Substructural Logic for Orientable and Non-Orientable Surfaces

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Topology and geometry appear as natural tools for studying Linear Logic proofs:

- Proofs may be represented as graphs where nodes are sub-formulas of the conclusions, edges are either axioms or cuts or introduction of connectives.
- Correctness of proofs is checked either sequentially by means of a sequent calculus, or geometrically by a global criterion on the corresponding graph (cyclicity, connexity, planarity, ...)

Two main variants of Linear Logic exist:

- variations on exponentials: *light* variants for studying complexity
- variations on structures: *cyclic* logic and extensions for studying (non-)commutativity

Permutative Logic is a non-commutative variant of linear logic.

Its logical status is based on a variety-presentation framework that models *orientable* structures.

➥The aim is here to characterize **orientable** as well as **non-orientable** topological surfaces.

- We prove that the system keeps standard logical properties: cut elimination and focussing.
- We give also a few comments on *relaxation*, the binary relation induced by structural transformations that may increase the topological genus of the transformed surface.

Presentation of a topological surface

$\mathscr{Q}$ is a closed and connected surface (2-manifold)

$\Downarrow \Uparrow$

set of triangles $\mathcal{T} = \{T_1, \ldots, T_n\}$ having the edges labelled (labels from $\mathcal{A} = \{a, b, c, \ldots\}$) and oriented

$\Downarrow \Uparrow$

polygon $P_{\mathcal{T}}$

$\Downarrow \Uparrow$

word $w_{P_{\mathcal{T}}}$ (over $\mathcal{A} \cup \overline{\mathcal{A}}$ with $\overline{\mathcal{A}} = \{\overline{a}, \overline{b}, \overline{c}, \ldots\}$) which 'reads' the perimeter of $P_{\mathcal{T}}$ being fixed an orientation

Remark

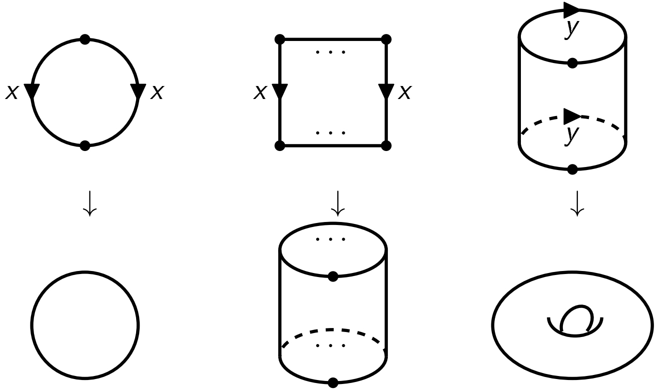
The geometrical information expressed by $\mathscr{Q}$ can be encoded (modulo homeomorphisms) in a *finite* and *discrete* combinatorial setting

Basic 2-Manifolds: *orientable* case

To go from polygons/words to surfaces:

gluing of edges wrt their labels and orientations

sphere: $x\overline{x}$ **cylinder**: $x \ldots \overline{x} \ldots$ **torus**: $xy\overline{xy}$

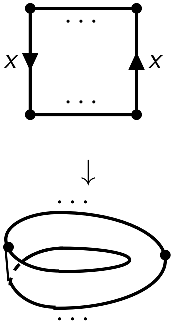


Basic 2-Manifolds: *non-orientable* case

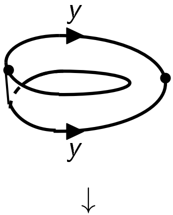
From polygons/words to surfaces:

gluing of edges wrt their labels and orientations

Möbius: $x \ldots x \ldots$



Klein: $xyx\overline{y}$



Basic 2-Manifolds: *non-orientable* case

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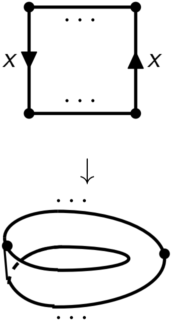

