* $\rightarrow np$

become a professor and wealthy

$$\dfrac{ap \Rightarrow np \vee ap \quad \dfrac{np \Rightarrow ap \vee np \quad \dfrac{ap \vee np \Rightarrow ap \vee np \quad vp \Rightarrow vp}{vp/ap \vee np \circ ap \vee np \Rightarrow vp}}{vp/ap \vee np \circ (np \circ ap \vee np\backslash ap \vee np) \Rightarrow vp}}{vp/ap \vee np \circ (np \circ ((ap \vee np\backslash ap \vee np)/ap \vee np \circ ap)) \Rightarrow vp}$$

Interpolation property

Lemma

If $\Phi \vdash_{\mathrm{NL}} \Gamma[\Delta] \Rightarrow A$, then there exists a formula D such that $\Phi \vdash_{\mathrm{NL}} \Delta \Rightarrow D$ and $\Phi \vdash_{\mathrm{NL}} \Gamma[D] \Rightarrow A$, where D is a subformula of some formulae appearing in $\Gamma[\Delta] \Rightarrow A$ and Φ.

- $\mathrm{NL}\Diamond$ (Jäger 2004) $\mathrm{NL}\wedge$(Farulewski 2008) DFNL (Buszkowski, and Farulewski 2009) $\mathrm{NL}_{\mathrm{S4}}$ (Plummer 2008).
- The consequence relation of NL is decidable in polynomial time (Buszkowski 2005)
- Context-freeness of $\mathrm{NL}\Diamond$ (Jäger 2004), $\mathrm{NL}_{\mathrm{S4}}$ (Plummer 2008), DFNL (Buszkowski, and Farulewski).
- FEP of Rgs, Dlrgs (Farulewski 2008, Buszkowski, and Farulewski 2009), FEP of RAs, distributive lattice-ordered RAs, boolean RAs, Heyting RAs and double RAs (Buszkowski 2010)

Question:

? interpolation property for $\mathrm{DNFL_{S4}}$ YES

Let T denote a set of formulas

- T-sequent: A sequent such that all formulas occurring in it belong to T.
- $\Phi \vdash_S \Gamma \Rightarrow_T A$: If $\Gamma \Rightarrow A$ has a deduction from Φ (in the given calculus S) which consists of T-sequents only (called a T-deduction).
- T-equivalent: Two formulae A and B are said to be T-equivalent in calculus S, if and only if $\vdash_S A \Rightarrow_T B$ and $\vdash_S B \Rightarrow_T A$.

Lemma

Let T be a set of formulae closed under $\vee$, $\wedge$. If $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma[\langle\Delta\rangle] \Rightarrow_T A$ then there exists a $D \in T$ such that $\Phi \vdash_{\mathrm{DFNL_{S4}}} \langle\Delta\rangle \Rightarrow_T D$, $\Phi \vdash_{\mathrm{DFNL_{S4}}} \langle D\rangle \Rightarrow_T D$, and $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma[D] \Rightarrow_T A$.

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Lemma

If T is set of formulas generated from a finite set and closed under $\wedge$, $\vee$, then T is finite up to the relation of T-equivalence in $\mathrm{DFNL_{S4}}$.

Let $\mathcal{M} = (M, \cdot, \Diamond)$ be a groupoid with a unary operation $\Diamond$.

- $U \odot V = \{a \cdot b \in G : a \in U, b \in V\}$ $U \backslash V = \{z \in G : U \odot \{z\} \subseteq V\}$, $V/U = \{z \in M; \{z\} \odot U \subseteq V\}$

$C : P(M) \to P(M)$ ($4\mathrm{T}$-closure operator on $\mathcal{M}$)

- (C1) $U \subseteq C(U)$. (C2) if $U \subseteq V$ then $C(U) \subseteq C(V)$

For any $U \subseteq M$: U is C-*closed*, if $C(U) = U$. $C(M)$: the family of all closed subsets of M. Operation on $C(M)$ are defined as follows:

$U \otimes V = C(U \odot V)$, $\blacklozenge U = C(\diamond U)$, $U \vee_C V = C(U \vee V)$, $\backslash$, $/$ $\Box\downarrow$, $\wedge$ as above.