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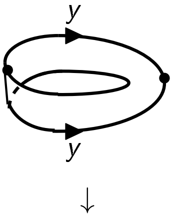


Well, as usual

Classification (Massey)

Theorem

Any closed and connected surface $\mathscr{Q}$ (possibly with boundary) is homeomorphic to:

- *the sphere $\mathscr{S}$ (orientable) or*
- *a connected sum of tori $\mathscr{T}_1 \# \ldots \# \mathscr{T}_n$ (orientable) or*
- *a connected sum of projective planes $\mathscr{P}_1 \# \ldots \# \mathscr{P}_n$ (non-orientable).*

The proof consists in:

- rewriting any word w into an equivalent one (i.e. denoting an homeomorphic surface) in so-called *canonical form* (i.e. explicitly denoting a surface homeomorphic to $\mathscr{T}_1 \# \ldots \# \mathscr{T}_n \# \mathscr{P}_1 \# \ldots \# \mathscr{P}_m$),
- stressing the basic homeomorphism $\mathscr{T} \# \mathscr{P} \sim \mathscr{P} \# \mathscr{P} \# \mathscr{P}$.

From Words to pq-Permutations

A canonical word has 3 parts:

$$w = \tau_p * \pi_q * d_1 v_1 \overline{d}_1 * \ldots * d_k v_k \overline{d}_k$$

- (decomposed tori)
  $\tau_p = a_1 b_1 \overline{a}_1 \overline{b}_1 \ldots a_p b_p \overline{a}_p \overline{b}_p \mapsto \mathscr{T}_1 \# \ldots \# \mathscr{T}_p$
- (decomposed proj. planes)
  $\pi_q = c_1 c_1 \ldots c_q c_q \mapsto \mathscr{P}_1 \# \ldots \# \mathscr{P}_q$
- (boundary made of k cycled parts) $v_1, \ldots, v_k$

It may be presented as a **pq-Permutation** $\alpha = \Sigma_{\langle p,q \rangle}$:

- permutation Σ (a set of cycles) denoting the boundary,
- double index $\langle p, q \rangle$ denoting the connected sum of p tori and q projective planes.

From Words to pq-Permutations

Example

Consider the word: $ab\bar{a}\bar{b} * cc * d_1 ef \bar{d}_1 * d_2 g \overline{hd}_2$

- The surface denoted
  - is homeomorphic to $\mathscr{T} \# \mathscr{P}$
  - with the boundary decomposed into 2 components ef and $g\bar{h}$
- the associated pq-permutation is $(e, f), (g, \bar{h})_{\langle 1,1 \rangle}$

Topology in (Linear) Logic

Linear Logic has a graph-theoretical representation of proofs:

Interpretation of proofs as topological objects

- ⇒ *surfaces drawn without crossing edges (Bellin and Fleury 98, Métayer 01, Mellies 04)*
- ⇒ *computation of surfaces (Gaubert 04)*

Topology in (Linear) Logic

Exchange rule as a topological operation:
➥(non-)commutative variants of Multiplicative Linear logic (MLL)

- planar logic (Mellies 04)
- the calculus of surfaces (Gaubert 04)
- permutative logic (PL) (Andreoli, Pulcini, Ruet 05)

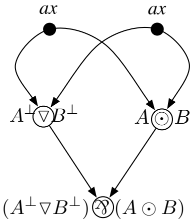
Topology in (Linear) Logic: Ribbon presentation (Mellies 04)

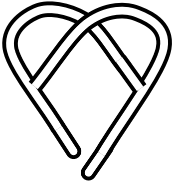
Proof structures may be represented by *ribbons*.

A non-commutative proof structure is correct when:

- (commutative criterion) the ribbon presentation of the commutative translation is homeomorphic to the disk
- the ribbon presentation is planar and has a unique external border σ
- σ contains all the conclusions

$$\nvdash (B \odot A) \multimap (A \odot B) \quad \equiv \quad \nvdash (A^{\perp} \triangledown B^{\perp}) \parr (A \odot B)$$

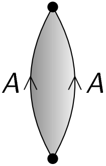




Topology in (Linear) Logic: Orientable surface presentation (Métayer 01)

Proofs are presented as the result of gluing edges (e.g. formulas) of a surface.

formula	$\rightarrow$	1-cell,
rule	$\rightarrow$	2-cell



$$\frac{}{\vdash A, A^{\perp}}\ (axiom)$$

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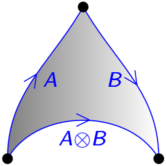

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \; (binary\;\; rule)$$

Topology in (Linear) Logic: Orientable surface presentation (Métayer 01)

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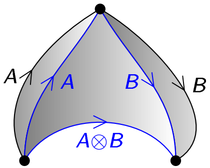
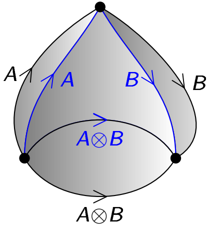


$$\dfrac{\dfrac{}{A \vdash A}\,ax \qquad \dfrac{}{B \vdash B}\,ax}{A, B \vdash A \otimes B}$$

Topology in (Linear) Logic: Orientable surface presentation (Métayer 01)

Proofs are presented as the result of gluing edges (e.g. formulas) of a surface.

formula → 1-cell,
rule → 2-cell



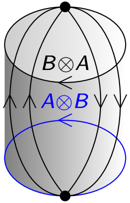
$$\dfrac{\dfrac{}{A \vdash A}\,ax \qquad \dfrac{}{B \vdash B}\,ax}{\dfrac{A, B \vdash A \otimes B}{A \otimes B \vdash A \otimes B}}$$

A cyclic proof is correct iff it is homeomorphic to a disk.

Topology in (Linear) Logic: Orientable surface presentation (Métayer 01)

Proofs are presented as the result of gluing edges (e.g. formulas) of a surface.

formula $\rightarrow$ 1-cell,
rule $\rightarrow$ 2-cell



$$\dfrac{\dfrac{\dfrac{\overline{A \vdash A}\ ax \qquad \overline{B \vdash B}\ ax}{A, B \vdash A \otimes B}}{B, A \vdash A \otimes B}\ Exchange}{B \otimes A \vdash A \otimes B}$$

-> topological cylinder.

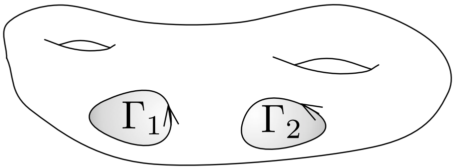
Topology in (Linear) Logic: Permutative Logic

The permutative logic PL,designed by Andreoli, Pulcini and Ruet concerns orientable surfaces: cylinder/divide and torus.

The general form of a sequent is $\vdash_q (\Gamma_1), \ldots, (\Gamma_n)$

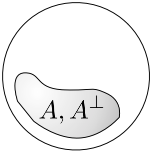
- q represents the number of "tori handles"
- Γ_i is a cyclic sequence of formulas built from atoms, $\otimes$ and $⅋$

Its structure is a $(p)q$-permutation, where p (number of projective planes) is not taken into account.

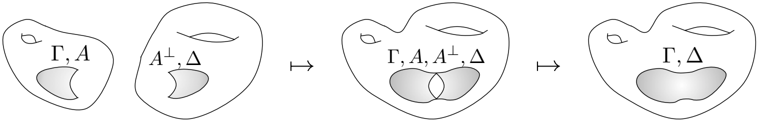


Permutative Logic: Axiom and Cut Rules

$$\frac{}{\vdash_0 (A, A^\perp)}\ (ax)$$

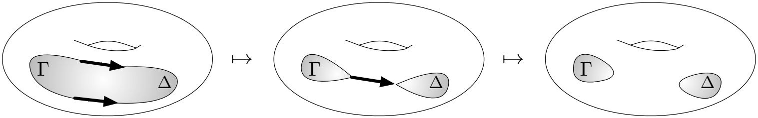


$$\frac{\vdash_q \Sigma, (\Gamma, A) \qquad \vdash_{q'} \Xi, (A^\perp, \Delta)}{\vdash_{q+q'} \Sigma, \Xi, (\Gamma, \Delta)}\ (Cut)$$