Theorem

$C(\mathcal{M}) = (C(M), \otimes, \backslash, /, \blacklozenge, \Box\downarrow, \wedge, \vee_C)$ is an $\mathrm{S4}$-lattice order residuated groupoid.

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- $U \vee V = U \cup V, U \wedge V = U \cap V$

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- (C1) $U \subseteq C(U)$. (C2) if $U \subseteq V$ then $C(U) \subseteq C(V)$
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Theorem

$C(\mathcal{M}) = (C(M), \otimes, \backslash, /, \blacklozenge, \Box\downarrow, \wedge, \vee_C)$ *is an* $\mathrm{S4}$-*lattice order residuated groupoid.*

Let $\mathcal{M} = (M, \cdot, \Diamond)$ be a groupoid with a unary operation $\Diamond$.

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- (C1) $U \subseteq C(U)$. (C2) if $U \subseteq V$ then $C(U) \subseteq C(V)$
- (C3) $C(C(U)) \subseteq C(U)$. (C4) $C(U) \odot C(V) \subseteq C(U \odot V)$
- (C5) $\diamond C(U) \subseteq C(\diamond U)$
- (C6) $C(\Diamond C(\Diamond C(U))) \subseteq C(\Diamond U)$. (C7) $C(U) \subseteq C(\Diamond U)$

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Theorem

$C(\mathcal{M}) = (C(M), \otimes, \backslash, /, \blacklozenge, \Box \downarrow, \wedge, \vee_C)$ is an $\mathrm{S4}$-lattice order residuated groupoid.

T: nonempty set of formulae containing all subformulae of formulae in Φ; T^*: all formula structures form out of formulae in T. Similarly; $T^*[\circ]$: all contexts in which all formulae belong to T.

Let $\Gamma[\circ] \in T^*$ and $A \in T$; $B(T)$: the family of all sets $[\Gamma[\circ], A]$

$$[\Gamma[\circ], A] = \{\Delta : \Delta \in T^* \quad \text{and} \quad \Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma[\Delta] \Rightarrow_T A\}$$

$$C_T(U) = \bigcap\{[\Gamma[\circ], A] \in B(T) : U \subseteq [\Gamma[\circ], A]\}$$

Lemma

C_T is a $\mathrm{S4}$-modal closed operator.

T: containing all formulae in Φ, closed under subformulae, $\wedge$ and $\vee$. $\mathcal{G}(\mathcal{T}^*) = (T^*, \circ, \langle\rangle)$: a groupoid, $\langle\rangle$ is an unary operation on T^*.

Lemma

$C_T(\mathcal{G}(\mathcal{T}^*))$ is a $\mathrm{S4}$-lrg

μ: $\mu(p) = [p]$.

$$[A] \otimes [B] = [A \cdot B], \quad [A] \backslash [B] = [A \backslash B], \quad [A]/[B] = [A/B] \tag{4}$$

$$\blacklozenge[A] = [\Diamond A] \quad \Box\!\downarrow [A] = [\Box\!\downarrow A] \tag{5}$$

$$[A] \cap [B] = [A \wedge B] \quad [A] \vee_C [B] = [A \vee B] \tag{6}$$

all formulas appearing in them belong to T.

Lemma

For any nontrivial closed set $U \in C_T(G(T^*))$, there exists a formula $A \in R$ such that $U = [A]$.

Lemma

$C_T(\mathcal{G}(\mathcal{T}^*))$ *is a finite* $4\mathrm{T} - dlrg$.

Lemma

T denotes a set of formulae, containing all formulae in Φ and closed under $\wedge$, $\vee$, and subformulae. Let μ be a valuation in $C_T(\mathcal{G}(\mathcal{T}^*))$ such that $\mu(p) = [p]$. For any T-sequent $\Gamma \Rightarrow A$, this sequent is true in ($C_T(\mathcal{G}(\mathcal{T}^*))$, μ) if and only if $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma \Rightarrow_T A$.

Theorem

Assume that $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma \Rightarrow A$ does not hold. Then there exist a finite distributive lattice ordered residuated groupoid with $4\mathrm{T}$-operators $\mathcal{G}$ and a valuation μ such that all sequents from Φ are true but $\Gamma \Rightarrow A$ is not true in ($\mathcal{G}$, μ).

Corollary

$\mathrm{S4} - dlrgs$ *has FEP.*

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Thank you