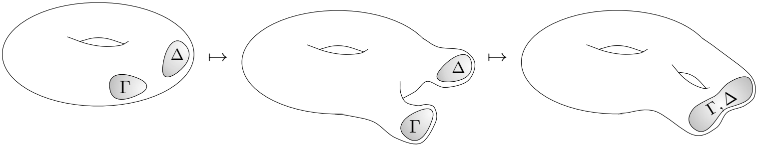


Permutative Logic: Structural Rules

$$\frac{\vdash_q \Sigma, (\Gamma, \Delta)}{\vdash_q \Sigma, (\Gamma), (\Delta)} \ (\mathit{Cylinder})$$



$$\frac{\vdash_q \Sigma, (\Gamma), (\Delta)}{\vdash_{q+1} \Sigma, (\Gamma, \Delta)} \ (\mathit{Torus})$$



Orientable and non-Orientable Surfaces

In order to take care of non-orientable surfaces, two additions are necessary in a sequent calculus for surfaces (sPL):

- A sequent has another index p to represent the number of "projective planes"
- Orientation is given by a unary operator on formulas

Definition

Formulas are inductively built from a countable infinite set of atoms $\mathcal{A} = \{a, b, c, \ldots, a^\perp, b^\perp, c^\perp, \ldots\}$ throughout the two usual multiplicative connectives $\parr$ and $\otimes$, together with an unary bar operation $(\bar{\_})$ $(a \in \mathcal{A})$:

$$F ::= a \mid \bar{F} \mid F_1 \parr F_2 \mid F_1 \otimes F_2$$

Orientable Structural Rules

Already in Permutative Logic:

$$\frac{\vdash^p_q \Sigma, (\Gamma, \Delta)}{\vdash^p_q \Sigma, (\Gamma), (\Delta)}\ \text{cylinder} \qquad \frac{\vdash^p_q \Sigma, (\Gamma), (\Delta)}{\vdash^p_{q+1} \Sigma, (\Gamma, \Delta)}\ \text{torus}$$

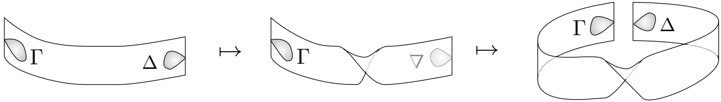
The shape is invariant wrt a global change of the orientation:

$$\frac{\vdash^p_q \Sigma}{\vdash^p_q \overline{\Sigma}}\ \text{invert}$$

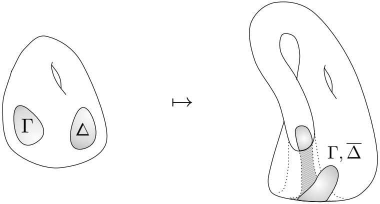
Non-Orientable Structural Rules

Two new rules for dealing with non-orientable surfaces:

$$\frac{\vdash^p_q \Sigma, (\Gamma, \Delta)}{\vdash^{p+1}_q \Sigma, (\Gamma, \overline{\Delta})} \text{ Möbius}$$



$$\frac{\vdash^p_q \Sigma, (\Gamma), (\Delta)}{\vdash^{p+2}_q \Sigma, (\Gamma, \overline{\Delta})} \text{ Klein}$$



IDENTITY GROUP

$$\frac{}{\vdash^0_0 (A, A^\perp)}\ \text{ax.}$$

$$\frac{\vdash^p_q \Sigma, (\Gamma, A) \qquad \vdash^{p'}_{q'} \Xi, (\Delta, A^\perp)}{\vdash^{p+p'}_{q+q'} \Sigma, \Xi, (\Gamma, \Delta)}\ \text{cut}$$

LOGICAL RULES

$$\frac{\vdash^p_q \Sigma, (\Gamma, A, B)}{\vdash^p_q \Sigma, (\Gamma, A \parr B)}\ \parr$$

$$\frac{\vdash^p_q \Sigma, (\Gamma, A) \qquad \vdash^{p'}_{q'} \Xi, (\Delta, B)}{\vdash^{p+p'}_{q+q'} \Sigma, \Xi, (\Gamma, A \otimes B, \Delta)}\ \otimes$$

Theorem (cut-elimination)

Any proof in sPL can be transformed into a proof without cut.

The proof is done by induction, studying the various cases of conmutation.
It may also be done via a cut-elimination proof of a focalized sequent calculus.

Theorem (focalization)

A focalized sequent calculus equivalent to sPL may be defined.

A focalized sequent is of the form $\vdash_q^p \Gamma | \Sigma$
where

- Γ is a cyclic sequence of formulas separated by ',',
- Σ is a multiset of cyclic sequences of formulas separated by ';',
- p and q are integers.

Γ as well as Σ may be eventually empty.

QUOTIENT RULES

$$\frac{\vdash^p_q \Gamma|\Sigma;(\Delta);(\Delta');\Xi}{\vdash^p_q \Gamma|\Sigma;(\Delta');(\Delta);\Xi}\text{ multiset} \qquad \frac{\vdash^p_q \Gamma,\Delta|\Sigma}{\vdash^p_q \Delta,\Gamma|\Sigma}\text{ cycle}$$

$$\frac{\vdash^p_q \Gamma|\Sigma}{\vdash^p_q \overline{\Gamma}|\overline{\Sigma}}\text{ invert}$$

$$\frac{\vdash^p_q |(\Gamma);\Sigma}{\vdash^p_q \Gamma|\Sigma}\text{ focus} \qquad \frac{\vdash^p_q \Gamma|\Sigma}{\vdash^p_q |(\Gamma);\Sigma}\text{ defocus}$$

Note that defocus rule may be viewed as a special case of the cylinder rule (with () as neutral w.r.t. ';').

IDENTITY GROUP

$$\frac{}{\vdash_0^0 A, A^\perp|}\text{ax.} \qquad \frac{\vdash_q^p \Gamma, A|\Sigma \qquad \vdash_{q'}^{p'} \Delta, A^\perp|\Xi}{\vdash_{q+q'}^{p+p'} \Gamma, \Delta|\Sigma;\Xi}\text{cut}$$

ORIENTABLE STRUCTURAL RULES

$$\frac{\vdash_q^p \Gamma, \Delta|\Sigma}{\vdash_q^p \Gamma|\Sigma;(\Delta)}\text{cylinder} \qquad \frac{\vdash_q^p \Gamma|\Sigma;(\Delta)}{\vdash_{q+1}^p \Gamma, \Delta|\Sigma}\text{torus}$$

NON-ORIENTABLE STRUCTURAL RULES

$$\frac{\vdash_q^p \Gamma, \Delta|\Sigma}{\vdash_q^{p+1} \Gamma, \overline{\Delta}|\Sigma}\text{Möbius} \qquad \frac{\vdash_q^p \Gamma|\Sigma;(\Delta)}{\vdash_q^{p+2} \Gamma, \overline{\Delta}|\Sigma}\text{Klein}$$

LOGICAL RULES

$$\frac{\vdash_q^p \Gamma, A, B|\Sigma}{\vdash_q^p \Gamma, A \parr B|\Sigma}\parr \qquad \frac{\vdash_q^p \Gamma, A|\Sigma \qquad \vdash_{q'}^{p'} \Delta, B|\Xi}{\vdash_{q+q'}^{p+p'} \Gamma, A\otimes B, \Delta|\Sigma;\Xi}\otimes$$

However, proving directly that the focalized system is equivalent to the intial one is not as easy as it seems because focussing is a strong constraint. For that purpose, we use an intermediary system fsPL where

- one deletes the defocus rule: $\dfrac{\vdash_q^p \Gamma|\Sigma}{\vdash_q^p |(\Gamma);\Sigma}$ defocus
- one adds the two following rules:

$$\dfrac{\vdash_q^p \Gamma,\Lambda,\Delta|\Sigma}{\vdash_{q+1}^p \Gamma,\Delta,\Lambda|\Sigma}\text{torus'} \qquad \dfrac{\vdash_q^p \Gamma,\Lambda,\Delta|\Sigma}{\vdash_q^{p+2} \Gamma,\Delta,\overline{\Lambda}|\Sigma}\text{Klein'}$$

The defocus rule is a special case of the cylinder rule and rules Klein' and torus' are derivable in the focalized sequent calculus.

Hence the following propositions may be proved by induction:

Proposition

A sequent is provable in fsPL iff it is provable in foc-sPL.

Definition (max-focalization)

A proof (in fsPL or foc-sPL) is *maximally focalized* iff there is no proof of the same sequent with longer sequences of cylinder rule applications.

Proposition

A sequent $\vdash_q^p \Gamma|\Sigma$ *is provable in fsPL iff it has a maximally focalized proof in foc-sPL.*
Cuts in a maximally focalized proof in foc-sPL of a sequent $\vdash_q^p \Gamma|\Sigma$ *may be eliminated.*

Phase Semantics

Phase semantics exist for Linear Logic and Non-Commutative Logic. What is the main difficulty when turning to a calculus of surfaces?

- The *orientation* has to be taken into account.
- The *context* cannot be neglected:
  - In NL, the non-commutative structure is an order variety. Hence a formula on which an operation is applied may be 'extracted' from its context: the structure of the semantics is close to what is required with Linear Logic.
  - This is no more true in the calculus of surfaces.

Hence,

- a *context phase space* Con(M) interprets sequents. Formulas are denoted by a subset Supp(M) of Con(M).
- orthogonality is defined wrt it.
- the denotation of the negation of a formula is the restriction to Supp(M) of its orthogonals.

The phase semantics is sound and complete.

Varieties and presentations

Any variety-presentation framework deals with two classes of objects: *varieties* and *presentations*, and with two basic operations of *composition* and *decomposition*.

A variety can always be decomposed into a presentation, simply by assuming a point x of its support as point of view. Conversely, two presentations α and β having disjoint supports, can always be composed in order to form a variety $\alpha \star \beta$.

- the composition $\star$ is associative and commutative with a neutral element.
- any variety-presentation framework induces a focalized system.

Relaxation

One can define on the set of varieties, a binary relation $\preccurlyeq$ called *relaxation*.

Relaxation aims to model transformations induced on sequents by structural rules:

➥A variety α relaxes a variety β if α can be rewritten into β through a series of structural rules.

Definition (relaxation on a system $\mathcal{S}$)

- **terms**: pq-permutations $\alpha, \beta, \gamma, \ldots$
- **rewriting rules**: *sPL* structural rules (cylinder, torus, Klein and Möbius)

Relaxation:

$$\alpha \preccurlyeq \beta \text{ iff } \alpha \rightsquigarrow^*_{\mathcal{S}} \beta$$

Relaxation

It induces a loss of information (hence relaxation):
the typical case is when α and β are two orders on the same set of points and β is obtained from α by weakening (relaxing) the structure of α.

- The decision of relaxation is essentially a trivial question for sets or partial orders,
- the problem of checking whether two *pq*-permutations are in relation of relaxation is not as trivial as before.

Relaxation

Example

$\alpha \preccurlyeq \beta$, where $\alpha = (a, b, c), (\overline{d})_{\langle 2,0 \rangle}$ and $\beta = (a, \overline{c}), (\overline{b}, \overline{d})_{\langle 3,1 \rangle}$.

$$\cfrac{\cfrac{\cfrac{(a, b, c), (\overline{d})_{\langle 2,0 \rangle}}{(a, \overline{c}, \overline{b}), (\overline{d})_{\langle 2,1 \rangle}}\text{ Möbius}}{(a, \overline{c}, \overline{b}, \overline{d})_{\langle 3,1 \rangle}}\text{ torus}}{(a, \overline{c}), (\overline{b}, \overline{d})_{\langle 3,1 \rangle}}\text{ cylinder}$$

Decision of Relaxation

$\Sigma_{\langle p,q\rangle} \preccurlyeq \Xi_{\langle p',q'\rangle}$? $\Rightarrow$ 'topologically minimal' path from the permutation Σ to the permutation Ξ

Definition (system $\mathcal{S}'$)

We aim to produce a chain $\Sigma \rightsquigarrow^*_{\mathcal{S}'} \Xi$.

- **terms**: permutations $\Sigma, \Xi, \ldots$
- **rules**: specific instances of the $\mathcal{S}$ rules:

$\Xi(a) = b$: $\quad \dfrac{\Sigma, (\Gamma, a, \Delta, b)}{\Sigma, (\Gamma, a, b), (\Delta)}$ cylinder $\quad \dfrac{\Sigma, (\Gamma, a), (\Delta, b)}{\Xi, (\Gamma, a, b, \Delta)}$ torus

$\Xi(a) = \overline{b}$: $\quad \dfrac{\Sigma, (\Gamma, a, \Delta, b)}{\Sigma, (\Gamma, a, \overline{b}, \overline{\Delta})}$ Möbius $\quad \dfrac{\Sigma, (\Gamma, a), (\Delta, b)}{\Sigma, (\Gamma, a, \overline{b}, \overline{\Delta})}$ Klein

Decision of Relaxation

Example

$\Sigma = (a, b, c), (\overline{d}) \rightsquigarrow^*_{\mathcal{S}'} \Xi = (a, \overline{c}), (\overline{b}, \overline{d})$.

$$\begin{array}{rl} \Xi(a) = \overline{c} & \dfrac{(a, b, c), (\overline{d})}{(a, \overline{c}, \overline{b}), (\overline{d})} \text{ Möbius} \\ \Xi(\overline{c}) = a & \dfrac{}{(a, \overline{c}), (\overline{b}), (\overline{d})} \text{ cylinder} \\ \Xi(\overline{b}) = \overline{d} & \dfrac{}{(a, \overline{c}), (\overline{b}, \overline{d})} \text{ torus} \end{array}$$

The chain $\mathscr{C} : \Sigma \rightsquigarrow^*_{\mathcal{S}'} \Xi$ 'topologically cost':

$$1 \text{ proj. plane (Möbius)} + 1 \text{ torus} \sim 3 \text{ proj. planes.}$$

Decision of Relaxation

Theorem

Any chain afforded by $\mathcal{S}'$ turns out to be minimal w.r.t. its 'topological cost'.

Proof.

Any chain $\Sigma \rightsquigarrow^*_{\mathcal{S}'} \Xi$ just 'mimics' the process of formation (through identification of paired edges) of the *quotient surface* $\mathscr{S}_\Sigma * \mathscr{S}_\Xi$, where:

- $\mathscr{S}_\Sigma$ is the surface denoted by $\Sigma_{\langle 0,0 \rangle}$
- $\mathscr{S}_\Xi$ is the surface denoted by $\Xi_{\langle 0,0 \rangle}$
- $\mathscr{S}_\Sigma * \mathscr{S}_\Xi$ is obtained by connecting $\mathscr{S}_\Sigma$ and $\mathscr{S}_\Xi$ through identification of a couple of paired edges

□

What do we have?

- We have a logical system that integrates **orientable** as well as **non-orientable** structural rules.
- We prove that the system keeps standard logical properties: cut elimination and focussing.
- We give also a few comments on *relaxation*, induced by structural transformations that may increase the topological genus of the transformed surface.

What remains to do?

- An extension of the correctness criterion (of Métayer or Melliès) for Permutative Logic as well as for sPL.
- A denotational semantics that (really !) relates logic and topology.
- A full study of rules, particularly singling out redundant rules (e.g. the Möbius rule).

Thanks for your attention